Journal of the Mathematical Society of Japan Vol. 8, No. 2, April, 1956

Harmonic analysis of the axially symmetrical incompressible viscous flow.

By Prem PRAKASH

(Received Dec. 8, 1955)

Introduction: The present paper has been inspired by a similar one for the two-dimensional case*. For an axially symmetrical three-dimensional flow, the paper gives the differential equations connecting the Fourier transforms of the velocity components, the stream function and the vorticity. Equivalent integro-differential equation of the Navier-Stokes equation of motion has been given and thus the Fourier transform Z of the vorticity being known, those of the velocity components and the stream function are given; also the spectral function of the kinetic energy of the flow is determined.

Flow being axially symmetrical it will be the same in all the planes passing through the axis. We therefore consider the flow in half of one of these planes bounded by a domain D in this plane, velocity vanishing on the boundary B of D. We take x along the axis and ϖ perpendicular to it as the coordinates of any point in this half-plane. If the fluid extends up to infinity, the velocity is zero there. If ψ be the stream function the two components u, v of the velocity are given by

$$u(x, \varpi, t)=-\frac{1}{\varpi}\frac{\partial\psi}{\partial\varpi}, \tag{1}$$

$$v(x, \varpi, t)=\frac{1}{\varpi}\frac{\partial\psi}{\partial x}. \tag{2}$$

According to boundary conditions we have

* J. Kampé de Fériet, Harmonic analysis of the two-dimensional flow of an incompressible viscous fluid, Quart. Appl. Math., (1) VI, 1–13 (1948).

$$u(x, \varpi, t)=0, \; v(x, \varpi, t)=0 \text{ on } B, \tag{3}$$

i. e.

$$\psi(x, \varpi, t)=0, \qquad \frac{\partial \psi}{\partial x}=0 \text{ on } B. \tag{4}$$

The vorticity is given by

$$\zeta(x, \varpi, t)=\frac{1}{\varpi}\left(\frac{\partial^2 \psi}{\partial x^2}+\frac{\partial^2 \psi}{\partial \varpi^2}-\frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}\right)=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial \varpi}. \tag{5}$$

The cross-differentiated Navier-Stokes equation of motion is

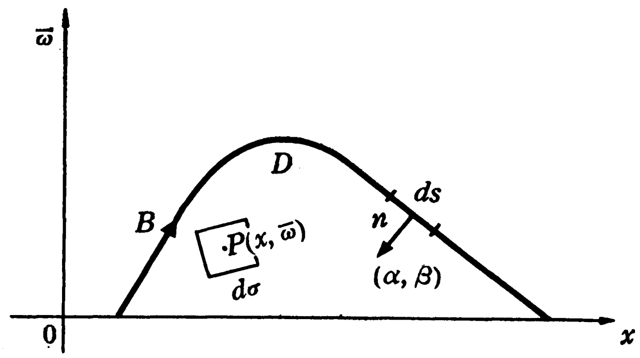
$$\frac{\partial \zeta}{\partial t}+u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial \varpi}-\frac{v\zeta}{\varpi}=\nu\left(\frac{\partial^2 \zeta}{\partial x^2}+\frac{\partial^2 \zeta}{\partial \varpi^2}+\frac{1}{\varpi}\frac{\partial \zeta}{\partial \varpi}-\frac{\zeta}{\varpi^2}\right), \tag{6}$$

where ν is the kinematic coefficient of viscosity.

Substituting the values (1), (2) and (5) into (6) we obtain a partial (non-linear) differential equation for $\psi(x, \varpi, t)$ which is characterestic for the three-dimensional axially symmetrical flow of an incompressible viscous fluid. As in the two-dimensional problem, we shall only deal with regular flows i. e. with stream functions such that the functions

$$\psi, u, v, \zeta, \frac{\partial \psi}{\partial t}, \frac{\partial u}{\partial t}, \frac{\partial v}{\partial t}, \frac{\partial \zeta}{\partial t}, \frac{\partial \zeta}{\partial x}, \frac{\partial \zeta}{\partial \varpi}, \nabla^2 \zeta \tag{7}$$

are continuous and finite in x, ϖ, t for (x, ϖ) in $D+B$ and $t_1 \leqslant t \leqslant t_2$. Besides these (as pointed out very kindly by Porf. Kampé de Fériet) the appearance of ϖ in the denominators of ψ, ζ etc. might cause difficulty in operating Green's formulae and taking Fourier transforms in some cases if the region occupied by the fluid includes the axis $\varpi=0$. For this we have to impose the following restrictions; $\frac{\zeta}{\varpi}$ together with its first derivative with respect to ϖ is finite on the axis, so are $\frac{\psi}{\varpi}$ and $\frac{\psi}{\varpi^{1/2}}$ together with its first and second derivatives.



This paper is divided into five parts. In the first part the Fourier transforms $\Psi(\omega_1, \omega_2, t)$, $U(\omega_1, \omega_2, t)$, $V(\omega_1, \omega_2, t)$ and $Z(\omega_1, \omega_2, t)$ of the stream function ψ, the velocity components u, v and the vorticity ζ are introduced and simple linear partial differential equations are obtained connecting Ψ, U, V, Z. This part of the paper uses only the definitions of ψ, u, v, ζ, their continuity condition and the boundary condition; the results are valid whether the fluid is viscous or not.

In the second part the kinetic energy of the fluid

$$E = \frac{\pi}{2}\int_D \varpi(u^2+v^2)d\sigma, \quad (d\sigma = dx d\varpi) \tag{8}$$

and its spectral decomposition

$$E = \int_\Omega \gamma(\omega_1, \omega_2, t)d\omega, \quad (d\omega = d\omega_1 d\omega_2) \tag{9}$$

are considered and it is shown that the spectral function $\gamma(\omega_1, \omega_2, t)$ has a simple expression in terms of Ψ and Z.

In the third part an integro-differential equation equivalent to the Navier-Stokes equation of motion (6) has been derived in terms of the Fourier transforms Z, U, V.

In the fourth part the case that the fluid occupies the entire half-plane

$$X(-\infty < x < \infty, \quad 0 < \varpi < \infty)$$

has been considered as the limit of the flow in a bounded domain, the fluid being at rest at infinity. The integro-differential equation simplifies considerably in this case.

In the fifth part a further special class of flows characterised by $\zeta = \varpi f(\psi)$, also called the self-superposable flows, has been studied and one application of the foregoing theory has been given.

If $f(x, \varpi)$ is a real function of the real variables x and ϖ defined and continuous in $D+B$, its Fourier transform is given by

$$F(\omega_1, \omega_2) = \frac{1}{4\pi^2}\int_D f(x, \varpi) e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma \, , \tag{10}$$

where the frequencies ω_1 and ω_2 are real. We shall use the following well-known properties of the Fourier transforms.

(a) $F(\omega_1, \omega_2)$ is a complex function of two real variables ω_1, ω_2 defined in the entire plane

$$\Omega(-\infty < \omega_1 < \infty, \quad -\infty < \omega_2 < \infty) .$$

(b) $F(\omega_1, \omega_2)$ is a continuous function of the variables ω_1, ω_2 at every point of Ω.

(c) If $|\omega_1| + |\omega_2| \to \infty$, then $F(\omega_1, \omega_2) \to 0$.

(d) If $F(\omega_1, \omega_2)$ is absolutely integrable

$$\int_\Omega |F(\omega_1, \omega_2)| d\omega < \infty \, ,$$

the integral

$$\int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 \varpi)} d\omega$$

defines a continuous function of x, ϖ for all values of x and ϖ in $D+B$:

$$f(x, \varpi) = \int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 \varpi)} d\omega \, ,$$

while

$$0 = \int_\Omega F(\omega_1, \omega_2) e^{i(\omega_1 x + \omega_2 \varpi)} d\omega$$

outside $D+B$. Since the integral is continuous for all values of x and ϖ the function $f(x, \varpi)$ must vanish on B. Thus if a given function $f(x, \varpi)$ does not vanish on B its Fourier transform cannot be absolutely integrable.

(e) If $f(x, \varpi)$ has continuous derivatives $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial \varpi}$ in $D+B$ (which is always the case for the functions considered here), one has

$$f(x, \varpi)=\int_{\Omega} F(\omega_1, \omega_2)e^{i(\omega_1 x+\omega_2 \varpi)}d\omega \tag{11}$$

at every point of D, the integral being now an improper integral defined as the limit

$$\lim_{\lambda\to\infty}\int_{C\lambda} F(\omega_1, \omega_2)e^{i(\omega_1 x+\omega_2 \varpi)}d\omega \tag{12}$$

$C\lambda$ being the circle $\omega_1^2+\omega_2^2 \leqslant \lambda^2$ (Cauchy's principal value). As a rule (11) does not hold on the boundary B.

(f) $f(x, \varpi)$, $g(x, \varpi)$ being continuous functions with continuous derivatives in $D+B$ and $F(\omega_1, \omega_2)$, $G(\omega_1, \omega_2)$ their Fourier transforms, we have

$$\frac{1}{4\pi^2}\int_D f(x, \varpi)g(x, \varpi)e^{-i(\theta_1 x+\theta_2 \varpi)}d\sigma=\int_{\Omega} F(\omega_1, \omega_2)\overline{G}(\omega_1+\theta_1, \omega_2+\theta_2)d\omega\,, \tag{13}$$

the meaning of the integral being the same as in (11). Here θ_1 and θ_2 are arbitrary real variables; in particular for $\theta_1=\theta_2=\theta$ one has Parseval's formula

$$\frac{1}{4\pi^2}\int_D f(x, \varpi)g(x, \varpi)d\sigma=\int_{\Omega} F(\omega_1, \omega_2)\overline{G}(\omega_1, \omega_2)d\omega\,, \tag{14}$$

$\overline{G}$ denoting the conjugate of G.

1. We now introduce the Fourier transforms of the stream function, the velocity components and the vorticity

$$\Psi(\omega_1, \omega_2, t)=\frac{1}{4\pi^2}\int_D \psi(x, \varpi, t)e^{-i(\omega_1 x+\omega_2 \varpi)}d\sigma \tag{15}$$

$$U(\omega_1, \omega_2, t)=\frac{1}{4\pi^2}\int_D u(x, \varpi, t)e^{-i(\omega_1 x+\omega_2 \varpi)}d\sigma \tag{16}$$

$$V(\omega_1, \omega_2, t)=\frac{1}{4\pi^2}\int_D v(x, \varpi, t)e^{-i(\omega_1 x+\omega_2 \varpi)}d\sigma \tag{17}$$

$$Z(\omega_1, \omega_2, t)=\frac{1}{4\pi^2}\int_D \zeta(x, \varpi, t)e^{-i(\omega_1 x+\omega_2 \varpi)}d\sigma\,. \tag{18}$$

This part of the paper deals with the purely kinematical significance of ψ, u, v, ζ, all computations being supposed to be made at a given time t. For the sake of brevity we shall therefore write

$\psi(x, \varpi)$, $\Psi(\omega_1, \omega_2)$ etc. for $\psi(x, \varpi, t)$, $\Psi(\omega_1, \omega_2, t)$ etc.

THEOREM I. *The following partial differential equations connect the Fourier transforms of the stream function, velocity components and the vorticity.*

$$\frac{\partial U}{\partial \omega_2} = -\omega_2 \Psi , \tag{19}$$

$$\frac{\partial V}{\partial \omega_2} = \omega_1 \Psi , \tag{20}$$

$$\frac{\partial^2 Z}{\partial \omega_2^2} = 3i\omega_2 \Psi + i(\omega_1^2 + \omega_2^2) \frac{\partial \Psi}{\partial \omega_2} . \tag{21}$$

Substituting (1) into (16) we get

$$U = -\frac{1}{4\pi^2}\int_D \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma$$

so that, since the integrand concerned is continuous

$$\frac{\partial U}{\partial \omega_2} = \frac{i}{4\pi^2}\int_D \frac{\partial \psi}{\partial \varpi} e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma$$

$$= \frac{i}{4\pi^2}\left\{\int_D \frac{\partial}{\partial \varpi}(\psi e^{-i(\omega_1 x + \omega_2 \varpi)}) d\sigma + i\omega_2 \int_D \psi e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma\right\}$$

$$= \frac{i}{4\pi^2}\left\{-\int_B \beta \psi e^{-i(\omega_1 x + \omega_2 \varpi)} ds + 4\pi^2 \omega_2 i \Psi\right\} \tag{21a}$$

by the Green's formula.

$= -\omega_2 \Psi$, since $\psi = 0$ on B by (4). Thus we get (19). Substituting (2) into (17) we get (20) by similar considerations. Substituting (5) into (18) we get

$$Z = \frac{1}{4\pi^2}\int_D \frac{1}{\varpi}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}\right) e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma$$

so that,

$$\frac{\partial Z}{\partial \omega_2} = -\frac{i}{4\pi^2}\int_D \left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial^2 \varpi}\right) e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma + \frac{i}{4\pi^2}\int \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi} e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma . \tag{22}$$

We now make use of Green's formula

$$\int_D (f\nabla^2 g - g\nabla^2 f)d\sigma = -\int_B \left(\frac{\partial g}{\partial x} - g \frac{\partial f}{\partial x} \right) ds\,; \quad \nabla^2 \equiv \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial \varpi^2} \tag{22a}$$

Putting $f=\psi$, $g=e^{-i(\omega_1 x+\omega_2 \varpi)}$ we get on account of (4) and the fact

$$\nabla^2 e^{-i(\omega_1 x+\omega_2 \varpi)} = -(\omega_1^2+\omega_2^2)e^{-i(\omega_1 x+\omega_2 \varpi)}\,, \tag{23}$$

$$-(\omega_1^2+\omega_2^2)\int_D \psi e^{-i(\omega_1 x+\omega_2 \varpi)}d\sigma = \int_D \nabla^2 \psi e^{-i(\omega_1 x+\omega_2 \varpi)}d\sigma\,,$$

so that (22) gives

$$\frac{\partial Z}{\partial \omega_2} = i(\omega_1^2+\omega_2^2)\Psi - iU\,.$$

Differentiating once more with respect to ω_2 and using (19) we get (21). Since by (11)

$$\zeta(x, \varpi) = \int_\Omega Z(\omega_1, \omega_2)e^{-i(\omega_1 x+\omega_2 \varpi)}d\omega$$

$$= \lim_{\lambda\to\infty}\int_{C\lambda} Z(\omega_1, \omega_2)e^{-i(\omega_1 x+\omega_2 \varpi)}d\omega \tag{24}$$

we can interpret $Z(\omega_1, \omega_2)d\omega_1 d\omega_2$ as the contribution of vortices the frequencies of which are between ω_1 and $\omega_1+d\omega_1$ and ω_2 and $\omega_2+d\omega_2$.

2. Since the kinetic energy of an element of fluid contained between the two semi-cylindrical surfaces of length dx, $\varpi=\varpi$ and $\varpi=\varpi+d\varpi$ is

$$\frac{1}{2}\pi\varpi(u^2+v^2)d\sigma\,,$$

the kinetic energy of the entire fluid is

$$\frac{\pi}{2}\int_D \varpi(u^2+v^2)d\sigma = \frac{\pi}{2}\int_D \frac{1}{\varpi}\left\{ \left(\frac{\partial\psi}{\partial x}\right)^2 + \left(\frac{\partial\psi}{\partial\varpi}\right)^2 \right\} d\sigma\,.$$

This can be written as

$$E = \frac{\pi}{2}\int_D \left[\left\{ \frac{\partial}{\partial x}\left(\frac{\psi}{\varpi^{1/2}}\right)\right\}^2 + \left\{ \frac{\partial}{\partial \varpi}\left(\frac{\psi}{\varpi^{1/2}}\right)\right\}^2 + \frac{1}{\varpi^2}\psi\frac{\partial\psi}{\partial\varpi} - \frac{\psi^2}{4\varpi^3} \right] d\sigma$$

$$= \frac{\pi}{2}\int_D \left[\left\{ \frac{\partial}{\partial x}\left(\frac{\psi}{\varpi^{1/2}}\right)\right\}^2 + \left\{ \frac{\partial}{\partial \varpi}\left(\frac{\psi}{\varpi^{1/2}}\right)\right\}^2 \right] d\sigma$$

$$+ \frac{\pi}{2}\int_D \left(\frac{1}{\varpi^2}\psi\frac{\partial\psi}{\partial\varpi} - \frac{\psi^2}{4\varpi^3} \right) d\sigma\,. \tag{25}$$

To the first integral on the right-hand side of (25) we apply Green's formula (which requires that $\frac{\psi}{\varpi^{1/2}}$ together with its derivatives should be finite).

$$\int_D f\nabla^2 f d\sigma + \int_D \left\{\left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial \varpi}\right)^2\right\} d\sigma = -\int_B f \frac{\partial f}{\partial x} ds .$$

Putting $f = \frac{\psi}{\varpi^{1/2}}$ in this we get

$$\int_D \left[\left\{\frac{\partial}{\partial x}\left(\frac{\psi}{\varpi^{1/2}}\right)\right\}^2 + \left\{\frac{\partial}{\partial \varpi}\left(\frac{\psi}{\varpi^{1/2}}\right)\right\}^2\right] d\sigma = -\int_D \frac{\psi}{\varpi^{1/2}} \nabla^2 \left(\frac{\psi}{\varpi^{1/2}}\right) d\sigma$$

$$-\int_B \frac{\psi}{\varpi^{1/2}} \frac{\partial}{\partial x}\left(\frac{\psi}{\varpi^{1/2}}\right) ds \qquad \text{(25a)};$$

by the boundary condition (4) the last integral on the right-hand side of this equation is zere. Hence

$$\int_D \left[\left\{\frac{\partial}{\partial x}\left(\frac{\psi}{\varpi^{1/2}}\right)\right\}^2 + \left\{\frac{\partial}{\partial \varpi}\left(\frac{\psi}{\varpi^{1/2}}\right)\right\}^2\right] d\sigma = -\int_D \frac{\psi}{\varpi^{1/2}} \left(\frac{1}{\varpi^{1/2}} \frac{\partial^2 \psi}{\partial x^2}\right.$$

$$\left. + \frac{1}{\varpi^{1/2}} \frac{\partial^2 \psi}{\partial \varpi^2} - \frac{1}{\varpi^{3/2}} \frac{\partial \psi}{\partial \varpi} + \frac{3\psi}{4\varpi^{5/2}}\right) d\sigma .$$

Substituting this in (25) we get

$$E = -\frac{\pi}{2}\int_D \psi \frac{1}{\varpi}\left(\frac{\partial^2 \psi}{\partial x^2} + \frac{\partial^2 \psi}{\partial \varpi^2} - \frac{1}{\varpi}\frac{\partial \psi}{\partial \varpi}\right) d\sigma + \frac{\pi}{2}\int_D \left(\frac{1}{\varpi^2} \psi \frac{\partial \psi}{\partial \varpi} - \frac{\psi^2}{\varpi^3}\right) d\sigma$$

$$= -\frac{\pi}{2}\int_D \psi\zeta d\sigma + \frac{\pi}{4}\int_D \frac{\partial}{\partial \varpi}\left(\frac{\psi^2}{\varpi^2}\right) d\sigma ; \qquad \text{by (5)}$$

$$= -\frac{\pi}{2}\int_D \psi\zeta d\sigma - \frac{\pi}{4}\int_B \beta \frac{\psi^2}{\varpi^2} ds , \qquad \text{(25b)}$$

by Green's formula, provided $\frac{\psi}{\varpi}$ is finite and continuous.

$$= -2\pi^3 \int_\Omega \Psi \bar{Z} d\omega$$

by applying Parseval's formula to the first and boundary condition (4) to the second integral on the right-hand side. Hence, by (9) we have

$$\gamma(\omega_1, \omega_2, t) = -2\pi^3 \Psi(\omega_1, \omega_2, t)\bar{Z}(\omega_1, \omega_2, t)\,. \tag{26}$$

Given Z, Ψ is known from (21), γ from (26) and U and V from (19) and (20). Thus theorem I and (26) reduce the harmonic analysis of the flow to the study of the Fourier transform Z of the vorticity.

From (9) we can interpret $\gamma(\omega_1, \omega_2, t)d\omega_1 d\omega_2$ as the amount of kinetic energy coming from the vortices with frequencies between ω_1 and $\omega_1 + d\omega_1$, and ω_2 and $\omega_2 + d\omega_2$. The Fourier transforms Ψ and Z being continuous and bounded in the entire plane Ω, we see from (26) that $\gamma(\omega_1, \omega_2)$ is also continuous and bounded in the entire Ω plane.

3. To make any study of the Fourier transforms Z, U, V, Ψ and the spectral function with time and following the motion of the fluid, we must obtain the equivalent relation between Z, U, V etc. to the equation of motion (6). To this end we take the Fourier transform of both sides of (6) and rearrange as

$$\frac{1}{4\pi^2}\int_D \frac{\partial\zeta}{\partial t} e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma + \frac{1}{4\pi^2}\int_D \left(u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial\varpi} - \frac{v\zeta}{\varpi}\right) e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma$$
$$= \frac{\nu}{4\pi^2}\int_D \left(\frac{\partial^2\zeta}{\partial x^2} + \frac{\partial^2\zeta}{\partial\varpi^2}\right) e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma + \frac{\nu}{4\pi^2}\int_D \frac{1}{\varpi}\left(\frac{\partial\zeta}{\partial\varpi} - \frac{\zeta}{\varpi}\right) e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma\,. \tag{27}$$

Since ζ and $\frac{\partial\zeta}{\partial t}$ are continuous in t, the functions Z and $\frac{\partial Z}{\partial t}$ are also continuous in t. Thus

$$\frac{\partial}{\partial t} Z(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D \frac{\partial\zeta}{\partial t} e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma\,. \tag{28}$$

To compute the second Fourier transform on the left-hand side of (27) we start from the equation

$$\left(u\frac{\partial\zeta}{\partial x} + v\frac{\partial\zeta}{\partial\varpi}\right) e^{-i(\omega_1 x + \omega_2 \varpi)} = \frac{\partial}{\partial x}\{u\zeta e^{-i(\omega_1 x + \omega_2 \varpi)}\}$$
$$+ \frac{\partial}{\partial\varpi}\{v\zeta e^{-i(\omega_1 x + \omega_2 \varpi)}\} + i(\omega_1 u\zeta + \omega_2 v\zeta) e^{-i(\omega_1 x + \omega_2 \varpi)}$$
$$-\zeta\left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial\varpi}\right) e^{-i(\omega_1 x + \omega_2 \varpi)}\,.$$

By the equation of continuity in axially symmetrical flows

$$\frac{\partial u}{\partial x}+\frac{\partial v}{\partial \varpi}+\frac{v}{\varpi}=0\,,$$

the above equation reduces to

$$\left(u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial \varpi}-\frac{v\zeta}{\varpi}\right)e^{-i(\omega_1 x+\omega_2\varpi)}=\frac{\partial}{\partial x}\{u\zeta e^{-i(\omega_1 x+\omega_2\varpi)}\}$$

$$+\frac{\partial}{\partial \varpi}\{v\zeta e^{-i(\omega_1 x+\omega_2\varpi)}\}+i(\omega_1 u\zeta+\omega_2 v\zeta)e^{-i(\omega_1 x+\omega_2\varpi)}\,.$$

We integrate this equation over D and note that by Green's formula

$$\int_D\frac{\partial}{\partial x}\{u\zeta e^{-i(\omega_1 x+\omega_2\varpi)}\}d\sigma=-\int_B \alpha u\zeta e^{-i(\omega_1 x+\omega_2\varpi)}ds=0 \tag{29}$$

$$\int_D\frac{\partial}{\partial \varpi}\{v\zeta e^{-i(\omega_1 x+\omega_2\varpi)}\}\,d\sigma=-\int_B \beta v\zeta e^{-i(\omega_1 x+\omega_2\varpi)}ds=0 \tag{30}$$

on account of the boundary condition (3). Accordingly $\left(\frac{\zeta}{\varpi}\right.$ being finite)

$$\int_D\left(u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial \varpi}-\frac{v\zeta}{\varpi}\right)e^{-i(\omega_1 x+\omega_2\varpi)}d\sigma$$

$$=i\omega_1\int_D u\zeta e^{-i(\omega_1 x+\omega_2\varpi)}d\sigma+i\omega_2\int_D v\zeta e^{-i(\omega_1 x+\omega_2\varpi)}$$

$$=4\pi^2 i\omega_1\int_\Theta U(\theta_1,\theta_2)\bar{Z}(\theta_1+\omega_1,\theta_2+\omega_2)d\theta+4\pi^2 i\omega_2\int_\Theta V(\theta_1,\theta_2)\bar{Z}(\theta_1+\omega_1,\theta_2+\omega_2)d\theta$$

by the formula (13), where the variables of integration are now θ_1, θ_2 and $d\theta=d\theta_1 d\theta_2$, the integral being exteneded to the entire plane Θ. Thus we have

$$\frac{1}{4\pi^2}\int_D\left(u\frac{\partial \zeta}{\partial x}+v\frac{\partial \zeta}{\partial \varpi}-\frac{v\zeta}{\varpi}\right)e^{-i(\omega_1 x+\omega_2\varpi)}d\sigma$$

$$=i\int_\Theta\{\omega_1 U(\theta_1,\theta_2)+\omega_2 V(\theta_1,\theta_2)\}\bar{Z}(\theta_1+\omega_1,\theta_2+\omega_2)d\theta\,. \tag{31}$$

To compute the first Fourier transform on the right-hand side of (27) let us put

$$C(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_D \nabla^2\zeta \cdot e^{-i(\omega_1 x+\omega_2\varpi)}d\sigma . \tag{32}$$

Applying Green's formula

$$\int_D\{\nabla^2\zeta \cdot e^{-i(\omega_1 x+\omega_2\omega)} - \zeta\nabla^2 e^{-i(\omega_1 x+\omega_2\varpi)}\}d\sigma$$

$$= -\int_B\left\{\frac{\partial\zeta}{\partial x} e^{-i(\omega_1 x+\omega_2\varpi)} - \zeta\frac{\partial}{\partial x}(e^{-i(\omega_1 x+\omega_2\varpi)})\right\}ds ,$$

and putting

$$\varphi_1(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_B\left\{\frac{\partial\zeta}{\partial x} e^{-i(\omega_1 x+\omega_2\varpi)} + i(\omega_1\alpha+\omega_2\beta)\zeta e^{-i(\omega_1 x+\omega_2\varpi)}\right\}ds . \tag{33}$$

we get

$$C(\omega_1, \omega_2, t) = -(\omega_1^2+\omega_2^2)Z(\omega_1, \omega_2, t) - \varphi_1(\omega_1, \omega_2, t) . \tag{34}$$

To compute the second Fourier transform on the right-hand side of (27) we observe that

$$\frac{\nu}{4\pi^2}\int_D\left(\frac{1}{\varpi}\frac{\partial\zeta}{\partial\varpi} - \frac{\zeta}{\varpi^2}\right)e^{-i(\omega_1 x+\omega_2\varpi)}d\sigma = \frac{\nu}{4\pi^2}\int_D\frac{\partial}{\partial\varpi}\left(\frac{\zeta}{\varpi}e^{-i(\omega_1 x+\omega_2\varpi)}\right)d\sigma$$

$$+\frac{\nu i\omega_2}{4\pi^2}\int_D\frac{\zeta}{\varpi}e^{-i(\omega_1 x+\omega_2\varpi)}d\sigma .$$

By Green's formula this is

$$= -\frac{\nu}{4\pi^2}\int_B\beta\frac{\zeta}{\varpi}e^{-i(\omega_1 x+\omega_2\varpi)}ds + \frac{\nu i\omega_2}{4\pi^2}\int_D\frac{\zeta}{\varpi}e^{-i(\omega_1 x+\omega_2\varpi)}d\sigma .$$

Putting

$$\varphi_2(\omega_1, \omega_2, t) = \frac{1}{4\pi^2}\int_B\beta\frac{\zeta}{\varpi}e^{-i(\omega_1 x+\omega_2\varpi)}ds , \tag{35}$$

we get

$$\frac{\nu}{4\pi^2}\int_D\left(\frac{1}{\varpi}\frac{\partial\zeta}{\partial\varpi} - \frac{\zeta}{\varpi^2}\right)e^{-i(\omega_1 x+\omega_2\varpi)}d\sigma = -\nu\varphi_2(\omega_1, \omega_2, t)$$

$$+\frac{\nu i\omega_2}{4\pi^2}\int_D\frac{\zeta}{\varpi}e^{-i(\omega_1 x+\omega_2\varpi)}d\sigma . \tag{36}$$

Now substituting from (28), (31)$\cdots$(36) into (27) we get

$$\frac{\partial}{\partial t} Z(\omega_1, \omega_2. t) + i\int \{\omega_1 U(\theta_1, \theta_2) + \omega_2 V(\theta_1, \theta_2)\}\bar{Z}(\theta_1 + \omega_1, \theta_2 + \omega_2) d\theta$$

$$+ \nu(\omega_1^2 + \omega_2^2) Z(\omega_1, \omega_2, t) + \nu\varphi_1(\omega_1, \omega_2, t) + \nu\varphi_2(\omega_1, \omega_2, t)$$

$$- \frac{\nu i \omega_2}{4\pi^2}\int_D \frac{\zeta}{\varpi} e^{-i(\omega_1 x + \omega_2 \varpi)} d\sigma = 0 \tag{36a}$$

Differentiating this equation twice with respect to ω_2, we get

THEOREM II:—

$$\frac{\partial^3}{\partial t \partial \omega_2^2} Z(\omega_1, \omega_2, t) + i \frac{\partial^2}{\partial \omega_2^2}\int \{\omega_1 U(\theta_1, \theta_2) + \omega_2 V(\theta_1, \theta_2)\}\bar{Z}(\theta_1 + \omega_1, \theta_2 + \omega_2) d\theta$$

$$(37)$$

$$+ \nu \frac{\partial^2}{\partial \omega_2^2}(\varphi_1 + \varphi_2) + \nu(\omega_1^2 + \omega_2^2) \frac{\partial^2}{\partial \omega_2^2} Z(\omega_1, \omega_2, t) + 3\nu\omega_2 \frac{\partial}{\partial \omega_2} Z(\omega_1, \omega_2, t) = 0 .$$

This integro-differential equation is fully equivalent to the Navier-Stokes equation of motion and therefore, as remarked by Kampé de Fériet, is the rational starting point of any rigorous study of the Fourier transform of the vorticity and hence of the velocity components, the stream function and the spectral function. As regards the scope of application of this equation, we refer to the remark of Kampé de Fériet upon equation (50) of his paper cited above.

4. We now consider the case where the flow extends over the entire half plane

$$X(-\infty < x < \infty, \qquad 0 < \varpi < \infty) .$$

This case may be regarded as a limiting case of the previous problem. The boundary condition (3) now means that the fluid is at rest at infinity:

$$\lim_{r\to\infty} u(x, \varpi, t) = 0, \quad \lim_{r\to\infty} v(x, \varpi, t) = 0 , \tag{38}$$

$$r = (x^2 + \varpi^2)^{1/2} .$$

If the kinetic energy is to remain finite

$$E = \frac{\pi}{2}\int_X \varpi(u^2 + v^2) d\sigma < \infty . \tag{39}$$

To ensure the existence of the Fourier transforms Z, Ψ, U, V it is sufficient to assume that ζ, ψ, u, v are absolutely integrable that is, $\int_X |\zeta| d\sigma$ etc. are finite. This condition is satified provided ζ, ψ, u, v are at the most of order r^{-3} for large values of r. And in that case (38) and (39) are automatically satisfied. Also in that case all the properties (a) to (f) enumerated for the Fourier transform (10) still hold for the Fourier transform of ζ, ψ, u, v when the fluid occupies the entire half plane X; also the relations (19) to (21) and (26) hold between them by virtue of the limit considerations of the following discussion.

To compute the Fourier transform of (6) we can use the same process as before, taking for D the semi-circle $x^2+\varpi^2<R^2$ and then letting R tend to ∞. We must note however that we no longer have $u=v=0$ on the boundary B; we must examine therefore some terms in our equations.

In the evaluation of the Fourier transform of $u\frac{\partial\zeta}{\partial x}+v\frac{\partial\zeta}{\partial\varpi}-\frac{v\zeta}{\varpi}$, the two terms (29) and (30) no longer vanish. According to Schwarz' inequality we have

$$\left|\int_B \alpha u\zeta e^{-i(\omega_1 x+\omega_2\varpi)}ds\right| \leqq \int_B |u||\zeta| ds \leqq \left[\int_B u^2 ds\right]^{1/2}\left[\int_B \zeta^2 ds\right]^{1/2}.$$

From our assumption concerning the decrease of u and ζ,

$$\lim_{R\to\infty}\int_B u^2 ds=0, \quad \lim_{R\to\infty}\int_B \zeta^2 ds=0.$$

Similar considerations apply to (30) and with a little modification to (25a). Thus in the limit the exprssion (31) for the Fourier transform of (6) and (25b) still hold.

In the evaluation of the Fourier transform of $\nabla^2\zeta$, we have from (33)

$$|\varphi_1| \leqq \frac{1}{4\pi^2}\int_B \left\{\left|\frac{\partial\zeta}{\partial x}\right| + (|\omega_1|+|\omega_2|)|\zeta|\right\} ds.$$

The assmption of decreasing of ζ yields

$$\lim_{R\to\infty}\int_B \left|\frac{\partial\zeta}{\partial x}\right| ds=0, \quad \lim_{R\to\infty}\int_B |\zeta| ds=0;$$

thus

$$\lim_{R\to\infty} \varphi_1 = 0 .$$

The Fourier transforms C and Z being extended to the whole plane, we have thus now

$$C = -(\omega_1^2 + \omega_2^2) Z .$$

By the same considerations we have from (35)

$$\lim_{R\to\infty} \varphi_2 = 0 ,$$

and the first terms on the right-hand side of (21a) and (22a) (with $f=\psi$, $g=e^{-i(\omega_1 x+\omega_2 \bar{\omega})}$) tend to zero as $R\to\infty$, so that in the limit the expressions (19), (20) and (21) still hold. Also by the same limiting process the second integral on the right-hand side of (25b) tends to zero and the expression (26) holds for the case of fluid occupying the entire half plane.

THEOREM III. *In the case of a flow extending over the entire half plane X, where the fluid is at rest at infinity and where ζ, ψ, u, v decrease so rapidly that their Fourier transforms exist, equation* (37) *reduces to*

$$\frac{\partial^3}{\partial t \partial \omega_2^2} Z(\omega_1, \omega_2, t) + i \frac{\partial^2}{\partial \omega_2^2} \int_{\Theta} \{\omega_1 U(\theta_1, \theta_2) + \omega_2 V\theta(_1, \theta_2)\} \bar{Z}(\theta_1 + \omega_1, \theta_2 + \omega_2) d\theta$$

$$+ \nu(\omega_1^2 + \omega_2^2) \frac{\partial^2}{\partial \omega_2^2} Z(\omega_1, \omega_2, t) + 3\nu\omega_2 \frac{\partial}{\partial \omega_2} Z(\omega_1, \omega_2, t) = 0 . \quad (40)$$

5. The equation (40) further simplifies in the case of self-superposable flows characterised by $\zeta = \bar{\omega} f(\psi)$; in that case the non-linear terms $\left(u \frac{\partial \zeta}{\partial x} + v \frac{\partial \zeta}{\partial \bar{\omega}} - \frac{v \zeta}{\bar{\omega}} \right)$ on the left-hand side of (31) identically vanish and so does their Fourier transform. Hence (40) further reduces to

$$\frac{\partial^3}{\partial t \partial \omega_2^2} Z(\omega_1, \omega_2, t) + \nu(\omega_1^2 + \omega_2^2) \frac{\partial^2}{\partial \omega_2^2} Z(\omega_1, \omega_2, t) + 3\nu\omega_2 \frac{\partial}{\partial \omega_2} Z(\omega_1, \omega_2, t) = 0 . \quad (41)$$

The order of (41) can further be reduced by putting $\chi = \frac{\partial Z}{\partial \omega_2}$ we get in that case

$$\frac{\partial^2 \chi}{\partial t \partial \omega_2} + \nu(\omega_1^2 + \omega_2^2) \frac{\partial \chi}{\partial \omega_2} + 3\nu\omega_2 \chi = 0 \,. \tag{42}$$

Assuming that Z should decrease with time let us take

$$\chi = \chi_0 e^{-\nu\lambda(\omega_1, \omega_2)t} \,, \tag{43}$$

where χ_0 is a function of ω_1 and ω_2 only. Substituting in (42) we get

$$\{\omega_1^2 + \omega_2^2 - \lambda\} \frac{\partial \chi_0}{\partial \omega_2} + \left\{ 3\omega_2 - \frac{\partial \lambda}{\partial \omega_2} + \nu t \frac{\partial \lambda}{\partial \omega_2} (\lambda - \omega_1^2 - \omega_2^2) \right\} \chi_0 = 0 \,,$$

or

$$\frac{1}{\chi_0} \frac{\partial \chi_0}{\partial \omega_2} = \nu t \frac{\partial \lambda}{\partial \omega_2} - \frac{\frac{\partial \lambda}{\partial \omega_2} + 3\omega_2}{\omega_1^2 + \omega_2^2 - \lambda} \,.$$

Since $\frac{\partial \chi_0}{\partial \omega_2} \Big/ \chi_0$ cannot be a function of t, $\frac{\partial \lambda}{\partial \omega_2} = 0$, i. e. λ must be independent of ω_2. So that we get

$$\chi_0 = k_1(\omega_1) \cdot (\omega_1^2 + \omega_2^2 - \lambda)^{-3/2} \,,$$

where $k_1(\omega_1)$ is an arbitrary complex function of ω_1. Hence from (43) we have

$$\chi = k_1 e^{-\nu\lambda t} (\omega_1^2 + \omega_2^2 - \lambda)^{-3/2} \,,$$

and from the relation $\chi = \frac{\partial Z}{\partial \omega_2}$

$$Z = e^{-\nu\lambda t} \left\{ \frac{k_1}{\omega_1^2 - \lambda} \cdot \frac{\omega_2}{(\omega_1^2 + \omega_2^2 - \lambda)^{1/2}} + k_2(\omega_1) \right\} .$$

But since Z has to satisfy ultimately Eq. (36a) modified for the self-superposable flows extending over the entire plane, $k_2(\omega_1) = 0$ or $\lambda = \omega_1^2$. But $\lambda = \omega_1^2$ makes Z independent of ω_2 which contradicts the property (a) of the Fourier transform Z. Hence

$$Z = e^{-\nu\lambda(\omega_1)t} k(\omega_2) \cdot \frac{\omega_2}{(\omega_1^2 + \omega_2^2 - \lambda)^{1/2}} \,, \tag{44}$$

or if Z_0 be an arbitrary value of Z at $t=0$, (44) can also be written as

$$Z=Z_0 e^{-\nu\lambda(\omega_1)t}. \tag{45}$$

Now, since it is essential to differentiate Eq. (36a) at least once to get a differential equation in Z alone, Eq. (42) contains derivative with respect to ω_2 and hence whatever solution we might get instead of (44) or (45) Z_0 will be arbitrary only as regards its behaviour with respect to ω_1. Also as regards the exponential decay of Z with time, we know that λ must be independent of ω_2. These two features for the case of axially symmetrical three-dimensional flow are in contrast to the solution obtained by Kampé de Fériet for the two-dimensional problem.

Thus (45) exhibits the decrease of Z with t, ω_1, due to the viscosity of the fluid.

I am grateful to Prof. R. Ballabh for guidance in this work. I also express my indebtedness to Prof. Kampé de Fériet who pointed out to me a shortcoming in an earlier version of the paper and suggested its remedy.

Lucknow University, India.