Journal of the Mathematical Society of Japan Vol. 9, No. 1, January, 1957

Topological structures in ordered linear spaces.

By Ichiro AMEMIYA and Tuyosi MORI

(Received Oct. 31, 1955)

Introduction. An adequate tool for the treatment of the integration theory is, as is well-known, the theory of ordered linear spaces (cf. [3]). These spaces have several topological structures, each of which has its own significance, and it is of interest from the view point of general analysis to investigate the relation among them. Such investigations have been made by H. Nakano [11] and [13], and by G. Köthe [9] in some special cases, among others. The purpose of the present paper is to generalize the previously obtained results in this regard and present them in a simplified form.

This paper is divided into 8 articles. In § 1, we shall give a certain property of complete lattices, which is fundamental for our paper. In § 2, the *intrinsic* topology of lattices is defined and determined for some particular cases, Boolean lattices etc. § 3 is devoted to the proof of the completeness of some uniform structures of lattices. We confine ourselves to ordered linear spaces in the next three articles, where the weakest and strongest compatible topologies are given and the compatibility of Mackey's topology in Nakano's duality is proved. In § 7, we given a topological characterization of some atomic lattices, for instance, the spaces considered in G. Köthe [9]. The last article presents some detailed considerations and pathological examples.

Preliminaries. Let L be a lattice. A subset $\{a_\lambda; \lambda \in \Lambda\}$ of L is said to be *upper* (resp. *lower*) *directed*, if for any $\lambda, \lambda' \in \Lambda$, there exists $\lambda'' \in \Lambda$ such that $a_{\lambda''} \geqq a_\lambda, a_{\lambda'}$ (resp. $a_{\lambda''} \leqq a_\lambda, a_{\lambda'}$) and we indicate it by $a_\lambda \uparrow_{\lambda \in \Lambda}$ (resp. $a_\lambda \downarrow_{\lambda \in \Lambda}$). If, moreover, $\bigcup_{\lambda \in \Lambda} a_\lambda = a$ (resp. $\bigcap_{\lambda \in \Lambda} a_\lambda = a$) we write $a_\lambda \uparrow_{\lambda \in \Lambda} a$ (resp. $a_\lambda \downarrow_{\lambda \in \Lambda} a$). The interval $\{x; a \leqq x \leqq b\}$ of L is denoted by $[a, b]$. $A \subset L$ is said to be *order-bounded*, if $A \subset [a, b]$ for some a and b. A is said to be *order-convex*, if $a, b \in A$ imply $[a, b] \subset A$. When L is complete, the greatest (resp. least) element is denoted by 1 (resp. 0). When L is a Boolean lattice, the complement of x and symmetric difference between x and y are denoted resp. by $1-x$ and $x \Delta y$.

All lattices (ordered linear spaces, Boolean lattices etc.) which we are to consider hereafter are supposed to be *conditionally complete.* For a filter Φ of L with an order-bounded basis Φ', we shall define the *order-convergence* $\lim_{\Phi} x=a$, if $\overline{\lim}_{\Phi} x=\underline{\lim}_{\Phi} x=a$ where $\overline{\lim}_{\Phi} x=\bigcap_{X\in\Phi'}\bigcup_{x\in X} x$ and $\underline{\lim}_{\Phi} x=\bigcup_{X\in\Phi'}\bigcap_{x\in X} x$. Then we can introduce a topology $\mathfrak{T}_0$ which is induced by this pseudo-topology in usual way. A topology on L is said to be *compatible with the lattice structure* if every element has a $\mathfrak{T}_0$-closed order-convex basis of neighbourhoods and the meet and join are continuous on $L\times L$.

In an ordered linear space E over the reals $\boldsymbol{R}$ ($\boldsymbol{R}$ denotes the space of real numbers throughout this paper), a subset A of E is said to be *normal,* if A is order-convex and $x\in A$ implies $|x|\in A$. (When A is a subspace, our definition means the semi-normalcy in H. Nakano's terminology [11]). The least normal set including A is called the *normal hull* of A. The Boolean lattice of all the $\mathfrak{T}_0$-closed normal subspaces is canonically isomorphic to the projection lattice of E. A topology on E which is compatible with both structures of linear space and of lattice is simply said to be *compatible.* (Our definition is more restrictive than that of N. Bourbaki [3]). A compatible topology of E is said to be *monotone complete,* if every topologically bounded directed set $0\leqq a_\lambda\uparrow_{\lambda\in\Lambda}$ is order-bounded.

For two ordered linear spaces E and F, the totality $\mathfrak{L}(E,F)$ of the $\mathfrak{T}_0$-continuous linear operators on E to F is an ordered linear space. In particular, $\mathfrak{L}(E,\boldsymbol{R})$ is denoted by E'. If E' is *total* on E, we have Nakano's duality between E and E'. (We consider the duality *only* in such cases). Besides the topologies $\sigma(E,E')$, $\tau(E,E')$ and $\beta(E,E')$ in Bourbaki's notation [5], that is, that of the simple convergence on E', of the uniform convergence on the convex $\sigma(E',E)$-compact sets, and that of the uniform convergence on the $\sigma(E',E)$-bounded sets respectively, we can introduce the topology $\sigma^*(E,E')$ of the uniform convergence on the order-bounded sets. $\sigma^*(E,E')$ is the weakest compatible topology, such that E' is dual of E. Then the main theorem of H. Nakano [10] can be described as follows: *$\sigma^*(E',E)$ is monotone complete, and E is $\mathfrak{T}_0$-dense and normal in E''* (by the natural inclusion). $\beta(E,E')$ and $\beta(E',E)$ are compatible, since every $\sigma(E,E')$-bounded set is $\sigma^*(E,E')$-bounded as is known in general theory of topological linear spaces. The compatibility of $\tau(E,E')$ and $\tau(E',E)$ will be proved in the following. If $E''=E$, then E is said to be *reflexive.*

For an abstract set X, we use the notations $S(X)$, $D(X)$, $B(X)$ and $\hat{B}(X)$, to mean respectively the ordered linear space of all the real-valued functions on X, that of all the real-valued functions which vanish except for finite subsets of X, the lattice of all the subsets of X, and that of all the finite subsets of X.

1. The binary relations, join and meet, of the lattice L are naturally extended to the subsets of L and then to filters of L. Filters Φ which are idempotent by these operations, i. e. such that $\Phi \cup \Phi = \Phi$, $\Phi \cap \Phi = \Phi$, are called *idempotent*. Filters with $\mathfrak{T}_0$-closed basis will be called simply $\mathfrak{T}_0$*-closed*. Furthermore we write $K(\Phi) = \bigcap_{A \in \Phi} A$ for a filter Φ. Then we have:

If L is complete, and Φ is a $\mathfrak{T}_0$-closed idempotent filter of L, then $K(\Phi) \neq \phi$ and every $\mathfrak{T}_0$-open order-convex set including $K(\Phi)$ belongs to Φ.

PROOF. Obviously $K(\Phi)$ is either void, or a $\mathfrak{T}_0$-closed idempotent subset and so a complete sublattice of L. On the other hand, Φ is the union of all the $\mathfrak{T}_0$-closed idempotent filters Ψ having countable bases and including in Φ, and $K(\Phi)$ is the intersection of all the $K(\Psi)$ for such Ψ's. Therefore it is sufficient to prove our proposition in the following two cases: (i) Φ is generated by complete sublattices, (ii) Φ has a countable basis. The case (i) is clear. In case (ii), we can find for Φ a $\mathfrak{T}_0$-closed basis A_ν $(\nu = 1, 2, \cdots)$ such that the "join", in the extended meaning, of the set A_ν with itself is included in $A_{\nu-1}$ for every $\nu = 2, 3, \cdots$. Then $x_\nu \in A_\nu$ $(\nu = 1, 2, \cdots)$ imply $\overline{\lim_{\nu\to\infty}} x_\nu \in A_\nu$ $(\nu = 1, 2, \cdots)$, and hence $K(\Phi)$ is non-void. If $K(\Phi) \subset B$, B being $\mathfrak{T}_0$-open and order-convex, then the union C of all the intervals $[x, 1]$ which are disjoint with B, is disjoint with A_ν for some ν, because otherwise there exists $x_\nu \in A_\nu \cap C$ $(\nu = 1, 2, \cdots)$ and this would imply $\overline{\lim_{\nu\to\infty}} x_\nu \in K(\Phi)$ which is a contradiction since $\overline{\lim_{\nu\to\infty}} x_\nu$ is not in B. The same is true for the union C' of all the intervals $[0, x]$ which are disjoint with B. Since the order-convexness of B implies $B \cup C \cup C' = L$, we have $A_\nu \subset B$ for some ν. Thus the proof is completed.

2. The topology $\mathfrak{T}_0$ is not, in general, compatible with the lattice structure and so it is important for our purpose whether L has a separated topology weaker than $\mathfrak{T}_0$ and compatible with the lattice structure. We shall say that L is *topologically ordered* if L has such a topology.

If a complete lattice L is topologically ordered, then there exists only one separated compatible topology weaker than $\mathfrak{T}_0$, and the totality of the $\mathfrak{T}_0$-open and order-convex sets constitutes an open basis of it. In fact the filter Φ generated by all the closed neighbourhoods of an element $a \in L$ for such a topology is obviously $\mathfrak{T}_0$-closed and idempotent, and since $K(\Phi)$ consists of one element a, every $\mathfrak{T}_0$-open order-convex set containing a is in Φ, as is shown in § 1.

This unique topology on a complete topologically ordered lattice L is called the *intrinsic topology* of L.

The complete lattice L with the intrinsic topology is always regular and has the Baire property, that is, every open set of L is of second category. The regularity was shown in the proof above and the latter fact can be seen easily as in the case of locally compact spaces. The product of topologically ordered lattices is also topologically ordered, the intrinsic topology of the product being the product topology of the intrinsic topologies of the factors.

Now we shall deal with some particular cases.

Boolean lattices: Let B be a Boolean lattice, then, among all idempotent filters in B generated by $\mathfrak{T}_0$-open order-convex sets containing 0, there exists the largest one, say Φ, which is $\mathfrak{T}_0$-closed, and B is topologically ordered if and only if $K(\Phi)=\{0\}$. In fact, it is obvious that Φ exists and is $\mathfrak{T}_0$-closed, and that $K(\Phi)=\{0\}$ when B is topologically ordered. If, conversely, $K(\Phi)=\{0\}$, then the intrinsic topology of B is given with $\Phi \Delta x=\{A \Delta x; A \in \Phi\}$, where $A \Delta x$ means $\{a \Delta x; a \in A\}$, as the filter of neighbourhoods of x for every $x \in B$. By the way this intrinsic topology is also compatible with the group structure of B with the operation Δ. In general, $K(\Phi)$ is an interval $[0, p]$ which includes no topologically ordered interval and B is the product of the topologically ordered lattice $[0, 1-p]$ and $[0, p]$. If B has a (completely additive finite) measure μ such that $\mu(x)=0$ implies $x=0$, then B is topologically ordered and its intrinsic topology is the usual metric topology by μ. B is also topologically ordered if there exist sufficiently many measures, because, for every orthogonal system $p_\lambda \in B$ $(\lambda \in \Lambda)$ such that $\bigcup_{\lambda \in \Lambda} p_\lambda = 1$, B is isomorphic to the product of the intervals $[0, p_\lambda]$. If for any idempotent filter Ψ generated by $\mathfrak{T}_0$-open order-convex sets containing 0, we have $K(\Psi)=[0,p]$, then Ψ can be considered as the filter of neighbourhoods of 0 by the intrinsic topology on $[0,1-p]$. Therefore the intrinsic topology is sequential in some intervals and the union of such intervals is dense in B. If B contains only two elements, 0 and 1, B is obviously topologically ordered, and hence every product of such lattices, i. e. $B(X)$ for every set X is topologically ordered. The Boolean lattice of all the regularly open set in $\boldsymbol{R}$ provides an example of the lattice in which no interval is topologically ordered.

Intervals in ordered linear spaces: Let E be an ordered linear space, and $\mathfrak{T}$ a separated topology of E which is compatible with the structure of E and weaker than $\mathfrak{T}_0$. Then $\mathfrak{T}$ induces obviously the intrinsic topology on every interval of E. Hence, if $\mathfrak{T}'$ is another topology of E with the same properties as $\mathfrak{T}$ above, $\mathfrak{T}, \mathfrak{T}'$ will coincide on every order-bounded set of E, though it may happen $\mathfrak{T}\neq\mathfrak{T}'$. If there exists at least one such $\mathfrak{T}$, then the projection lattice $B(E)$ of E is topologically ordered, since, in every interval $[0,a]$ of E, there exists a sublattice isomorphic to some interval of $B(E)$. We shall now show the converse: i. e. E has a topology $\mathfrak{T}$ with above said properties, if $B(E)$ is topologically ordered. So, in order to show the existence of a topology with certain properties, hereditary to subspaces, of the ordered linear space E with the given projection lattice $B(E)$, we have only to introduce a topology with these properties to S for a given B. Let B be any Boolean lattice, then the totality S of the continuous functions on the Stonean space of B, which are finite on some dense open set, can be considered naturally as an ordered linear space. Then B is the projection lattice of S, and any ordered linear space whose projection lattice is isomorphic to B is isomorphic to a $\mathfrak{T}_0$-dense normal subspace of S. In case B is topologically ordered, we can introduce on S a separated compatible topology weaker than $\mathfrak{T}_0$. For this, take a neighbourhood U of 0 in B by the intrinsic topology and for $\varepsilon>0$ let (U,ε) be the totality of the elements x in S such that $(1-p)|x|\leqq\varepsilon\cdot 1$ for some $p\in U$ where 1 means the constant 1 considered as a function, then the totality of (U,ε) constitutes a basis of neighbourhoods of 0 in S for the topology wanted, because we can prove easily that, for every directed set $a_\lambda\downarrow_{\lambda\in\Lambda}0$ in S, there exists $p\neq 0$ in B such that pa_λ converges to 0 uniformly as functions. (As to topologies of this type, cf. § 4).

Continuous geometries: Let L be a continuous geometry and B its centre, then the dimension-lattice D of L is isomorphic to an interval of S which is constructed from B as above. If L is topologically ordered then B is also topologically ordered as it is a complete sublattice of L. Conversely, if B and hence D is topologically ordered, then, for every $x \in L$, the totality of the sets $\{y; d(x \cup y) - d(x \cap y) \in U\}$, where U is an neighbourhood of 0 in D for the intrinsic topology and $d(x)$ means the dimension of x, constitutes a basis of neighbourhoods of x in L for a separated compatible topology, which can be seen easily to be weaker than $\mathfrak{T}_0$ because of the $\mathfrak{T}_0$-continuity of the mapping: $x \rightarrow d(x)$. If L is irreducible, that is, if D is isomorphic to the interval $[0, 1]$ of $\boldsymbol{R}$, then the intrinsic topology of L is nothing other than the known topology of L as the metric lattice. The lattice of all the projections in a ring of operators of finite class is a continuous geometry and is always topologically ordered since there exist sufficiently many measures on its centre.

3. In all above cases, the intrinsic topologies introduced are induced by uniform structures in an obvious way. We shall show that these structures are complete. This is based on the completenss of the lattice structures.

A uniform structure $\mathfrak{U}$ *on a complete lattice* L *is complete if* $\mathfrak{U}$ *has a* $\mathfrak{T}_0$*-closed basis (in* $L \times L$*) and the join and meet are uniformly continuous on* $L \times L$. In fact, we can see easily that under our conditions every Cauchy filter is equivalent to a $\mathfrak{T}_0$-closed and idempotent filter Φ, and hence it converges to every element in $K(\Phi)$, which was proved to be non-void in § 1.

For a conditionally complete lattice L, *if* $\mathfrak{U}$ *satisfies, besides the above conditions, also the condition that every Cauchy filter induced by a directed system* $a_\lambda \uparrow_{\lambda \in \Lambda}$ *or* $a_\lambda \downarrow_{\lambda \in \Lambda}$ *is convergent, then* $\mathfrak{U}$ *is complete.* We can easily prove this, making use of the mapping: $x \rightarrow (x \cup a) \cap b$ of L onto $[a, b]$ which is uniformly continuous and hence by which the image of a Cauchy filter is convergent.

Thus, in the ordered linear space, every compatible topology is complete in closed order-bounded sets, and if it is moreover monotone complete it is complete in whole space. These facts were also proved by H. Nakano in [12]. In the duality between E and E', $\sigma^*(E', E)$ is complete and E'' is a completion of E by $\sigma^*(E, E')$. In § 2, we have constructed the ordered linear space S from a Boolean lattice B. If B is topologically ordered, then the topology on S which was defined there is complete. More generally we can prove that every separated compatible topology on S is monotone complete. To the end, it is sufficient to prove that, for every directed set $0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda}$, either it is order-bounded or its normal hull includes a subspace different from $\{0\}$. Let ϕ be the well-known lattice-isomorphism of S into the sublattice $[-1, 1]$ of S, then either $b = \bigcup_{\lambda \in \Lambda} \varphi(a_\lambda)$ is in $\varphi(S)$ or there exists a projection $p \neq 0$ in B such that $pb = p1$. In the latter case, we can find $0 \neq q \leqq p$ such that $q\varphi(a_\lambda)$ converges uniformly to $q1$, and this completes the proof.

4. On a topologically ordered linear space E, we shall consider the weakest separated compatible topology.

From the projection lattice B of E we construct S in § 2. (Then S is the universal completion of E in the sence of H. Nakano [10]). For any separated compatible topology $\mathfrak{T}_1$ on E considered as a normal subspace of S, we can introduce a compatible topology on S as follows. For a neighbourhood U of 0 in B and a neighbourhood V of 0 in E, let (U, V) be the set of all the elements x in S such that $(1-p)x \in V$ for some $p \in U$, then the totality of (U, V) constitutes a neighbourhood system of 0 for a compatible topology on S. (If E is the totality of the finite continuous functions on the Stonean space and the topology is the usual one, we have the topology on S introduced in § 2). Such a topology $\mathfrak{T}$ on S is complete, separated and weaker than $\mathfrak{T}_0$. This topology $\mathfrak{T}$ on S is determined by E and the topology $\mathfrak{T}_1$ on E. Now we shall prove that $\mathfrak{T}$ is in reality independent of E and of $\mathfrak{T}_1$, and is completely determined as the weakest separated compatible topology on S. $\mathfrak{T}$ induces moreover to every normal subspace E of S the weakest separated compatible topology on E. Let $\mathfrak{T}'$ be any separated compatible topology on E, then every directed set $0 \leqq a_\lambda \uparrow_{\lambda \in \Lambda}$ in E which is $\mathfrak{T}'$-bounded is order-bounded in S as was shown in § 3, and hence it is $\mathfrak{T}$-bounded. This shows that $\mathfrak{T}$ is weaker than $\mathfrak{T}'$ so long as $\mathfrak{T}'$ is sequential. In general, for every neighbourhood U of 0 by $\mathfrak{T}$, there exists a sequential compatible topology $\mathfrak{T}''$ weaker than $\mathfrak{T}'$ such that the closure of $\{0\}$ by $\mathfrak{T}''$ is included in U. Since this closure is pE for some $p \in B$, $\mathfrak{T}''$ is separated in $(1-p)E$ where it is stronger than $\mathfrak{T}$, and hence there exists a neighbourhood V of 0 for $\mathfrak{T}''$ for which $(1-p)V \subset U$, $V \subset U + U$ and this shows that $\mathfrak{T}'$ is stronger than $\mathfrak{T}$ on E.

Any linear functional x' on E which is continuons for this topology $\mathfrak{T}$ vanishes in a direct sum factor of finite co-dimension, since x' vanishes in $VE=\bigcup_{p\in V} pE$ for some neighbourhood V of 0 in B. Therefore, this topology $\mathfrak{T}$ is convex if and only if S is isomorphic to a $S(X)$, and then $\mathfrak{T}$ coincides with the topology of the simple convergence on X. For function spaces in the integration theory, the topology of this type means that of the measure convergence. As to the ring of operators M of finite class, J. v. Neumann used in [14] a similar topology $\mathfrak{T}$ constructed by means of the intrinsic topology of the lattice of all the projections in M and the uniform topology on M, and we can see easily that the completion of M by $\mathfrak{T}$ is the totality of the closed operators belonging to M and having dense domains.

5. On an ordered linear space E, we shall consider the strongest convex compatible topology.

Let $\mathfrak{T}$ be a complete separated compatible topology on E, and x_ν $(\nu=1,2,\cdots)$ any $\mathfrak{T}$-bounded sequence, then $\frac{1}{2^\nu} x_\nu$ $(\nu=1,2,\cdots)$ is order-bounded. So every $\mathfrak{T}$-bounded set is also bounded for any other separated compatible topology. Hence a complete sequential (or bornologic) separated compatible topology, if it exists, is the strongest among the separated compatible topologies of E.

In the duality between E and E', $\beta(E,E')$ is the strongest convex compatible topology. First we prove that *every $\mathfrak{T}_0$-closed normal subset $A\subset E$ is closed for any separated compatible topology $\mathfrak{T}$.* Let a be in the closure of A for $\mathfrak{T}$. Then $|a|$ is also in the same closure and hence in the closure of $[0,|a|]\cap A$ which is the image of A by the continuous mapping $x\rightarrow(x\cup 0)\cap|a|$. If $|a|\notin A$, then $[0,|a|]-[0,|a|]\cap A$ is $\mathfrak{T}_0$-open order-convex subset of the complete lattice $[0,|a|]$ and contains $|a|$. Since the filter Φ of all neighbourhoods of $|a|$ for $\mathfrak{T}$ is a $\mathfrak{T}_0$-closed idempotent filter for which $K(\Phi)=\{|a|\}$, we have $[0,|a|]-[0,|a|]\cap A\in\Phi$, a contradiction.

Therefore every $\mathfrak{T}_0$-closed normal convex neighbourhood U of 0 for a compatible convex topology $\mathfrak{T}$ is $\sigma^*(E,E')$-closed and hence $\sigma(E,E')$-closed. This shows that U is a neighbourhood of 0 for $\beta(E,E')$.

Since every $\sigma^*(E',E)$-bounded set is $\beta(E',E)$-bounded and hence the monotone completeness of $\sigma^*(E',E)$ implies that of $\beta(E',E)$, $\beta(E',E)$

is complete.

6. Here we shall prove that, *in the duality between E and E', $\tau(E, E')$ and $\tau(E', E)$ are compatible and hence $\tau(E', E)$ is complete.* For this, it is sufficient to show that *the normal hull of every $\sigma(E, E')$-compact set is also $\sigma(E, E')$-compact.* (We can exchange E and E'). A set A in E is said to be equi-continuous if every order-convergent filter in E' converges uniformly on A. We have proved in [1] that a set A is equi-continuous if every order-convergent sequence in E' converges uniformly on A. Since we can see easily that the normal hull of every equi-continuous set is equi-continuous, we have only to prove the equivalency of the equi-continutiy and relative $\sigma(E, E')$-compactness. The relative $\sigma(E, E')$-compactness follows from the equi-continuity immediately by Tychonoff's theorem. The converse was proved first by H. Nakano in [11], and then by J. L. B. Cooper in [6] in a more general case. We give here another proof of this fact.

Let A be a $\sigma(E, E')$-compact set and a'_ν $(\nu=1, 2,\cdots)$ be an order-convergent sequence in E'. Since $\{a'_\nu\}$ is order-bounded, we can suppose, without loss of generality, that all a'_ν are in an interval $[0, a']$. If the sequence does not converge uniformly on A, then there exists a sequence a_ν in A such that the convergence of a'_ν is not uniform on every infinite subsequence of a_ν. Then our proof is based on the following facts:

(i) We can find a subsequence of a_ν, which we shall again denote by a_ν for simplicity's sake, such that $x'(a_\nu)$ is convergent for every $x' \in [0, a']$.

(ii) If $x'(a_\nu)$ is convergent for every $x' \in [0, a']$, then, for some $x' \in [0, a']$, a_ν is convergent uniformly *at* x' as continuous functions on $[0, a']$ with the topology $\sigma(E', E)$.

(iii) Every relatively $\sigma(E', E)$-open set in $[0, a']$ contains almost all $y' + a'_\nu$ $(\nu=1, 2,\cdots)$ (i. e. except for finite number), for some $y' \in [0, a']$.

In fact, suppose (i), (ii) and (iii) hold. As a'_ν is not convergent uniformly on $\{a_\nu\}$ in (i), we may suppose, for instance, $a'_\nu(a_\nu) > 1$ $(\nu = 1, 2,\cdots)$. Then, since on some relatively $\sigma(E', E)$-open set containing x' in (ii) the values of a_ν are smaller than $\frac{1}{2}$ for every sufficiently large ν, we can find by (iii) $y' \in [0, a']$ for which $y'(a_\nu)$ is not convergent, and this is a contradiction.

Now, since $[0, a']$ is $\sigma(E', E)$-compact, (i) is nothing but Šmulian's theorem in the form generalized by A. Grothendieck in [7]. (ii) is an immediate consequence of Baire's theorem which asserts that, on a topological space, every sequence of continuous functions which is simply convergent is convergent uniformly *at* every point outside of some set of first category. (iii) is principally based on the lattice structure and can be seen easily.

More generally, if $u_\lambda \downarrow_{\lambda \in \Lambda} 0$ in $\mathfrak{L}(E, F)$, then u_λ converges to 0 uniformly on every $\sigma(E, E')$-compact set, where the topology on F is $\sigma^*(F, F')$.

7. The topologically ordered Boolean lattice B can be considered as a separated commutative group. As such, if B has no atomic element it has no non-trivial continuous character, because, for every non-trivial continuous character, a minimal element of the inverse image of -1 must be atomic. In paticular, *if the intrinsic topology of B is compact, B is isomorphic to a $B(X)$, and its dual group is $\hat{B}(X)$.*

When X is countably infinite, $L^2(B(X))$, defined by Haar's measure, is isomorphic to $L^2([0, 1])$, and $\hat{B}(X)$ constitutes Walsh-Rademacher's orthonormal basis.

Applying this to the duality between E and E', we see that *if every order-bounded set in E is $\sigma^*(E, E')$-precompact, then E is isomorphic to a $\mathfrak{T}_0$-dense normal subspace of a $S(X)$ for some set X.* Conversely, in such a space, every $\sigma(E, E')$-compact set is $\sigma^*(E, E')$-compact. (In these assertions we can of course exchange E and E'). These facts were stated by I. Halperin and H. Nakano in [8] in the following form: E is isomorphic to a $\mathfrak{T}_0$-dense normal subspace of a $S(X)$ if and only if $\sigma(E, E')\text{-}\lim_{\nu\to\infty} x_\nu = 0$ implies $\sigma^*(E, E')\text{-}\lim_{\nu\to\infty} x_\nu = 0$. More generally, we can prove by the last assertion in § 6 that, if $0 \leqq u_\lambda \uparrow_{\lambda \in \Lambda} u$ in $\mathfrak{L}(E, F)$ and u_λ have finite-dimensional ranges, then u maps every $\sigma(E, E')$-compact set to a $\sigma^*(F, F')$-compact set. But the converse of this assertion although it seems plausible to us, is not yet proved.

Applying our results to rings of operators, we see that a ring of operators whose unit sphere is strongly compact is a (usual) direct sum of factors of type (I_n).

Since every $\sigma(E, E')$-neighbourhood of 0 includes a subspace of finite co-dimension, $\sigma^*(E, E')$ coincides with $\sigma(E, E')$ if and only if E' is isomorphic to a $D(X)$, and hence, for the reflexive space only if E is isomorphic to $S(X)$. But the last assertion is inexact when E is not reflexive, because there is a normal subspace $E \neq S(X)$ of $S(X)$ whose dual is $D(X)$, for instance, $E = \{x; \lim_{\Phi} x(t) = 0\}$ for a ultrafilter Φ on X which can not be generated by one point. We can not obtain any such counter-example considering normal subspaces generated by countable elements, according to the classical du Bois-Reimond's theorem.

8. A topology on an ordered linear space E is said to be normal if it is compatible with the structure of linear space and has an order-convex basis of neighbourhoods.

Now we want to know whether the results obtained above about compatible topologies on E remain valid or not for normal topologies (less restricted than compatible ones).

Every monotone complete sequential normal topology is also complete, as was shown in § 2, but this is inexact in non-sequential cases, since this is reduced to the well-known Ulam's problem, when we consider $S(X)$ and the strongest convex normal topology on it. The proof of the monotone completeness of every separated compatible topology on S is also valid for normal topology.

The topology of § 4 is weaker than every sequential separated normal topology since that part of the proof remains valid, but this is inexact in non-sequential cases. For instance, we can represent $L^\infty([0, 1])$ as a $C(X)$ and the topology of the simple convergence on X gives a normal topology of $L^\infty([0, 1])$ which is not comparable with the topology of § 4 of this space.

The first assertion in § 5 remain valid for normal topologies, and hence if E is reflexive and the strongest convex normal topology $\mathfrak{T}$ on E is sequential, then $\mathfrak{T}$ coincides with $\beta(E, E')$, but if E is not reflexive this is not true. To see this, we consider the normal subspace $E = \{x; \sum_{\nu=1}^{\infty} \varphi_\nu(|x|) < +\infty\}$ of l^∞ where φ_ν are homomorphisms of l^∞ to $\boldsymbol{R}$ such that $\varphi_\nu \neq \varphi_{\nu'}$ for $\nu \neq \nu'$ and that $\varphi_\nu(1) = 1$ and $\varphi_\nu(x) = 0$ for every $x \in c_0$. In E, the norm $\|x\| = \|x\|_\infty + \sum_{\nu=1}^{\infty} \varphi_\nu(|x|)$ is complete and hence defines the strongest normal topology on E, but it is not equivalent with $\|x\|_\infty$ which defines $\beta(E, E')$. If $\mathfrak{T}$ is non-sequential, then we can not decide if $\mathfrak{T}$ coincides with $\beta(E, E')$ even if E is reflexive. In fact this question is equivalent with Ulam's problem, when $E=S(X)$.

Department of Mathematics
University of Hokkaido

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