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On the number of prime factors of integers II.

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1. Introduction.

Let P be the set of all rational prime numbers, and $\{\pi_1,\cdots,\pi_k\}$ a family of subsets of P satisfying the following conditions:

(C_1) The sets $\pi_1,\cdots,\pi_k$ are mutually disjoint;

(C_2) The series $\sum_{p\in\pi_i}\frac{1}{p}$ $(i=1,\cdots,k)$ are divergent.

We need not suppose $\pi_1\cup\cdots\cup\pi_k=P$ for the following development. We shall suppose, except for in the last section, the family $\{\pi_1,\cdots,\pi_k\}$ as given once for all. The letter i will always represent one of the integers $1,\cdots,k$.

We denote by $\omega_i(n)$ the number of distinct prime factors of a positive integer n which belong to the set π_i:

$$\omega_i(n)=\sum_{p|n,p\in\pi_i}1.$$

We also put

$$y_i(n)=\sum_{p\leqq n,p\in\pi_i}\frac{1}{p},$$

and denote by n_0 the least positive integer for which $y_i(n_0)>0$ $(i=1,\cdots,k)$.[1] We further put, for $n\geqq n_0$,

$$u_i(n)=\frac{\omega_i(n)-y_i(n)}{\sqrt{y_i(n)}}.$$

Then, to each integer $n\geqq n_0$, there corresponds a point $U(n)=(u_1(n),\cdots,u_k(n))$ in the space R^k of k dimensions. Let E be a Jordan-measurable set, bounded or unbounded, in R^k, and let $A(x;E)$ denote the number of integers $n, n_0\leqq n\leqq x$, for which the corresponding points $U(n)$ belong to the set E.

1) When it is desirable to emphasize that we are considering the relevant formulas for $i=1,\cdots,k$ simultaneously, we add the expression '$(i=1,\cdots,k)$' to indicate the simultaneousness.

Now the purpose of this prper is to prove the following Main Theorem:

THEOREM A.

$$\lim_{x\to\infty}\frac{A(x;E)}{x}=(2\pi)^{-\frac{k}{2}}\int_E \exp\left(-\frac{1}{2}\sum_{i=1}^{k}u_i^2\right)du_1\cdots du_k.$$ [2)]

This is a generalization of a result of Erdös and Kac [3], of which we have given another generalization in a different direction in our previous paper I.[3)] Our method of proof is based on Brun's sieve method like in Erdös [1] and [2], and the probability theory will be nowhere used, whereas Erdös and Kac [3] makes essential use of this theory. We could prove our Theorem A without using the inequalities such as Lemmas 1 and 2 below, if we impose some additional condition on our family $\{\pi_1,\cdots,\pi_k\}$.[4)] But, in order to prove our Theorem A in the present form, we had to extend the inequalities (our Lemma 1), used by Erdös [1] and Landau [5], to our Lemma 2, on ground of which we could then proceed along the same line as in Erdös [2].

We shall, in section 2, prove Theorem A, and, in section 3, refer to some special cases of Theorem A.

This paper is self-contained; it may be read independently of Erdös [1], [2], and I; we shall only quote the well-known formula (8) during the proof of Theorem A in section 2.[5)]

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2. The proof of the main theorem.

We shall first prove some inequalities involving binomial coefficients which will be used in Brun's sieve method.

LEMMA 1.[6)] *Let a and b be non-negative integers. Then*

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2) The letter π without subscript denotes, as usual, the number 3.14⋯

3) I. e. Tanaka [6].

4) Such as the following:

$$(C_3)\quad \log y_i(n)=o\{\sqrt{y_j(n)}\}\quad \text{for } i,j=1,\cdots,k\,;\, i\neq j.$$

5) We quote also the formula (18), but this is not used in the proof of Theorem A.

6) Cf. Erdös [1], p. 536, and Landau [5], p. 71, Satz 116.

$$\sum_{c=0}^{b}(-1)^c\binom{a}{c}^{7)}\begin{cases} =1, & \text{when } a=0\,, \\ \geqq 0, & \text{when } a>0 \text{ and } b \text{ is even}\,, \\ \leqq 0, & \text{when } a>0 \text{ and } b \text{ is odd}\,. \end{cases}$$

PROOF. The case $a=0$ is trivial. The cases $a>0$ follow at once from the formula

$$\sum_{c=0}^{b}(-1)^c\binom{a}{c}=(-1)^b\binom{a-1}{b}\,.$$

LEMMA 2. *Let $a_i(i=1,\cdots,k)$ be non-negative integers, and $b_i(i=1,\cdots,k)$ be non-negative even integers. Let*

$$\begin{aligned}\gamma&=\gamma(a_1,\cdots,a_k\,;\,b_1,\cdots,b_k)\\ &=\sum_{j=1}^{k}\left\{\sum_{c_j=0}^{b_j+1}(-1)^{c_j}\binom{a_j}{c_j}\cdot\prod_{\substack{i=1\\ i\neq j}}^{k}\sum_{c_i=0}^{b_i}(-1)^{c_i}\binom{a_i}{c_i}\right\}\\ &\quad-(k-1)\prod_{i=1}^{k}\sum_{c_i=0}^{b_i}(-1)^{c_i}\binom{a_i}{c_i}\,.\end{aligned}$$

Then

$$\gamma\begin{cases} =1, & \text{when } a_i=0\ (i=1,\cdots,k)\,, \\ \leqq 0, & \text{when at least one of the } a_i \text{ is positive}\,. \end{cases}$$

PROOF. The case $a_i=0$ $(i=1,\cdots,k)$ follows at once from the case $a=0$ of Lemma 1.

Now suppose that at least one of the a_i is positive. Without loss of generality, we can assume that $a_i>0$ $(i=1,\cdots,\kappa)$ and $a_i=0$ $(i=\kappa+1,\cdots,k)$. Then, applying again the case $a=0$ of Lemma 1, we have

$$\gamma=\sum_{j=1}^{\kappa}\left\{\sum_{c_j=0}^{b_j+1}(-1)^{c_j}\binom{a_j}{c_j}\cdot\prod_{\substack{i=1\\ i\neq j}}^{\kappa}\sum_{c_i=0}^{b_i}(-1)^{c_i}\binom{a_i}{c_i}\right\}-(\kappa-1)\prod_{i=1}^{\kappa}\sum_{c_i=0}^{b_i}(-1)^{c_i}\binom{a_i}{c_i}\,,$$

from which, applying this time the cases $a>0$ of Lemma 1, we see that $\gamma\leqq 0$. Thus the lemma is proved.

Henceforth, let x be a positive variable which will be taken sufficiently large as occasion demands. Now we define some functions and sets which will be used in the sequel.

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7) $\binom{a}{0}=1$, and $\binom{a}{c}=0$ for integers a, c for which $0\leqq a<c$.

We put

$$y_i(x) = \sum_{p \leqq x, p \in \pi_i} \frac{1}{p}.$$

This coincides with the definition of $y_i(n)$ in section 1, and the condition (C_2) is equivalent with: '$y_i(x)$ $(i=1,\cdots,k)$ tend to infinity with x.'

We define $\pi_i'(x)$ to be the set consisting of the p's for which

$$p \in \pi_i \text{ and } e^{4y_i(x)} < p < x^{1/\{20ky_i(x)\}}.$$

We denote by $\omega_i'(n;x)$ the number of distinct prime factors of a positive integer n which belong to the set $\pi_i'(x)$:

$$\omega_i'(n;x) = \sum_{p|n, p \in \pi_i'(x)} 1.$$

We put

$$z_i(x) = \sum_{p \in \pi_i'(x)} \frac{1}{p}.$$

We obviously have $z_i(x) \leqq y_i(x)$ $(i=1,\cdots,k)$ for sufficiently large values of x. Henceforth, we consider only such values of x.

For any positive integer t, we define $\mathfrak{M}_i(x;t)$ to be the set consisting of positive integers m which satisfy the following conditions:

m is composed only of primes belonging to the set $\pi_i'(x)$;

m is squarefree;

m has t prime factors.

For any positive integers t_i $(i=1,\cdots,k)$, we denote by $G(x;t_1,\cdots,t_k)$ the number of positive integers $n \leqq x$ for which $\omega_i'(n;x) = t_i$ $(i=1,\cdots,k)$.

For any positive integers m_i $(i=1,\cdots,k)$ such that $m_i \in \mathfrak{M}_i(x;t_i)$ $(i=1,\cdots,k)$ with some positive integers $t_i (i=1,\cdots,k)$, we denote by $H(x;m_1,\cdots,m_k)$ the number of positive integers $n \leqq x$ for which

$$\prod_{p|n, p \in \pi_i'(x)} p = m_i \qquad (i=1,\cdots,k).$$

For any positive integers $m_i (i=1,\cdots,k)$ such that $m_i \in \mathfrak{M}_i(x;t_i)$ $(i=1,\cdots,k)$ with some positive integers $t_i(i=1,\cdots,k)$, and for any positive integers $T_i(i=1,\cdots,k)$, we put

$$K_0(x;m_1,\cdots,m_k;T_1,\cdots,T_k)$$
$$= \sum_{\tau_1=0}^{2T_1} \cdots \sum_{\tau_k=0}^{2T_k} (-1)^{\tau_1+\cdots+\tau_k} L(x;m_1,\cdots,m_k;\tau_1,\cdots,\tau_k),$$

where

$$L(x\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k)$$

$$=\sum_{\substack{\mu_1\in\mathfrak{M}_1(x;\tau_1)\\(\mu_1,m_1)=1}}\cdots\sum_{\substack{\mu_k\in\mathfrak{M}_k(x;\tau_k)\\(\mu_k,m_k)=1}}{}^{8)}\left[\frac{x}{m_1\cdots m_k\mu_1\cdots\mu_k}\right].$$

Here we denote by the square brackets $[*]$ the largest integer not exceeding $*$. (Gauss's notation.) Also we put

$$K_i(x\,;m_1,\cdots,m_k\,;T_1,\cdots,T_k)$$

$$=\sum_{\tau_1=0}^{2T_1}\cdots\sum_{\tau_i=0}^{2T_i+1}\cdots\sum_{\tau_k=0}^{2T_k}(-1)^{\tau_1+\cdots+\tau_k}L(x\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k)\,,$$

where the summation-variables $\tau_j(j=1,\cdots,k\,;j\neq i)$ run through the integers $0,\cdots,2T_j$ respectively, and in particular the summation-variable τ_i runs through the integers $0,\cdots,2T_i+1$.

Now we prove

LEMMA 3. *Let $m_i(i=1,\cdots,k)$ be positive integers such that $m_i\in\mathfrak{M}_i(x\,;t_i)$ $(i=1,\cdots,k)$ with some positive integers $t_i(i=1,\cdots,k)$, and let $T_i(i=1,\cdots,k)$ be any positive integers. Then*

$$\sum_{i=1}^{k}K_i(x\,;m_1,\cdots,m_k\,;T_1,\cdots,T_k)-(k-1)K_0(x\,;m_1,\cdots,m_k\,;T_1,\cdots,T_k)$$

$$\leqq H(x\,;m_1,\cdots,m_k)\leqq K_0(x\,;m_1,\cdots,m_k\,;T_1,\cdots,T_k)\,.$$

PROOF. (By Brun's sieve method.)

If we write

$$\left[\frac{x}{m_1\cdots m_k\mu_1\cdots\mu_k}\right]=\sum_{\substack{n\leqq x\\ m_1\cdots m_k\mu_1\cdots\mu_k|n}}1$$

in the definition of $L(x\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k)$, then we have

$$L(x\,;m_1,\cdots,m_k\,;\tau_1,\cdots,\tau_k)=\sum_{\substack{\mu_1\in\mathfrak{M}_1(x;\tau_1)\\(\mu_1,m_1)=1}}\cdots\sum_{\substack{\mu_k\in\mathfrak{M}_k(x;\tau_k)\\(\mu_k,m_k)=1}}\sum_{\substack{n\leqq x\\ m_1\cdots m_k\mu_1\cdots\mu_k|n}}1$$

$$=\sum_{\substack{n\leqq x\\ m_1\cdots m_k|n}}\sum_{\substack{\mu_1\in\mathfrak{M}_1(x;\tau_1)\\(\mu_1,m_1)=1\\ \mu_1|n}}\cdots\sum_{\substack{\mu_k\in\mathfrak{M}_k(x;\tau_k)\\(\mu_k,m_k)=1\\ \mu_k|n}}1=\sum_{\substack{n\leqq x\\ m_1\cdots m_k|n}}\prod_{i=1}^{k}\binom{\omega_i'(n\,;x)-t_i}{\tau_i}\,.$$

8) We mean by $\mathfrak{M}_i(x\,;0)$ the set consisting only of the number 1.

Hence

$$K_0(x; m_1, \cdots, m_k; T_1, \cdots, T_k) = \sum_{\substack{n \leqq x \\ m_1 \cdots m_k | n}} \delta(n; x),$$

where

$$\delta(n; x) = \prod_{i=1}^{k} \sum_{\tau_i=0}^{2T_i} (-1)^{\tau_i} \binom{\omega_i'(n; x) - t_i}{\tau_i}.$$

Also

$$K_j(x; m_1, \cdots, m_k; T_1, \cdots, T_k)$$
$$= \sum_{\substack{n \leqq x \\ m_1 \cdots m_k | n}} \left\{ \sum_{\tau_j=0}^{2T_j+1} (-1)^{\tau_j} \binom{\omega_j'(n; x) - t_j}{\tau_j} \cdot \prod_{\substack{i=1 \\ i \neq j}}^{k} \sum_{\tau_i=0}^{2T_i} (-1)^{\tau_i} \binom{\omega_i'(n; x) - t_i}{\tau_i} \right\},$$

so that

$$\sum_{j=1}^{k} K_j(x; m_1, \cdots, m_k; T_1, \cdots, T_k) - (k-1) K_0(x; m_1, \cdots, m_k; T_1, \cdots, T_k)$$
$$= \sum_{\substack{n \leqq x \\ m_1 \cdots m_k | n}} \delta'(n; x),$$

where

$$\delta'(n; x)$$
$$= \sum_{j=1}^{k} \left\{ \sum_{\tau_j=0}^{2T_j+1} (-1)^{\tau_j} \binom{\omega_j'(n; x) - t_j}{\tau_j} \cdot \prod_{\substack{i=1 \\ i \neq j}}^{k} \sum_{\tau_i=0}^{2T_i} (-1)^{\tau_i} \binom{\omega_i'(n; x) - t_i}{\tau_i} \right\}$$
$$-(k-1) \prod_{i=1}^{k} \sum_{\tau_i=0}^{2T_i} (-1)^{\tau_i} \binom{\omega_i'(n; x) - t_i}{\tau_i}.$$

The functions $\delta(n; x)$ and $\delta'(n; x)$ are defined for positive integers $n \leqq x$ such that $m_1 \cdots m_k | n$ and, as to their values, we can conclude from Lemmas 1 and 2 as follows:

$\delta(n; x) = \delta'(n; x) = 1$ for positive integers $n \leqq x$ such that $m_1 \cdots m_k | n$ and $\omega_i'(n; x) = t_i$ $(i = 1, \cdots, k)$, that is,

$$\prod_{p | n, p \in \pi_i'(x)} p = m_i \qquad (i = 1, \cdots, k).$$

$\delta(n; x) \geqq 0$ and $\delta'(n; x) \leqq 0$ for positive integers $n \leqq x$ such that $m_1 \cdots m_k | n$ and $\omega_i'(n; x) > t_i$ for at least one i.

The lemma now follows from this fact and the definition of $H(x; m_1, \cdots, m_k)$.

LEMMA 4. *Let m_i, t_i $(i=1,\cdots,k)$ be positive integers such that $m_i \in \mathfrak{M}_i(x; t_i)$, $t_i < 2y_i(x)$ $(i=1,\cdots,k)$. Then*

$$H(x; m_1,\cdots, m_k) = \frac{xe^{-\{z_1(x)+\cdots+z_k(x)\}}}{\varphi(m_1\cdots m_k)} \{1+o(1)\},$$

where $\varphi(m_1\cdots m_k)$ is Euler's function, and the term $o(1)$ tends to zero, as $x\to\infty$, uniformly in $m_i \in \mathfrak{M}_i(x; t_i)$ with $t_i < 2y_i(x)$ $(i=1,\cdots,k)$.[9]

PROOF. We put

$$L'(x; m_1,\cdots, m_k; \tau_1,\cdots, \tau_k)$$

$$= \sum_{\substack{\mu_1\in\mathfrak{M}_1(x;\tau_1)\\(\mu_1,m_1)=1}} \cdots \sum_{\substack{\mu_k\in\mathfrak{M}_k(x;\tau_k)\\(\mu_k,m_k)=1}} \frac{x}{m_1\cdots m_k\mu_1\cdots\mu_k},$$

removing the square brackets of the summands of $L(x; m_1,\cdots, m_k; \tau_1,\cdots, \tau_k)$, and put further

$$K_0'(x; m_1,\cdots, m_k; T_1,\cdots, T_k)$$

$$= \sum_{\tau_1=0}^{2T_1} \cdots \sum_{\tau_k=0}^{2T_k} (-1)^{\tau_1+\cdots+\tau_k} L'(x; m_1,\cdots, m_k; \tau_1,\cdots, \tau_k).$$

For a while, T_i $(i=1,\cdots,k)$ may be any positive integers, and will be specified later on as suitable functions of x.

Since $[*] \leqq * < [*]+1$,

$$L(x; m_1,\cdots, m_k; \tau_1,\cdots, \tau_k) \leqq L'(x; m_1,\cdots, m_k; \tau_1,\cdots, \tau_k)$$

$$\leqq L(x; m_1,\cdots, m_k; \tau_1,\cdots, \tau_k) + \sum_{\substack{\mu_1\in\mathfrak{M}_1(x;\tau_1)\\(\mu_1,m_1)=1}} \cdots \sum_{\substack{\mu_k\in\mathfrak{M}_k(x;\tau_k)\\(\mu_k,m_k)=1}} 1$$

$$= L(x; m_1,\cdots, m_k; \tau_1,\cdots, \tau_k) + \prod_{i=1}^{k} \binom{|\pi_i'(x)|-t_i}{\tau_i}$$

$$\leqq L(x; m_1,\cdots, m_k; \tau_1,\cdots, \tau_k) + \prod_{i=1}^{k} \{|\pi_i'(x)|-1\}^{\tau_i},$$

where $|\pi_i'(x)|$ denotes the number of primes belonging to the set $\pi_i'(x)$. Hence

9) More precisely we mean the following by this expression: Since the term $o(1)$ depends on x and m_i $(i=1,\cdots,k)$, we shall put $o(1)=\delta(x; m_1,\cdots, m_k)$. Then we mean that we can take, corresponding to an arbitrarily given $\varepsilon>0$, a positive number $x_0=x_0(\varepsilon)$ such that, when $x>x_0$ and $m_i \in \mathfrak{M}_i(x; t_i)$ with $t_i < 2y_i(x)$ $(i=1,\cdots,k)$, we have $|\delta(x; m_1,\cdots, m_k)| < \varepsilon$. The uniformity in Lemma 5 is to be interpreted in the similar way.

$$(1)\qquad |K_0(x;m_1,\cdots,m_k;T_1,\cdots,T_k)-K_0'(x;m_1,\cdots,m_k;T_1,\cdots,T_k)|$$
$$\leqq \prod_{i=1}^{k}\sum_{\tau_i=0}^{2T_i}\{|\pi_i'(x)|-1\}^{\tau_i}\leqq \prod_{i=1}^{k}|\pi_i'(x)|^{2T_i}$$

Thus we have estimated the error introduced in the value of $K_0(x;m_1,\cdots,m_k;T_1,\cdots,T_k)$ by reason of removing the square brackets of the summands of $L(x;m_1,\cdots,m_k;\tau_1,\cdots,\tau_k)$.

Now we put for brevity

$$M_i(x;m_i;\tau_i)=\sum_{\substack{\mu_i\in\mathfrak{M}_i(x;\tau_i)\\(\mu_i,m_i)=1}}\frac{1}{\mu_i},$$

Then

$$L'(x;m_1,\cdots,m_k;\tau_1,\cdots,\tau_k)=\frac{x}{m_1\cdots m_k}\prod_{i=1}^{k}M_i(x;m_i;\tau_i),$$

so that

$$(2)\qquad K_0'(x;m_1,\cdots,m_k;T_1,\cdots,T_k)$$
$$=\frac{x}{m_1\cdots m_k}\prod_{i=1}^{k}\sum_{\tau_i=0}^{2T_i}(-1)^{\tau_i}M_i(x;m_i;\tau_i).$$

Also we obviously have

$$\sum_{\tau_i=0}^{\infty}(-1)^{\tau_i}M_i(x;m_i;\tau_i)^{10)}=\prod_{\substack{p\in\pi_i'(x)\\p\nmid m_i}}\left(1-\frac{1}{p}\right),$$

so that

$$(3)\qquad \left|\sum_{\tau_i=0}^{2T_i}(-1)^{\tau_i}M_i(x;m_i;\tau_i)-\prod_{\substack{p\in\pi_i'(x)\\p\nmid m_i}}\left(1-\frac{1}{p}\right)\right|\leqq\sum_{\tau_i=2T_i+1}^{\infty}M_i(x;m_i;\tau_i).$$

Now, recalling the definition of $z_i(x)$, and that we are considering only so large values of x that $z_i(x)\leqq y_i(x)$ holds, we have

$$M_i(x;m_i;\tau_i)\leqq\frac{\{z_i(x)\}^{\tau_i}}{\tau_i!}\leqq\frac{\{y_i(x)\}^{\tau_i}}{\tau_i!},$$

which implies

$$\sum_{\tau_i=2T_i+1}^{\infty}M_i(x;m_i;\tau_i)\leqq\sum_{\tau_i=2T_i+1}^{\infty}\frac{\{y_i(x)\}^{\tau_i}}{\tau_i!}.$$

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10) This sum is substantially finite. In fact, when $\tau_i>|\pi_i'(x)|-t_i$, $M_i(x;m_i;\tau_i)=0$ as an empty sum.

Till now, T_i may be any positive integer. Here we put

(4) $$T_i=[4y_i(x)]+1.$$

Then

$$\sum_{\tau_i=2T_i+1}^{\infty}\frac{\{y_i(x)\}^{\tau_i}}{\tau_i!}=\frac{\{y_i(x)\}^{2T_i+1}}{(2T_i+1)!}\left\{1+\frac{y_i(x)}{2T_i+2}+\frac{y_i^2(x)}{(2T_i+2)(2T_i+3)}+\cdots\right\}$$

$$<\frac{\{y_i(x)\}^{2T_i+1}}{(2T_i+1)!}\left(1+\frac{1}{2}+\frac{1}{4}+\cdots\right)=\frac{2\{y_i(x)\}^{2T_i+1}}{(2T_i+1)!}$$

$$<\frac{2e^{2T_i+1}\{y_i(x)\}^{2T_i+1}}{(2T_i+1)^{2T_i+1}}{}^{11)}=\frac{2\{ey_i(x)\}^{2[4y_i(x)]+3}}{\{2[4y_i(x)]+3\}^{2[4y_i(x)]+3}}$$

$$<\frac{2\{ey_i(x)\}^{8y_i(x)+3}}{\{8y_i(x)\}^{8y_i(x)}}=2e^3y_i^3(x)\left(\frac{e}{8}\right)^{8y_i(x)}$$

$$<2e^3y_i^3(x)e^{-8y_i(x)}=o(e^{-y_i(x)}).$$

Thus we obtain, as the estimation of the right-hand side of (3),

(5) $$\sum_{\tau_i=2T_i+1}^{\infty}M_i(x;m_i;\tau_i)=o(e^{-y_i(x)}).$$

Here and in the rest of the proof of the present lemma, the positive integers T_i $(i=1,\cdots,k)$ are always considered as the functions of x defined by (4). Next we shall transform the product on the left-hand side of (3). Recalling the definition of the set $\pi_i'(x)$, we have

$$\sum_{p\in\pi_i'(x)}\frac{1}{p^2}<\sum_{p>\exp\{4y_i(x)\}}\frac{1}{p^2}=O(e^{-4y_i(x)})=o(1),$$

and hence

$$\prod_{p\in\pi_i'(x)}\left(1-\frac{1}{p}\right)=\exp\left\{\sum_{p\in\pi_i'(x)}\log\left(1-\frac{1}{p}\right)\right\}$$

$$=\exp\left\{-\sum_{p\in\pi_i'(x)}\frac{1}{p}+O\left(\sum_{p\in\pi_i'(x)}\frac{1}{p^2}\right)\right\}$$

$$=\exp\{-z_i(x)+o(1)\},$$

which implies that

$$\prod_{\substack{p\in\pi_i'(x)\\ p\nmid m_i}}\left(1-\frac{1}{p}\right)=\{1+o(1)\}e^{-z_i(x)}\prod_{p\mid m_i}\left(1-\frac{1}{p}\right)^{-1}.$$

11) By the well-known formula $t!>t^te^{-t}$ for positive integer t.

By this and (3) and (5),

$$\sum_{\tau_i=0}^{2T_i}(-1)^{\tau_i}M_i(x;m_i;\tau_i)=\{1+o(1)\}e^{-z_i(x)}\prod_{p|m_i}\left(1-\frac{1}{p}\right)^{-1}+o(e^{-y_i(x)})\,.$$

Moreover, since $z_i(x)\leqq y_i(x)$, the term $o(e^{-y_i(x)})$ in this formula can be absorbed in the first term of the right-hand side. Hence

$$\sum_{\tau_i=0}^{2T_i}(-1)^{\tau_i}M_i(x;m_i;\tau_i)=(1+o(1)\}e^{-z_i(x)}\prod_{p|m_i}\left(1-\frac{1}{p}\right)^{-1}.$$

Putting this in (2), we now obtain[12)]

$$K_0'(x;m_1,\cdots,m_k;T_1,\cdots,T_k)=\frac{xe^{-\{z_1(x)+\cdots+z_k(x)\}}}{\varphi(m_1\cdots m_k)}\{1+o(1)\}\,. \tag{6}$$

Our next step is to obtain, from (6), a similar formula for $K_0(x;m_1,\cdots,m_k;T_1,\cdots,T_k)$, and (1) will serve for this purpose. Now, since T_i is defined by (4), we have $T_i<5y_i(x)$ for sufficiently large values of x. Also $|\pi_i'(x)|<x^{1/20ky_i(x)}$ by the definition of the set $\pi_i'(x)$. Hence

$$\prod_{i=1}^{k}|\pi_i'(x)|^{2T_i}<\prod_{i=1}^{k}(x^{1/20ky_i(x)})^{10y_i(x)}=\sqrt{x}.$$

Hence, by (1) and (6),

$$K_0(x;m_1,\cdots,m_k;T_1,\cdots,T_k)=\frac{xe^{-\{z_1(x)+\cdots+z_k(x)\}}}{\varphi(m_1\cdots m_k)}\{1+o(1)\}+O(\sqrt{x})\,. \tag{7}$$

As a matter of fact, the term $O(\sqrt{x})$ in this formula can be absorbed in the first term on the right-hand side. To see this, we quote the well-known formula[13)]

$$\sum_{p\leqq x}\frac{1}{p}=\log\log x+O(1)\,. \tag{8}$$

By this formula and

$$z_i(x)\leqq y_i(x)\leqq\sum_{p\leqq x}\frac{1}{p}\,,$$

we have

$$z_i(x)\leqq\log\log x+O(1)\,,$$

12) Notice that $m_i(i=1,\cdots,k)$ are squarefree, and relatively prime in pairs

13) Cf., for instance, Landau [4], pp. 100-102, § 28.

which implies that

$$e^{-z_i(x)} > \frac{1}{c \log x},$$

where c is a suitable positive number independent of x. On the other hand, since m_i is assumed to belong to the set $\mathfrak{M}_i(x; t_i)$ with $t_i < 2y_i(x)$, we have

$$m_1 \cdots m_k < \prod_{i=1}^{k} (x^{1/20ky_i(x)})^{2y_i(x)} = x^{1/10},$$

recalling the definitions of the sets $\pi_i'(x)$, $\mathfrak{M}_i(x; t_i)$. Thus

$$\frac{xe^{-\{z_1(x)+\cdots+z_k(x)\}}}{m_1 \cdots m_k} > \frac{x^{9/10}}{c^k \log^k x},$$

and a fortiori

$$\frac{xe^{-\{z_1(x)+\cdots+z_k(x)\}}}{\varphi(m_1 \cdots m_k)} > \frac{x^{9/10}}{c^k \log^k x},$$

which now shows that we may omit the term $O(\sqrt{x})$ in (7), and write

$$K_0(x; m_1, \cdots, m_k; T_1, \cdots, T_k) = \frac{xe^{-\{z_1(x)+\cdots+z_k(x)\}}}{\varphi(m_1 \cdots m_k)} \{1 + o(1)\}. \tag{9}$$

During the above argument, I have not referred to the uniformity of the O aud o terms in m_i as yet. But, if we review the course through which (9) has been derived, then we easily see that the term $o(1)$ on the right-hand side of (9) tends to zero, as $x \to \infty$, uniformly in $m_i \in \mathfrak{M}_i(x; t_i)$ with $t_i < 2y_i(x)$ $(i = 1, \cdots, k)$.

Quite similarly we can derive

$$K_i(x; m_1, \cdots, m_k; T_1, \cdots, T_k) = \frac{xe^{-\{z_1(x)+\cdots+z_k(x)\}}}{\varphi(m_1 \cdots m_k)} \{1 + o(1)\}, \tag{10}$$

the term $o(1)$ tending uniformly to zero, as $x \to \infty$, in the same sense as in (9).

Our Lemma 3, which was proved by the sieve method, yields now at once Lemma 4 in view of (9) and (10).

LEMMA 5. *Let t_i $(i = 1, \cdots, k)$ be positive integers such that $t_i < 2y_i(x)$ $(i = 1, \cdots, k)$. Then*

$$G(x; t_1, \cdots, t_k) = \frac{x\{z_1(x)\}^{t_1} \cdots \{z_k(x)\}^{t_k} e^{-\{z_1(x)+\cdots+z_k(x)\}}}{t_1! \cdots t_k!} \{1 + o(1)\},$$

the term $o(1)$ *tending to zero, as* $x\to\infty$, *uniformly in* $t_i<2y_i(x)$ $(i=1,\cdots,k)$.

PROOF. We have

$$G(x;t_1,\cdots,t_k)=\sum_{m_1\in\mathfrak{M}_1(x;t_1)}\cdots\sum_{m_k\in\mathfrak{M}_k(x;t_k)}H(x;m_1,\cdots,m_k),$$

by the definitions of $G(x;t_1,\cdots,t_k)$ and $H(x;m_1,\cdots,m_k)$. Hence by Lemma 4,

$$G(x;t_1,\cdots,t_k)=\{1+o(1)\}xe^{-\{z_1(x)+\cdots+z_k(x)\}}\prod_{i=1}^{k}\sum_{m_i\in\mathfrak{M}_i(x;t_i)}\frac{1}{\varphi(m_i)},\tag{11}$$

where the term $o(1)$ tends to zero, as $x\to\infty$, uniformly in $t_i<2y_i(x)$ $(i=1,\cdots,k)$.

We shall be, for a while, concerned with the inner sums on the right-hand side of (11). Now by the multinomial theorem,

$$\sum_{m_i\in\mathfrak{M}_i(x;t_i)}\frac{1}{m_i}\leqq\frac{\{z_i(x)\}^{t_i}}{t_i!}\leqq\sum_{m_i\in\mathfrak{M}_i(x;t_i)}\frac{1}{m_i}+\sum_w{}'\frac{1}{w},\tag{12}$$

where the prime attached to the second summation on the right-hand side means that the summation-variable w runs through positive integers satisfying the following conditions:

w is composed only of primes belonging to the set $\pi_i'(x)$;

w is not squarefree;

w has t_i prime factors, multiple factors being counted multiply.

For each of these w, we can put $w=d^2q$ with positive integers d and q satisfying the following conditions:

d is composed only of primes belonging to the set $\pi_i'(x)$, and $d>1$, so that $d>e^{4y_i(x)}$ by the definition of the set $\pi_i'(x)$; q is composed only of primes belonging to the set $\pi_i'(x)$, and is squarefree.

Hence we have

$$\sum_w{}'\frac{1}{w}\leqq\sum_d\frac{1}{d^2}\sum_q\frac{1}{q},$$

where

$$\sum_d\frac{1}{d^2}\leqq\sum_{a=[\exp\{4y_i(x)\}]+1}^{\infty}\frac{1}{a^2}=O(e^{-4y_i(x)}),$$

and, by the definition of $z_i(x)$,

$$\sum_q\frac{1}{q}\leqq 1+z_i(x)+\frac{z_i^2(x)}{2!}+\cdots=e^{z_i(x)}\leqq e^{y_i(x)}.$$

Thus we obtain

(13) $$\sum_w{}' \frac{1}{w} = O(e^{-3y_i(x)}).$$

On the other hand, by (8) and by the definitions of $z_i(x)$ and of the set $\pi_i'(x)$, we have

(14) $$\begin{aligned} y_i(x) - z_i(x) &= \sum_{p \le x,\, p \in \pi_i - \pi_i'(x)} \frac{1}{p} \\ &\le \sum_{p \le \exp\{4y_i(x)\}} \frac{1}{p} + \sum_{\exp\{\log x/20ky_i(x)\} \le p \le x} \frac{1}{p} \\ &= \log 4y_i(x) + \log\log x - \log \frac{\log x}{20ky_i(x)} + O(1) \\ &= O\{\log y_i(x)\}. \end{aligned}$$

Hence, for sufficiently large values of x, the assumption $t_i < 2y_i(x)$ implies $t_i < ez_i(x)$, and therefore implies

$$\frac{\{z_i(x)\}^{t_i}}{t_i!} > \left(\frac{t_i}{e}\right)^{t_i} \cdot \frac{1}{t_i^{t_i}} = e^{-t_i} > e^{-2y_i(x)}.$$

Now, by this and (13), we can write

$$\sum_w{}' \frac{1}{w} = \frac{\{z_i(x)\}^{t_i}}{t_i!} \cdot O(e^{-y_i(x)}),$$

and a fortiori

$$\sum_w{}' \frac{1}{w} = \frac{\{z_i(x)\}^{t_i}}{t_i!} \cdot o(1),$$

which, combined with (12), gives

(15) $$\sum_{m_i \in \mathfrak{M}_i(x;\, t_i)} \frac{1}{m_i} = \frac{\{z_i(x)\}^{t_i}}{t_i!} \{1 + o(1)\}.$$

Here we can replace the summands $1/m_i$ by $1/\varphi(m_i)$. In fact, since we assume that $t_i < 2y_i(x)$, on recalling the definitions of the sets $\pi_i'(x)$ and $\mathfrak{M}_i(x;t_i)$, we see that the number of prime factors of $m_i \in \mathfrak{M}_i(x;t_i)$ is less than $2y_i(x)$, and each of the prime factors is greater than $e^{4y_i(x)}$. Hence

$$\begin{aligned} 1 \le \frac{m_i}{\varphi(m_i)} &= \prod_{p \mid m_i} \left(1 - \frac{1}{p}\right)^{-1} \le \prod_{p \mid m_i} \left(1 + \frac{2}{p}\right) \\ &< \{1 + 2e^{-4y_i(x)}\}^{2y_i(x)} = 1 + O\{y_i(x) e^{-4y_i(x)}\} = 1 + o(1). \end{aligned}$$

From this and (15) we now obtain

$$\sum_{m_i\in\mathfrak{M}_i(x;\,t_i)}\frac{1}{\varphi(m_i)}=\frac{\{z_i(x)\}^{t_i}}{t_i!}\{1+o(1)\}\,. \tag{16}$$

Furthermore, if we review the above process of deriving this formula, we easily see that the term $o(1)$ tends to zere, as $x\to\infty$, uniformly in $t_i<2y_i(x)$.

Finally, putting (16) in (11) we obtain the desired lemma.

LEMMA 6. *Let $\alpha_i<\beta_i$ $(i=1,\cdots,k)$ be arbitrarily given but fixed real numbers. Let t_i $(i=1,\cdots,k)$ be positive integers such that $t_i=z_i(x)+u_i\sqrt{z_i(x)}$ with $\alpha_i<u_i<\beta_i$ $(i=1,\cdots,k)$. Then*

$$G(x\,;t_1,\cdots,t_k)$$
$$=(2\pi)^{-\frac{k}{2}}x\{z_1(x)\cdots z_k(x)\}^{-\frac{1}{2}}e^{-\frac{1}{2}(u_1^2+\cdots+u_k^2)}\{1+o(1)\}\,,$$

the term $o(1)$ tending to zero, as $x\to\infty$, uniformly in u_i $(i=1,\cdots,k)$ with $\alpha_i<u_i<\beta_i$ $(i=1,\cdots,k)$.

PROOF. In the Stirling's formula

$$t\,!=\sqrt{2\pi}t^{t+\frac{1}{2}}e^{-t}\left\{1+O\left(\frac{1}{t}\right)\right\}\,,$$

we put $t=z+u\sqrt{z}$, and consider large values of z, leaving u contained in a finite interval, then easy calculations give

$$t\,!=\sqrt{2\pi}\,z^{z+u\sqrt{z}+\frac{1}{2}}\,e^{-z+\frac{u^2}{2}}\left\{1+O\left(\frac{1}{\sqrt{z}}\right)\right\}\,,$$

or

$$\frac{z^te^{-z}}{t\,!}=\frac{e^{-\frac{u^2}{2}}}{\sqrt{2\pi z}}\left\{1+O\left(\frac{1}{\sqrt{z}}\right)\right\}$$

Here we put $t=t_i$, $z=z_i(x)$, $u=u_i$, and combining thus obtained formulas for $i=1,\cdots,k$, we get

$$\frac{\{z_1(x)\}^{t_1}\cdots\{z_k(x)\}^{t_k}e^{-\{z_1(x)+\cdots+z_k(x)\}}}{t_1!\cdots t_k!}$$
$$=(2\pi)^{-\frac{k}{2}}\{z_1(x)\cdots z_k(x)\}^{-\frac{1}{2}}e^{-\frac{1}{2}(u_1^2+\cdots+u_k^2)}\{1+o(1)\}\,.$$

Now we have $t_i<2y_i(x)$ $(i=1,\cdots,k)$ for sufficiently large x, and therefore Lemma 5 can be applied to the present case. Thus, from the above formula and Lemma 5, we obtain Lemma 6, the term $o(1)$

tending uniformly to zero in the above-mentioned sense.

LEMMA 7. *Let* $\alpha_i < \beta_i$ $(i=1,\cdots,k)$, *and let* $A^{**}(x)=A^{**}(x;\alpha_1,\beta_1,\cdots,\alpha,\beta_k)$ *denote the number of positive integers* $n \leqq x$ *for which*

$$z_i(x)+\alpha_i\sqrt{z_i(x)} < \omega_i'(n;x) < z_i(x)+\beta_i\sqrt{z_i(x)} \qquad (i=1,\cdots,k)$$

simultaneously. Then

$$\lim_{x\to\infty}\frac{A^{**}(x)}{x}=(2\pi)^{-\frac{k}{2}}\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-\frac{u_i^2}{2}}du_i\,.$$

PROOF. We have

$$A^{**}(x)=\sum_{t_1,\cdots,t_k}G(x;t_1,\cdots,t_k)\,, \tag{17}$$

by the definitions of $A^{**}(x)$ and $G(x;t_1,\cdots,t_k)$, the summation extending over the systems of positive integers t_i $(i=1,\cdots,k)$ such that $z_i(x)+\alpha_i\sqrt{z_i(x)} < t_i < z_i(x)+\beta_i\sqrt{z_i(x)}$. Now let these values of t_i be t_{ij} $(j=1,\cdots,s_i)$, and let $t_{ij}=z_i(x)+u_{ij}\sqrt{z_i(x)}$, where $s_i=[(\beta_i-\alpha_i)\sqrt{z_i(x)}]$ or $[(\beta_i-\alpha_i)\sqrt{z_i(x)}]\pm 1$. Then

$$u_{i,j+1}-u_{ij}=\frac{1}{\sqrt{z_i(x)}}\,.$$

With these notations, from (17) and Lemma 6, we obtain

$$\frac{A^{**}(x)}{x}=\{1+o(1)\}(2\pi)^{-\frac{k}{2}}\prod_{i=1}^{k}\sum_{j=1}^{s_i}e^{-\frac{u_{ij}^2}{2}}(u_{i,j+1}-u_{ij})\,.$$

The lemma follows at once from this formula by making $x\to\infty$.

LEMMA 8. *Let* $\alpha_i < \beta_i$ $(i=1,\cdots,k)$, *and let* $A^*(x)=A^*(x;\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k)$ *denote the number of positive integers* $n \leqq x$ *for which*

$$z_i(x)+\alpha_i\sqrt{z_i(x)} < \omega_i(n) < z_i(x)+\beta_i\sqrt{z_i(x)} \qquad (i=1,\cdots,k)$$

simultaneously. Then

$$\lim_{x\to\infty}\frac{A^{*}(x)}{x}=(2\pi)^{-\frac{k}{2}}\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-\frac{u_i^2}{2}}du_i\,.$$

PROOF. We have

$$\sum_{n\leqq x}\{\omega_i(n)-\omega_i'(n;x)\}=\sum_{n\leqq x}\sum_{p|n,\,p\in\pi_i-\pi_i'(x)}1$$

$$=\sum_{p\leqq x,\,p\in\pi_i-\pi_i'(x)}\left[\frac{x}{p}\right]\leqq x\sum_{p\leqq x,\,p\in\pi_i-\pi_i'(x)}\frac{1}{p}\,,$$

and hence, by (14),

$$\sum_{n\leqq x}\{\omega_i(n)-\omega_i'(n\,;x)\}=O\{x\log y_i(x)\}\,.$$

Since $y_i(x)\sim z_i(x)$ as $x\to\infty$ by (14), this result can be rewritten as

$$\sum_{n\leqq x}\{\omega_i(n)-\omega_i'(n\,;x)\}=O\{x\log z_i(x)\}\,,$$

and a fortiori

$$\sum_{n\leqq x}\{\omega_i(n)-\omega_i'(n\,;x)\}=o\{x\sqrt{z_i(x)}\}\,.$$

Now it can easily be concluded from this estimation that we can take, for an arbitrarily given $\varepsilon>0$, a positive number $x_1=x_1(\varepsilon)$ such that, when $x>x_1$, the number of positive integers $n\leqq x$, for which at least one of the inequalities $\omega_i(n)-\omega_i'(n\,;x)>\varepsilon\sqrt{z_i(x)}$ $(i=1,\cdots,k)$ holds, is less than εx. Then, for $x>x_1$,

$$\begin{aligned}&A^{**}(x\,;\alpha_1,\beta_1-\varepsilon,\cdots,\alpha_k,\beta_k-\varepsilon)-\varepsilon x\\&\quad\leqq A^*(x\,;\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k)\\&\quad\leqq A^{**}(x\,;\alpha_1-\varepsilon,\beta_1,\cdots,\alpha_k-\varepsilon,\beta_k)+\varepsilon x\,.\end{aligned}$$

From this and Lemma 7, we obtain

$$\begin{aligned}(2\pi)^{-\frac{k}{2}}\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i-\varepsilon}e^{-\frac{u_i^2}{2}}du_i-\varepsilon&\leqq\liminf_{x\to\infty}\frac{A^*(x\,;\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k)}{x}\\&\leqq\limsup_{x\to\infty}\frac{A^*(x\,;\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k)}{x}\leqq(2\pi)^{-\frac{k}{2}}\prod_{i=1}^{k}\int_{\alpha_i-\varepsilon}^{\beta_i}e^{-\frac{u_i^2}{2}}du_i+\varepsilon\,,\end{aligned}$$

which, on making $\varepsilon\to 0$, gives the lemma.

LEMMA 9. *Let $\alpha_i<\beta_i$ $(i=1,\cdots,k)$ and let $A(x)=A(x\,;\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k)$ denote the number of positive integers $n\leqq x$ for which*

$$y_i(n)+\alpha_i\sqrt{y_i(n)}<\omega_i(n)<y_i(n)+\beta_i\sqrt{y_i(n)}\qquad(i=1,\cdots,k)$$

simultaneously. Then

$$\lim_{x\to\infty}\frac{A(x)}{x}=(2\pi)^{-\frac{k}{2}}\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-\frac{u_i^2}{2}}du_i$$

PROOF. If $\sqrt{x}<n\leqq x$, then by (8),

$$0\leqq y_i(x)-y_i(n)\leqq y_i(x)-y_i(\sqrt{x})\leqq\sum_{\sqrt{x}<p\leqq x}\frac{1}{p}=O(1)\,.$$

It follows easily from this and (14) that we can take, for an arbitrarily given $\varepsilon>0$, a positive number $x_2=x_2(\varepsilon)$ such that, when $x>x_2$ and $\sqrt{x}<n\leqq x$, we have

$$z_i(x)+(\alpha_i-\varepsilon)\sqrt{z_i(x)}<y_i(n)+\alpha_i\sqrt{y_i(n)}$$
$$<z_i(x)+(\alpha_i+\varepsilon)\sqrt{z_i(x)} \qquad (i=1,\cdots,k)$$
$$z_i(x)+(\beta_i-\varepsilon)\sqrt{z_i(x)}<y_i(n)+\beta_i\sqrt{y_i(n)}$$
$$<z_i(x)+(\beta_i+\varepsilon)\sqrt{z_i(x)} \qquad (i=1,\cdots,k).$$

Then, for $x>x_2$,

$$A^*(x;\alpha_1+\varepsilon,\beta_1-\varepsilon,\cdots,\alpha_k+\varepsilon,\beta_k-\varepsilon)-\sqrt{x}$$
$$\leqq A(x;\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k)$$
$$\leqq A^*(x;\alpha_1-\varepsilon,\beta_1+\varepsilon,\cdots,\alpha_k-\varepsilon,\beta_k+\varepsilon)+\sqrt{x}.$$

From this and Lemma 8, we obtain

$$(2\pi)^{-\frac{k}{2}}\prod_{i=1}^{k}\int_{\alpha_i+\varepsilon}^{\beta_i-\varepsilon}e^{-\frac{u_i^2}{2}}du_i\leqq\liminf_{x\to\infty}\frac{A(x;\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k)}{x}$$
$$\leqq\limsup_{x\to\infty}\frac{A(x;\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k)}{x}\leqq(2\pi)^{-\frac{k}{2}}\prod_{i=1}^{k}\int_{\alpha_i-\varepsilon}^{\beta_i+\varepsilon}e^{-\frac{u_i^2}{2}}du_i,$$

which, on making $x\to\infty$, gives the lemma.

Lemma 9 is the special case of Theorem A, when the set E is an interval.

THE PROOF OF THEOREM A. We are now in a position to accomplish the proof of theorem A with an arbitrarily given Jordan-measurable set E.

First we consider the case when the set E is bounded. We take two systems of intervals finite in number, say I_μ $(\mu=1,2,\cdots)$ and I'_μ $(\mu=1,2,\cdots)$, such that

$$\bigcup_\mu I_\mu\subset E\subset\bigcup_\mu I'_\mu$$

and any two of the intervals I_μ do not overlap. Then we obviously have

$$\sum_\mu A(x;I_\mu)\leqq A(x;E)\leqq\sum_\mu A(x;I'_\mu).$$

On applying Lemma 9 to the interval I_μ, I'_μ, we obtain

$$(2\pi)^{-\frac{k}{2}} \sum_\mu \int_{I_\mu} \exp\left(-\frac{1}{2}\sum_{i=1}^k u_i^2\right) du_1\cdots du_k \leqq \liminf_{x\to\infty} \frac{A(x;E)}{x}$$

$$\leqq \limsup_{x\to\infty} \frac{A(x;E)}{x} \leqq (2\pi)^{-\frac{k}{2}} \sum_\mu \int_{I_\mu'} \exp\left(-\frac{1}{2}\sum_{i=1}^k u_i^2\right) du_1\cdots du_k .$$

But, since the set E is supposed to be Jordan-measurable, we can take, corresponding to an arbitrarily given $\varepsilon>0$, the intervals I_μ, I'_μ such that

$$\int_E -\varepsilon < \sum_\mu \int_{I_\mu} \leqq \sum_\mu \int_{I_\mu'} < \int_E +\varepsilon ,$$

omitting the common integrand

$$(2\pi)^{-\frac{k}{2}} \exp\left(-\frac{1}{2}\sum_{i=1}^k u_i^2\right) .$$

Now, on combining the above inequalities, we obtain

$$\int_E -\varepsilon < \liminf_{x\to\infty} \frac{A(x;E)}{x} \leqq \limsup_{x\to\infty} \frac{A(x;E)}{x} < \int_E +\varepsilon ,$$

which, on making $\varepsilon\to 0$, leads to

$$\lim_{x\to\infty} \frac{A(x;E)}{x} = \int_E .$$

Next, we consider the case when the set E is not bounded. Again, let ε be an arbitrarily given positive number. If we take an interval I sufficiently large, and apply Lemma 9 to this interval, then we have

$$\lim_{x\to\infty} \frac{A(x;I)}{x} = \int_I > 1-\varepsilon ,$$

or

$$\lim_{x\to\infty} \frac{A(x;I^c)}{x} = \int_{I^c} < \varepsilon ,$$

which implies that

$$\limsup_{x\to\infty} \frac{A(x;E\cap I^c)}{x} < \varepsilon , \quad \int_{E\cap I} > \int_E -\varepsilon .$$

Also, since the set $E\cap I$ is bounded, it is already proved that

$$\lim_{x\to\infty}\frac{A(x;E\cap I)}{x}=\int_{E\cap I}.$$

Thus we have

$$\liminf_{x\to\infty}\frac{A(x;E)}{x}\geqq\lim_{x\to\infty}\frac{A(x;E\cap I)}{x}=\int_{E\cap I}>\int_{E}-\varepsilon,$$

$$\limsup_{x\to\infty}\frac{A(x;E)}{x}=\lim_{x\to\infty}\frac{A(x;E\cap I)}{x}+\limsup_{x\to\infty}\frac{A(x;E\cap I^c)}{x}$$

$$<\int_{E\cap I}+\varepsilon<\int_{E}+\varepsilon,$$

which, on making $\varepsilon\to 0$, leads to

$$\lim_{x\to\infty}\frac{A(x;E)}{x}=\int_{E},$$

and Theorem A is completely proved.

3. Some special cases.

We shall mention some special cases of Theorem A.

THEOREM 1. *Let m be a positive integer. Let C_i $(i=1,\cdots,k)$ denote the residue classes modulo m and prime to m in an arbitrary order, where $k=\varphi(m)$ is Euler's function of m, and let $\omega_i(n)$ denote the number of distinct prime factors of a positive integer n which belong to the class C_i. Let $\alpha_i<\beta_i$ $(i=1,\cdots,k)$, and let $A(x)=A(x;\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k)$ denote the number of integers n, $3\leqq n\leqq x$, for which*

$$\frac{1}{k}\log\log n+\frac{\alpha_i}{\sqrt{k}}\sqrt{\log\log n}<\omega_i(n)<\frac{1}{k}\log\log n+\frac{\beta_i}{\sqrt{k}}\sqrt{\log\log n}\qquad(i=1,\cdots,k)$$

simultaneously. Then

$$\lim_{x\to\infty}\frac{A(x)}{x}=(2\pi)^{-\frac{k}{2}}\prod_{i=1}^{k}\int_{\alpha_i}^{\beta_i}e^{-\frac{u^2}{2}}du.$$

THEOREM 2.[14] *Let $\omega_i(n)$ $(i=1,\cdots,k)$ have the same meaning as in Theorem 1, and let $B(x)$ denote the number of positive integers $n\leqq x$ for which*

14) In Erdös [2], a special case of this theorem is stated as Theorem 1 without proof.

$$\omega_1(n) < \omega_2(n) < \cdots < \omega_k(n) .$$

Then

$$\lim_{x\to\infty} \frac{B(x)}{x} = \frac{1}{k!} .$$

It is well-known that[15]

(18) $$\sum_{p \leq x, p \in C_i} \frac{1}{p} = \frac{1}{k} \log\log x + O(1) .$$

Theorems 1 and 2 follow easily from (18) and Theorem A.[16]

THEOREM 3. *Let all the primes be numbered in the order of their magnitudes; $p_1=2, p_2=3, p_3=5, \cdots$. Let k be a positive integer. Let C_i $(i=1,\cdots,k)$ denote the residue classes modulo k in an arbitrary order, and let $\omega_i(n)$ denote the number of distinct prime factors p_j of a positive integer n for which the number j belongs to the class C_i. Let $\alpha_i < \beta_i$ $(i=1,\cdots,k)$, and let $A(x)=A(x;\alpha_1,\beta_1,\cdots,\alpha_k,\beta_k)$ denote the number of integers n, $3 \leq n \leq x$ for which*

$$\frac{1}{k} \log\log n + \frac{\alpha_i}{\sqrt{k}} \sqrt{\log\log n} < \omega_i(n) < \frac{1}{k} \log\log n + \frac{\beta_i}{\sqrt{k}} \sqrt{\log\log n} \qquad (i=1,\cdots,k)$$

simultaneously. Then

$$\lim_{x\to\infty} \frac{A(x)}{x} = (2\pi)^{-\frac{k}{2}} \prod_{i=1}^{k} \int_{\alpha_i}^{\beta_i} e^{-\frac{u^2}{2}} du .$$

THEOREM 4. *Let $\omega_i(n)$ $(i=1,\cdots,k)$ have the same meaning as in Theorem 3, and let $B(x)$ denote the number of positive integers $n \leq x$ for which*

$$\omega_1(n) < \omega_2(n) < \cdots < \omega_k(n) .$$

Then

$$\lim_{x\to\infty} \frac{B(x)}{x} = \frac{1}{k!} .$$

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15) Cf. Landau [4], pp. 449–450, § 110.

16) If we aim at proving only Theorems 1 and 2, we had better proceed as follows: We first derive Theorem 1 from Lemma 8. Using (14) and (18), we can replace $z_i(x)$ in Lemma 8 by $\log\log n/k$ in a similar way as we have replaced $z_i(x)$ by $y_i(n)$ in the proof of Lemma 9. Next, we can derive, from Theorem 1, a general theorem similar to Theorem A, where $y_i(n)$ in the definition of $u_i(n)$ in section 1 is replaced by $\log\log n/k$, in just the same way as we have derived Theorem A from Lemma 9. Then Theorem 2 is a special case of thus obtained general theorem.

It easily follows from (8) that

$$\sum_{p_j \leqq x,\, j \in C_i} \frac{1}{p_j} = \frac{1}{k} \log\log x + O(1).$$

Theorems 3 and 4 follow easily from this and Theorem A.[17]

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17) The same remark as we have given on Theorems 1 and 2 in 16) may also be given on Theorems 3 and 4.

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