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An operator-theoretical integration of the wave equation.

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§ I. Introduction and the theorem. We consider the Cauchy problem for the wave equation in m-dimensional euclidean space E^m:

$$\frac{\partial^2 u(t, x)}{\partial t^2} = Au(t, x),\ u(0, x) = f(x),\ u_t(0, x) = g(x)\,, \tag{1.1}$$

$$A = a^{ij}(x)\frac{\partial^2}{\partial x_i \partial x_j} + b^i(x)\frac{\partial}{\partial x_i} + c(x),\ x = (x_1, \cdots, x_m)\,.$$

The problem is equivalent to the matricial equation

$$\frac{\partial}{\partial t}\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix},\quad \begin{pmatrix} u(0, x) \\ v(0, x) \end{pmatrix} = \begin{pmatrix} f(x) \\ g(x) \end{pmatrix}, \tag{1.1$'$}$$

and we may apply the theory of semi-group of linear operators[1] to the integration in the large of (1.1), by considering, in a suitable Banach space, the "resolvent equation"

$$\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - n^{-1}\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\right)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \quad \text{for large } |n|,\quad (n = \text{integer}) \tag{1.2}$$

and obtaining the estimate

$$\left\|\begin{pmatrix} u \\ v \end{pmatrix}\right\| \leqq (1 + |n^{-1}|\beta)\left\|\begin{pmatrix} f \\ g \end{pmatrix}\right\| \tag{1.3}$$

where β is a positive constant independent of n, f and g. The irrelevance of the sign of n implies that

$$\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \tag{1.4}$$

1) E. Hille: Functional Analysis and Semi-groups, New York (1948).
K. Yosida: On the differentiability and the representation of one-parameter semi-group of linear operators, J. Math. Soc. Japan, 1 (1948), 15-21.

generates a group $\{T_t\}_{-\infty<t<\infty}$ such that

$$T_t \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} = \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} \tag{1.5}$$

yields a solution of (1.1)$'$ when the initial functions $\{f(x), g(x)\}$ are prescribed appropriately.

In this way we may prove the solvability of the Cauchy problem in the large of (1.1) without appealing to the classical Cauchy-Kowalewski existence theorem or to the Laplace-Fourier transform theory[2]. Our result reads as the

THEOREM. *Let (i) the coefficients $a^{ij}(x)$, $b^i(x)$, $c(x)$ are real-valued C^∞ functions and let*

$$\max \left(\sup_x |a^{ij}(x)|, \ \sup_x |b^i(x)|, \ \sup_x |c(x)|, \ \sup_x \left|\frac{\partial a^{ji}(x)}{\partial x_k}\right|, \right. \tag{1.6}$$
$$\left. \sup_x \left|\frac{\partial b^i(x)}{\partial x_j}\right|, \ \sup_x \left|\frac{\partial^2 a^{ij}(x)}{\partial x_k \partial x_s}\right| \right) = \eta < \infty$$

Let (ii), moreover, there exist positive constants λ and μ such that

$$\mu \sum_i \xi_i^2 \geqq a^{ij}(x)\xi_i \xi_j \geqq \lambda \sum_i \xi_i^2 . \tag{1.7}$$

Then there exists a positive constant β such that, for sufficiently small positive constant α_0, the Cauchy problem for (1.1) is solvable in the following sense: For any pair $\{f(x), g(x)\}$ of C^∞ functions such that $(A^k f)(x)$, $(A^k g)(x)$ and their first partial derivatives are square integrable over E^m (for all $k=0, 1, \cdots$), the equation (1.1) admits a C^∞ solution $u(t, x)$ satisfying the estimate

$$((u-\alpha_0 Au, u)+\alpha_0(u_t, u_t))^{1/2} \leqq \exp(\beta|t|)\,((f-\alpha_0 Af, f)+\alpha_0(g, g))^{1/2}, \tag{1.8}$$

(h, k) denoting, as usual, the inner product

$$(h, k) = \int_{E^m} h(x)k(x)dx, \ dx = dx_1 dx_2 \cdots dx_m . \tag{1.9}$$

2) Cf. J. Schauder: Der Anfangswertproblem einer quasi-linearen hyperbolischen Differentialgleichungen, Fund. Math. 24 (1935), 213-246, and J. Leray: Symbolic Calculus with Several Variables, Projections and Boundary Value Problems for Differential Equations, Princeton (1952). The two authors ingeneously make use of the Cauchy-Kowalewski existence theorem in their treatment.

Before proceeding to the proof of the theorem, we must prepare some lemmas concerning the elliptic differential operators A and its formal adjoint A^*:

$$(1.10)\quad (A^*f)(x)=\frac{\partial^2}{\partial x_i \partial x_j}(a^{ij}(x)f(x))-\frac{\partial}{\partial x_i}(b^i(x)f(x))+c(x)f(x).$$

§2. Lemma 1 (concerning the partial integration). Let H be the space of real-valued C^∞ functions $f(x)$ for which

$$(2.1)\qquad ||f||_1=\left(\int_{E^m} f^2dx+\int_{E^m}\sum_i\left(\frac{\partial f}{\partial x_i}\right)^2dx\right)^{1/2},$$

and let H_1 be the completion of H with respect to the norm $||f||_1$. Let similarly H_0 be the completion of H with respect to the norm

$$(2.2)\qquad ||f||_0=\left(\int_{E^m} f^2dx\right)^{1/2}.$$

We have thus introduced two real Hilbert spaces H_1 and H_0, and H and H_1 are $||\ ||_0$-dense in H_0.

LEMMA 1. *Let $f, g\in H_0$. and let $Af\in H_0$. Then we have*

$$(2.3)\qquad (Af, g)=-\int_{E^m} a^{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}dx-\int_{E^m}\frac{\partial a^{ij}}{\partial x_j}\frac{\partial f}{\partial x_i}g\,dx$$
$$+\int_{E^m} b^i\frac{\partial f}{\partial x_i}g\,dx+\int_{E^m} cfg\,dx,$$

viz. we may, in (Af, g), partially integrate the terms containing the second order derivatives as if the integrated terms are nought.

PROOF. By (1.6), $Af\in H_0$ and the fact that f and g both belong to H_1 we see that $a^{ij}\dfrac{\partial^2 f}{\partial x_i\partial x_j}\cdot g$ is integrable over E^m. We have, by Fubini's theorem,

$$\int_{E^m} a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}\cdot g\,dx=\lim_{\substack{\delta_1\to\infty\\ \varepsilon_1\to-\infty}}\int_{-\infty}^{\infty}\cdots\int dx_2\cdots dx_m\int_{\varepsilon_1}^{\delta_1}a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}\cdot g\,dx_1,$$

and

$$\int_{\varepsilon_1}^{\delta_1}a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}\cdot g\,dx_1=\left[a^{1j}\frac{\partial f}{\partial x_j}g\right]_{x_1=\varepsilon_1}^{x_1=\delta_1}+\left\{\int_{\varepsilon_1}^{\delta_1}-a^{1j}\frac{\partial f}{\partial x}\frac{gx}{\partial x_1}dx_1\right.$$

$$-\int_{\varepsilon_1}^{\delta_1}\frac{\partial a^{1j}}{\partial x_1}\frac{\partial f}{\partial x_1}\cdot g\,dx_1\Big\}+\int_{\varepsilon_1}^{\delta_1}\sum_{i,j\neq 2}a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}g\,dx_1$$

$$=\kappa_1(\delta_1,\varepsilon_1\,x_2,\cdots,x_m)+\kappa_2(\delta_1,\varepsilon_1,x_2,\cdots,x_m)+\kappa_3(\delta_1,\varepsilon_1,x_2,\cdots,x_m)\,.$$

By (1.6) and Schwarz inequality we have

$$\left|\int_{-\infty}^{\infty}\cdots\int dx_2\cdots dx_m\,\kappa_1(\delta_1,\varepsilon_1,x_2,\cdots,x_m)\right|\leqq$$

$$\eta\sum_i\left(\int_{-\infty}^{\infty}\cdots\int\left|\frac{\partial f(\delta_1,x_2,\cdots,x_m)}{\partial x_j}\right|^2dx_2\cdots dx_m\times\int_{-\infty}^{\infty}\cdots\int g(\delta_1,x_2,\cdots,x_m)^2\,dx_2\cdots dx_m\right)^{1/2}$$

+similar terms pertaining to ε_1 instead of δ_1.

We have, by Fubini's theorem,

$$\int_{E^m}g^2\,dx=\int_{-\infty}^{\infty}dx_1\int_{-\infty}^{\infty}\cdots\int g(x_1,\cdots,x_m)^2\,dx_2\cdots dx_m\,.$$

Hence we see that

$$\lim\int_{-\infty}^{\infty}\cdots\int g(x_1,\cdots,x_m)^2\,dx_2\cdots dx_m=0$$

when x_1 tends to ∞ (or $-\infty$) without taking the values of x_1 which form a set of finite measure. The same reasoning applies also when we replace g by $\partial f/\partial x_j$. Therefore there exist two sequences $\{\delta_1^{(k)}\}$ and $\{\varepsilon_1^{(k)}\}$ such that

$$\lim_{\substack{\delta_1^{(k)}\to\infty\\ \varepsilon_1^{(k)}\to-\infty}}\int_{-\infty}^{\infty}\cdots\int\kappa_1(\delta_1^{(k)},\varepsilon_1^{(k)},x_2,\cdots,x_m)\,dx_2\cdots dx_m=0\,. \tag{2.4}$$

On the other hand, we see, remembering (1.6) and the fact that $f, g, \partial g/\partial x_1$ and $\partial f/\partial x_j$ belongs to H_0, that

$$\lim_{\substack{\delta_1\to\infty\\ \varepsilon_1\to-\infty}}\int_{-\infty}^{\infty}\cdots\int\kappa_2(\delta_1,\varepsilon_1,x_2,\cdots,x_m)\,dx_2\cdots dx_m$$

$$=\int_{E^m}\left\{-a^{1j}\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_1}-\frac{\partial a^{1j}}{\partial x_1}\frac{\partial f}{\partial x_j}g\right\}dx=\kappa_2$$

exists and is finite. Therefore

$$\int_{E^m} a^{ij}\frac{\partial^2 f}{\partial x_i \partial x_j} g\,dx = \kappa_2 + \lim_{\substack{\delta_1^{(k)}\to\infty\\ \varepsilon_1^{(k)}\to-\infty}} \int_{-\infty}^{\infty}\cdots\int \kappa_3(\delta_1^{(k)}, \varepsilon_1^{(k)}, x_2,\cdots, x_m)\,dx_2\cdots dx_m\,.$$

Thus

$$\int_{\varepsilon_1^{(k)}}^{\delta_1^{(k)}} \sum_{i,j\neq 1} a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j} g\,dx_1$$

is integrable over the domain defined by $-\infty < x_i < \infty$ $(i=2,\cdots,m)$. Hence

$$\kappa_3 = \lim_{\substack{\delta_1^{(k)}\to\infty\\ \varepsilon_1^{(k)}\to-\infty}} \int_{-\infty}^{\infty}\cdots\int \kappa_3(\delta_1^{(k)}, \varepsilon_1^{(k)}, x_2,\cdots, x_m)\,dx_2\cdots dx_m$$

$$= \lim_{\substack{\delta_1^{(k)}\to\infty\\ \varepsilon_1^{(k)}\to-\infty}} \lim_{\substack{\delta_2\to\infty\\ \varepsilon_2\to-\infty}} \int_{-\infty}^{\infty}\cdots\int dx_3\cdots dx_m \left\{\int_{\varepsilon_2}^{\delta_2} dx_2 \int_{\varepsilon_1^{(k)}}^{\delta_1^{(k)}} \sum_{i,j\neq 1} a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j} g\,dx_1\right\}.$$

However

$$\{\quad\} = \int_{\varepsilon_1^{(k)}}^{\delta_1^{(k)}} dx_1 \int_{\varepsilon_2}^{\delta_2} \sum_{i,j\neq 1} a^{ij}\frac{\partial^2 f}{\partial x_i \partial g_j} g\,dx_2$$

$$= \int_{\varepsilon_1^{(k)}}^{\delta_1^{(k)}} dx_1 \left[\left[a^{2j}\frac{\partial f}{\partial x_j} g\right]_{x_2=\varepsilon_2}^{x_2=\delta_2} + \int_{\varepsilon_2}^{\delta_2} -a^{2j}\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_2}\,dx_2 - \int_{\varepsilon_2}^{\delta_2}\frac{\partial a^{2j}}{\partial x_2}\frac{\partial f}{\partial x_j} g dx_2\right]$$

$$+ \int_{\varepsilon_1^{(k)}}^{\delta_1^{(k)}}\int_{\varepsilon_2}^{\delta_2} \sum_{i,j\neq 1,2} a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j} g\,dx_1\,dx_2\,.$$

By (1.6) and Schwarz inequality, we have

$$\left|\int_{-\infty}^{\infty}\cdots\int dx_3\cdots dx_m \int_{\varepsilon_1^{(k)}}^{\delta_1^{(k)}} a^{2j}\frac{\partial f}{\partial x_j} g\,dx_1\right|$$

$$\leqq \eta \sum_j \left(\int_{-\infty}^{\infty}\cdots\int \left(\frac{\partial f}{\partial x_j}\right)^2 dx_1\,dx_3\cdots dx_m \times \int_{-\infty}^{\infty}\cdots\int g^2\,dx_1\,dx_3\cdots dx_m\right)^{1/2}.$$

Since $\partial f/\partial x_j$ and g both belong to H_0 we see, as in the case of (2.4), that there exist two sequences $\{\delta_2^{(l)}\}$ and $\{\varepsilon_2^{(l)}\}$ such that

$$\lim_{\substack{\delta_2^{(l)}\to\infty \\ \varepsilon_2^{(l)}\to-\infty}} \int_{-\infty}^{\infty}\cdots\int dx_3\cdots dx_m \int_{\varepsilon_1}^{\delta_1}\left[a^{2j}\frac{\partial f}{\partial x_j}g\right]_{x_2=\varepsilon_2^{(l)}}^{x_2=\delta_2^{(l)}} dx_1=0$$

uniformly with respect to δ_1 and ε_1. We have also, by (1.6) and the fact that $\partial f/\partial x_j$, $\partial g/\partial x_2$ all belong to H_0,

$$\lim_{\substack{\delta_1^{(k)}\to\infty \\ \varepsilon_1^{(k)}\to-\infty}} \lim_{\substack{\delta_2\to\infty \\ \varepsilon_2\to-\infty}} \int_{-\infty}^{\infty}\cdots\int dx_3\cdots dx_m \int_{\varepsilon_1^{(k)}}^{\delta_1^{(k)}} dx_1 \int_{\varepsilon_2}^{\delta_2}\left(-a^{2j}\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_2}-\frac{\partial a^{2j}}{\partial x_2}\frac{\partial f}{\partial x_j}g\right)dx_2$$

$$=\int_{-\infty}^{\infty}\cdots\int\left(-a^{2j}\frac{\partial f}{\partial x_j}\frac{\partial g}{\partial x_2}-\frac{\partial a^{2j}}{\partial x_2}\frac{\partial f}{\partial x_j}g\right)dx_1\cdots dx_m.$$

Therefore

$$\int_{E^m} a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}g\,dx=-\left(\int_{E^m}\sum_{i\text{ or }j=1,2} a^{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}dx\right)$$

$$-\left(\int_{E^m}\sum_{i\text{ or }j=1,2}\frac{\partial a^{ij}}{\partial x_i}\frac{\partial f}{\partial x_j}g\,dx\right)$$

$$+\lim_{\substack{\delta_1^{(k)}\to\infty \\ \varepsilon_1^{(k)}\to-\infty}} \lim_{\substack{\delta_2^{(l)}\to\infty \\ \varepsilon_2^{(l)}\to-\infty}} \int_{-\infty}^{\infty}\cdots\int dx_3\cdots dx_m \int_{\varepsilon_1^{(k)}}^{\delta_1^{(k)}}\int_{\varepsilon_2^{(l)}}^{\delta_2^{(l)}}\sum_{i,j\neq 1,2} a^{ij}\frac{\partial^2 f}{\partial x_i\partial x_j}g\,dx_1\,dx_2.$$

Repeating the same argument we obtain (2.3).

REMARK. If $f, g\in H$ and if $A^*f\in H_0$, we may, in (A^*f, g), partally integrate the terms containing the second order derivatives as if the integrated terms are nought:

$$(2.3)'\quad (A^*f,g)=-\int_{E^m}a^{ij}\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial x_j}dx-\int_{E^m}\frac{\partial a^{ij}}{\partial x_i}f\frac{\partial g}{\partial x_i}dx$$

$$-\int_{E^m}b^i f\frac{\partial g}{\partial x_i}dx+\int_{E^m}cfg\,dx.$$

COROLLARY. *There exist a positive constant* κ *and, for sufficiently small* $\alpha>0$, *positive constants* γ *and* δ *such that*,

$$(2.5)\qquad \delta\|f\|_1^2\leqq \begin{matrix}(f-\alpha Af,f)\\ (f-\alpha A^*f,f)\end{matrix}\leqq(1+\alpha\gamma)\|f\|_1^2\quad \begin{matrix}\text{if } f\in H \text{ and } Af\in H_0,\\ \text{if } f\in H \text{ and } A^*f\in H_0,\end{matrix}$$

$$(2.6)\qquad \begin{aligned} &|(f-\alpha Af, g)| && \text{if } f, g\in H \text{ and } Af\in H_0, \\ &\qquad\qquad \leqq (1+\alpha\gamma)\,\|f\|_1\cdot\|g\|_1 && \\ &|(f-\alpha A^*f, g)| && \text{if } f, g\in H \text{ and } A^*f\in H_0, \end{aligned}$$

$$(2.7)\qquad |(Af, g)-(f, Ag)|\leqq\kappa\|f\|_1\cdot\|g\|_0 \qquad \text{if } f, g\in H \text{ and } Af, Ag\in H_0.$$

PROOF. (2.5)-(2.6) may be proved by (1.6)-(1.7) and (2.3)-(2.3)′ remembering the inequality

$$2\alpha\,|ab|\leqq\alpha\,(\varepsilon|a|^2+\varepsilon^{-1}|b|^2)\,, \qquad (\alpha \text{ and } \varepsilon>0)\,,$$

since f and g both belong to H_1. Similarly we obtain (2.7) from

$$(Af, g)-(f, Ag)=-\int_{E^m}\Big(2\frac{\partial a^{ij}}{\partial x_i}\frac{\partial f}{\partial x_j}g + \frac{\partial^2 a^{ij}}{\partial x_i\partial x_j}fg-2b^i\frac{\partial f}{\partial x_i}g-\frac{\partial b^i}{\partial x_i}fg\Big)\,dx\,.$$

The right hand side is obtained by (2.3) and the expression obtained from (2.3) corresponding to (f, Ag) in which we have partially integrated the terms containing the factors like $f\times(\partial g/\partial x_j)$.

§ 3. Lemma 2 (concerning the existence of solutions of $u-n^{-2}Au=f$).

We invoke to Milgram-Lax theorem[3] for the proof of the Lemma 2 below. For the sake of completeness, we here give the full statement of the theorem together with its proof.

MILGRAM-LAX THEOREM. *Let a bilinear functional $B(u, v)$ defined on the Hilbert space H_1 satisfy the conditions:*

$$(3.1)\qquad |B(u, v)|\leqq\gamma'\,\|u\|_1\cdot\|v\|_1, \qquad 0<\gamma'<\infty\,,$$

$$(3.2)\qquad \delta'\,\|u\|_1^2\leqq B(u, u)\,, \qquad 0<\delta'<\infty\,,$$

Then, to any $v\in H_1$ there corresponds a uniquely determined $v^*=Sv\in H_1$ such that

(3.3) $(u, v)_1=B(u, Sv)$ *for all* $u\in H_1$ ($(u, v)_1$ *denotes the inner product in* H_1),

$$(3.4)\qquad \delta'\,\|Sv\|_1\leqq\|v\|_1\,.$$

3) P. D. Lax and A. N. Milgram: Parabolic Equations in "Contributions to the Theory of Partial Differential Equations", Princeton (1954), 167-190.

PROOF. Let $\{v, v^*\}$ be a pair of elements of H_1 for which we have $(u, v)_1 = B(u, v^*)$ for every $u \in H_1$. v^* is determined uniquely by v, since $B(u, v^*) = 0$ for all $u \in H_1$ implies

$$\delta' \| v^* \|_1^2 \leqq B(v^*, v^*) = 0 .$$

Moreover, the operator $S(v^* = Sv)$ is continuous and (3.4) holds good since

$$\delta' \| Sv \|_1^2 \leqq B(Sv, Sv) = (Sv, v)_1 \leqq \| Sv \|_1 \cdot \| v \|_1 .$$

Thus the domain $D(S)$ of the operator S is a closed linear subspace of H_1. Assume $D(S) \neq H_1$. Then there exists $w_0 \in H_1$ such that

$$(3.5) \qquad (w_0, v)_1 = 0 \quad \text{for every} \quad v \in D(S) \quad \text{and} \quad \| w_0 \| \neq 0 .$$

We consider the linear functional $F(z) = B(z, w_0)$ on H_1. It is a bounded functional since

$$| F(z) | = | B(z, w_0) | \leqq \gamma' \| z \|_1 \cdot \| w_0 \|_1 ,$$

and hence, by Riesz theorem, there exists $w_0' \in H_1$ such that $F(z) = B(z, w_0) = (z, w_0')_1$. Therefore $w_0' \in D(S)$ and $Sw_0' = w_0$. This is a contradiction, because of (3.5) and (3.2):

$$\delta' \| w_0 \|_1^2 \leqq B(w_0, w_0) = (w_0, w_0')_1 = 0 .$$

Therefore $D(S) = H_1$ and the theorem is proved.

LEMMA 2. *Let a positive number α_0 be chosen so small that the Corollary of the Lemma 1 is valid for $0 < \alpha \leqq \alpha_0$. Then, for any function $f(x) \in H$, the equation*

$$(3.6) \qquad u - \alpha A u = f \qquad (0 < \alpha \leqq \alpha_0)$$

admits a uniquely determined solution $u_f(x) \in H$.

PROOF. Let us define a bilinear functional

$$\hat{B}(u, v) = (u - \alpha A^* u, v)$$

for functions $u, v \in H$ satisfying $A^* u \in H_0$. From the Corollary of the Lemma 1, we have

$$(3.7) \qquad | \hat{B}(u, v) | \leqq (1 + \alpha\gamma) \| u \|_1 \cdot \| v \|_1 , \quad \delta \| u \|_1^2 \leqq \hat{B}(u, u) .$$

Hence, by continuity, $\hat{B}(u, v)$ may be extended to the bilinear functional $B(u, v)$ defined on H_1 satisfying

(3.7)′ $$|B(u, v)| \leqq (1+\alpha\gamma)\, \|u\|_1 \cdot \|v\|_1 , \quad \delta\|u\|_1^2 \leqq B(u, u) .$$

Consider the linear functional $F(u)=(u, f)$ defined on H_1. It is a bounded functional since

$$|(u, f)| \leqq \|u\|_0 \cdot \|f\|_0 \leqq \|u\|_1 \cdot \|f\|_1 ,$$

and hence, by Riesz theorem, there exists a uniquely determined $v=v(f)\in H_1$ such that $(u, f)=(u, v(f))_1$. Thus, by Milgram-Lax theorem, we have

(3.8) $$(u, f)=B(u, Sv(f)) \quad \text{for all} \quad u\in H_1 .$$

Let u run over C^∞ functions with compact supports, and let $v_n\in H$ be such that

$$\lim_{n\to\infty} \|v_n - Sv(f)\|_1 = 0 .$$

Then

$$B(u, Sv(f)) = \lim_{n\to\infty} B(u, v_n) = \lim_{n\to\infty} \hat{B}(u, v_n) = \lim_{n\to\infty} (u-\alpha A^* u, v_n)$$
$$= (u-\alpha A^* u, Sv(f)) ,$$

since the norm $\| \ \|_1$ is larger than the norm $\| \ \|_0$. Hence

(3.8)′ $$(u, f)=(u-\alpha A^* u, Sv(f)) .$$

$f(x)$ being any C^∞ function with compact support and $(I-\alpha A^*)$ being an elliptic differential operator with C^∞ coefficients, we see, by L. Schwartz theorem[4], that $u_f = Sv(f) \in H_1$ is a C^∞ solution of (3.6).

The proof of the uniqueness of the solution of (3.6). Let a function $u\in H$ satisfy

$$u-\alpha A u = 0 ,$$

Then Au belongs to H and hence to H_0. Thus, by the Corollary of the Lemma 1, we obtain

$$0=(u-\alpha A u, u) \geqq \delta\|u\|_1^2, \quad \text{viz.} \quad u=0 .$$

§ 4. Proof of the Theorem.

We first prove the

LEMMA 3. *Let the integer n be such that $|n^{-1}|$ is sufficiently small.*

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4) L. Schwartz: Théorie des Distributions, Paris (1950), 136.

Then, for any pair $\{f, g\}$ *of elements* $\in H$ *such that* $Af \in H_0$, *the resolvent equation*

$$\left(\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - n^{-1}\begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix}\right)\begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f \\ g \end{pmatrix} \tag{1.2}$$

admits a uniquely determined solutions $\{u, v\}$, u *and* $v \in H$, *satisfying*

$$((u-\alpha_0 Au, u)+\alpha_0(v, v))^{1/2} \leqq (1+\beta|n^{-1}|)\,((f-\alpha_0 Af, f)+\alpha_0(g, g))^{1/2}, \tag{4.1}$$

with a positive constont β *independent of* n *and* $\{f, g\}$.

PROOF. Let $u_1 \in H$ and $v_2 \in H$ respectively be the solutions of

$$u_1 - n^{-2}Au_1 = f \quad \text{and} \quad v_2 - n^{-2}Av_2 = g.$$

The existence of such solutions was proved in the Lemma 2. Then

$$u = u_1 + n^{-1}v_2, \quad v = n^{-1}Au_1 + v_2$$

satisfies (1.2).

The proof of (4.1). We first remark that

$$Au = n(v-g) \in H \subseteqq H_0 \quad \text{and hence} \quad Av = n(Au - Af) \in H_0.$$

Therefore we may apply the Corollary of the Lemma 1. Thus, by (1.2),

$$\begin{aligned}(f-\alpha_0 Af, f) &= (u-n^{-1}v-\alpha_0 A(u-n^{-1}v), u-n^{-1}v) \\ &= (u-\alpha_0 Au, u) - 2n^{-1}(u, v) + \alpha_0 n^{-1}(Au, v) + \alpha_0 n^{-1}(Av, u) \\ &\quad + n^{-2}(v-\alpha_0 Av, v)\end{aligned}$$

and

$$\begin{aligned}\alpha_0(g, g) &= \alpha_0(v-n^{-1}Au, v-n^{-1}Au) \\ &= \alpha_0(v, v) - \alpha_0 n^{-1}(v, Au) - \alpha_0 n^{-1}(Au, v) + \alpha_0 n^{-2}(Au, Au)\end{aligned}$$

imply that there exists a positive constant β satisfying

$$\begin{aligned}(f-\alpha_0 Af, f) + \alpha_0(g, g) &\geqq (u-\alpha_0 Au, u) + \alpha_0(v, v) \\ &\quad - \alpha_0|n^{-1}|\,|(Av, u)-(Au, v)| - 2|n^{-1}|\,|(u, v)| \\ &\geqq (1+\beta|n^{-1}|)^{-2}\,((u-\alpha_0 Au, u)+\alpha_0(v, v))\end{aligned}$$

for sufficiently large $|n|$.

The above estimate for the solutions $\{u, v\}$ belonging to H shows that the solutions are uniquely determined by $\{f, g\}$. Q. E. D.

The product space $H_1 \otimes H_0$ of vectors

$$\begin{pmatrix} u \\ v \end{pmatrix} = \{u, v\}', \quad \text{where} \quad u \in H_1 \quad \text{and} \quad v \in H_0, \tag{4.2}$$

is a Banach space by the norm

$$\left\| \begin{pmatrix} u \\ v \end{pmatrix} \right\| = \| \{u, v\}' \| = ((u - \alpha_0 A u, u) + \alpha_0 (v, v))^{1/2}. \tag{4.3}$$

Let the domain $D(\mathfrak{A})$ of the operator

$$\mathfrak{A} = \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \tag{4.4}$$

be the vectors $\{u, v\}' \in H_1 \otimes H_0$ such that

$$u, v \in H \quad \text{and} \quad A(u - n^{-1} v) \in H_0, \; v - n^{-1} A u \in H.$$

Then, the Lemma 3 shows that the range of the additive operator $\begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - n^{-1}\mathfrak{A}$ coincides with the set of vectors $\{f, g\}'$ in the Lemma 3. Moreover, it is easy to see that the set of such vectors $\{f, g\}'$ is $\| \;\|$-dense in the Banach space $H_1 \otimes H_0$. Hence we have the

COROLLARY. *The smallest closed extension* $\overline{\mathfrak{A}}$ *of the operator* $\mathfrak{A}$ *is such that the operator*

$$\mathfrak{J} - n^{-1}\overline{\mathfrak{A}} = \begin{pmatrix} I & 0 \\ 0 & I \end{pmatrix} - n^{-1}\overline{\mathfrak{A}}, \qquad (n = \textit{integer}), \tag{4.5}$$

admits, for sufficiently large $|n|$, *everywhere (in* $H_1 \otimes H_0$*) defined inverse* $\mathfrak{J}_n = (\mathfrak{J} - n^{-1}\overline{\mathfrak{A}})^{-1}$ *satisfying*

$$\| \mathfrak{J}_n \| \leqq (1 + \beta |n^{-1}|). \tag{4.6}$$

Hence, by the semi-group theory[1] and the irrelevance of the sign of n, there exists a uniquely determined group T_t:

$$T_t \begin{pmatrix} f \\ g \end{pmatrix} = \text{strong} \lim_{n \to \infty} \exp(t \overline{\mathfrak{A}} \mathfrak{J}_n) \begin{pmatrix} f \\ g \end{pmatrix} \tag{4.7}$$

of linear bounded operators T_t on $H_1 \otimes H_0$ into $H_1 \otimes H_0$ such that

$$T_t T_s = T_{t+s} \; (-\infty < t, s < \infty), \quad T_0 = \text{the identity I}, \tag{4.8}$$

$$\| T_t \| \leqq \exp(\beta |t|), \quad \text{strong} \lim_{t \to t_0} T_t \begin{pmatrix} f \\ g \end{pmatrix} = T_{t_0} \begin{pmatrix} f \\ g \end{pmatrix}, \tag{4.9}$$

(4.10) if $\begin{pmatrix} f \\ g \end{pmatrix}$ is in the domain of the "infinitesimal generator" $\overline{\mathfrak{A}}$,

we have strong $\lim_{h\to 0} h^{-1}(T_{t+h}-T_t)\begin{pmatrix} f \\ g \end{pmatrix} = \overline{\mathfrak{A}}T_t\begin{pmatrix} f \\ g \end{pmatrix} = T_t\overline{\mathfrak{A}}\begin{pmatrix} f \\ g \end{pmatrix}$.

Now, by the assumption of the Theorem,

$$A^k f \in H \quad \text{and} \quad A^k g \in H \quad (k=0, 1, \cdots). \tag{4.11}$$

Hence we see that

$$\overline{\mathfrak{A}}^k\begin{pmatrix} f \\ g \end{pmatrix} = \mathfrak{A}^k\begin{pmatrix} f \\ g \end{pmatrix} \in H_1 \otimes H_0 \qquad (k=0, 1, \cdots), \tag{4.12}$$

viz. the vector $\{f, g\}'$ is in the domain of the every power of $\overline{\mathfrak{A}}$. Therefore, by (4.10), the vectors

$$\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} = T_t\begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \tag{4.13}$$

are in the domain of every power of $\overline{\mathfrak{A}}$ and

$$\overline{\mathfrak{A}}^k\begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix}$$

belongs to $H_1 \otimes H_0$. Therefore, the "distribution"

(4.14) $U_t \cdot \varphi = \int_{E^m} u(t, x)\, \varphi(x)\, dx$ (the testing functions φ run over C^∞ functions with compact supports)

is such that, for every $k=0, 1, \cdots$, the "distribution"

$$A^k U_t \tag{4.15}$$

is the "distribution" defined by a function which is locally summable (in the truth, this function belongs to H_0). A being an elliptic differential operator with C^∞ coefficients, we see, by a theorem due to L. Schwartz[5], that $u(t, x)$ is a C^∞ function in x.

Thus $u(t, x)$ is, for fixed t, not only belongs to H_1 but also belongs

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5) L. Schwartz: Théorie des Distributions, II, Paris (1951), 47. Actually, the theorem is proved for the case when A=the Laplacian. However, since the proof is based upon the fact that the parametrix of the iterated Laplacian Δ^k becomes more smooth as k becomes large, the theorem may be extended to general elliptic differential operator A with C^∞ coefficients.

to H. Hence the value at $(t, x)=(t, x_1,\cdots, x_m)$ of $u(t, x)$ is determined without ambiguity. We also see, from (4.7), that this function $u(t, x)$ is measurable in (t, x). And, by the estimate (4.9), we see that the function $u(t, x)$ is locally summable in $t-x$ space. To this function we may apply every power of A and hence every power of

(4.16) $\quad \partial_{t^2}=$ the strong second order derivative with respect to t,

and

$$(\partial_{t^2})^k u(t, x)=A^k u(t, x) \qquad (k=0, 1,\cdots). \tag{4.17}$$

This we see by (4.10) and the fact that (4.12) holds good for our initial functions $\{f, g\}'$. Thus the "distribution"

(4.18) $\quad U\psi=\int_{E^m}\int_{-\infty}^{\infty} u(t, x)\,\psi(t, x)\,dxdt$ (the testing functions ψ run over C^∞ functions in (t, x) with compact supports)

is such that, for any $k=0, 1,\cdots$, the "distribution"

$$\left(\frac{\partial^2}{\partial t^2}+A\right)^k U=(2A)^k U \tag{4.19}$$

is a "distribution" defined by a locally summable function in (t, x). The operator

$$\left(\frac{\partial^2}{\partial t^2}+A\right)$$

being elliptic in (t, x), we see, again by making use of Schwartz theorem[5], that $u(t, x)$ is a C^∞ function in (t, x). Thus it is easy to see that $u(t, x)$ is a C^∞ solution of (1.1).

Finally the inequality (1.8) is identical with the estimate $\|T_t\| \leqq \exp(\beta|t|)$ in (4.9).

REMARK 1. We may prove

$$\begin{aligned}&((A^k u-\alpha_0 A^{k+1}u, A^k u)+\alpha_0(A^k u_t, A^k u_t))^{1/2}\\ &\leqq \exp(\beta|t|)((A^k f-\alpha_0 A^{k+1}f, A^k f)+\alpha_0(A^k g, A^k g))^{1/2}, \quad (k=0, 1,\cdots),\end{aligned} \tag{1.8$'$}$$

since $(A^k u)(t, x)$ is the solution of the original wave equation (1.1) with the initial condition

$$(A^k u)(0, x)=(A^k f)(x), (A^k u)(0, x)=(A^k g)(x),$$

to be obtained by our method.

REMARK 2. The above obtained solution $u(t, x)$ together with $v(t, x)=u_t(t, x)$ satisfy, by (4.10) and (4.9),

$$\left\| h^{-1}\begin{pmatrix} u(t+h, x)-u(t, x) \\ v(t+h, x)-v(t, x) \end{pmatrix} - \begin{pmatrix} 0 & I \\ A & 0 \end{pmatrix} \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} \right\| \to 0 \text{ as } h \to 0, \tag{4.20}$$

$$\left\| \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} - \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \right\| \to 0 \text{ as } t \to 0,$$

$$\left\| \begin{pmatrix} u(t, x) \\ v(t, x) \end{pmatrix} \right\| \leq \exp(\beta |t|) \left\| \begin{pmatrix} f(x) \\ g(x) \end{pmatrix} \right\|.$$

As was proved by E. Hille[6], such solution is unique since the resolvent $\mathfrak{J}_n=(\mathfrak{J}-n^{-1}\mathfrak{A})^{-1}$ exists and satisfies (4.6) for sufficiently large $|n|$, n denoting integers.

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6) A note on Cauchy's problem, Ann. Soc. Polonaise de Math., 25 (1952), 59.