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Logarithmic order of free distributive lattice

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1.—Introduction.—The problem to determine the order $f(n)$ of the free distributive lattice $FD(n)$ generated by n symbols $\gamma_1, \cdots, \gamma_n$ was first proposed by Dedekind, but very little is known about this number [1, p. 146]. Only the first six values of $f(n)$ are computed, and enumerations of further $f(n)$ appear to lie beyond the scope of any reasonable methods known today. It might, however, be pointed out that Morgan Ward, who found $f(6)$ by the help of computing machines, stated [2] an asymptotic relation

$$\log_2 \log_2 f(n) \sim n$$

and that the present author proved in a previous note [3] that

$$f(n) \equiv 0 \pmod 2 \qquad \text{if} \qquad n \equiv 0 \pmod 2).$$

An inspection of numerical results $f(n)$, $n \leqq 6$ suggests strongly the following asymptotic equivalence

$$(*) \qquad \log_2 f(n) \sim \sqrt{\frac{2}{\pi}}\, 2^n n^{-\frac{1}{2}}.$$

The author cannot prove or disprove this interesting relation, but he proves in the present paper that

$$\sqrt{\frac{2}{\pi}}\, 2^n n^{-\frac{1}{2}}(1+O(n^{-1})) < \log_2 f(n) < \sqrt{\frac{2}{\pi}}\, 2^n n^{-\frac{1}{2}} \log_2 \sqrt{\frac{n\pi}{2}}\,(1+O(n^{-1}))$$

(Theorem 2), which in particular implies that for an arbitrary positive constant δ

$$2^n n^{-\frac{1}{2}-\delta} < \log_2 f(n) < 2^n n^{-\frac{1}{2}+\delta}$$

if n is sufficiently large, and that

$$\log_2 \log_2 f(n) = n - \frac{1}{2} \log_2 n + O(\log_2 \log_2 n)$$

an improvement of Ward's result, whereas our conjecture (*) will take the form

$$\log_2 \log_2 f(n) = n - \frac{1}{2} \log_2 n + \left(\frac{1}{2} - \frac{1}{2} \log_2 \pi\right) + o(1).$$

2.—Although the problem of Dedekind seems exceedingly difficult, the lattice-theoretical version of the problem was completely solved by Th. Skolem. (Cf. [1, pp. 145–6].) He has shown that if the greatest element I and the least element O are adjoined, $FD(n)$ is simply isomorphic with 2^{2^n}. We assume in this paper that I and O are contained in $F = FD(n)$.

For the sake of brevity of notations we denote the two lattice operations in F in the ring-theoretical manner, i.e., we write join as a sum and meet as a product.

3.—The join-irreducible elements of F are the products

$$\sigma_i = \gamma_{k_1} \cdots \gamma_{k_i}$$

of distinct generators. A product of i distinct generators will be called an *i-simplex*, the 0-simplex being defined as I, the greatest element. Now form sums from among these simplexes, then the totality of such sums will constitute F itself ([1, pp. 145–6]), the empty sum corresponding to O, the least element. We can moreover reduce the number of summands in each sum to a minimum, by the absorptive law. A reduced sum will be called a *complex*. F is again identified with the totality of complexes, but the correspondence is, this time, biunique.

A reduced sum ξ_i of i-simplexes will be called an *i-cochain*, the empty sum being denoted by O_i, the null i-cochain. Any complex is a unique sum of cochains

$$\xi = \xi_0 + \xi_1 + \cdots + \xi_n,$$

the i-cochain ξ_i here being called the i-th *component* of ξ. If ξ_i consists of a_i simplexes for $i = 0, \cdots, n$, we say that ξ has the *length type* $(a_0, \cdots, a_n)$. Then least integer i such that $a_i > 0$ will be called the *co-degree* of ξ, and dually the greatest integer j with $a_j > 0$ will be called its *degree*. The only element defficient of co-degree or/and degree is O. We further define the i-th *co-segment* $\xi^{(i)}$ and the i-th *segment* $\xi_{(i)}$ of ξ as the sum of the j-th components of ξ such that $j<i$, or $j>i$, respectively. Obviously $\xi=\xi^{(i)}+\xi_i+\xi_{(i)}$.

4.—Let us define the *coboundary operators* ∇_i for $i=0,1,\cdots,n-1$ and the *boundary operators* Δ_i for $i=1,2,\cdots,n$ as follows.

1°. $\nabla_i\xi=\xi$, unless ξ has co-degree i.

2°. If ξ is of co-degree i and $\xi=\xi_i+\xi_{i+1}+\cdots$ then

$$\nabla_i\xi=\nabla_i\xi_i+\xi_{i+1}+\cdots,$$

where $\nabla_i\xi_i$ is defined as the *reduced* sum of those $(i+1)$-simplexes which are incident with some i-simplex in ξ_i. The Δ_i will be defined dually.

LEMMA 1. *$\nabla_i\xi$ and $\Delta_j\xi$ are reduced. This means that the sum defined in* 2°. *above is reduced already.*

PROOF. We have only to consider the former case of $\nabla_i\xi$. By 2°. the reduced property asserted would only be violated by possible incidence relations between an $(i+1)$-simplex σ'_{i+1} in $\nabla_i\xi_i$ and some j-simplex σ_j in ξ_j with $j>i$. The incidence must be $\sigma'_{i+1}\geqq\sigma_j$, but on the other hand there should be an i-simplex σ_i incident with, i. e. containing, σ'_{i+1}. Then we would have $\sigma_i>\sigma_j$, contrary to the reduced hypothesis on ξ.

LEMMA 2. *For $1\leqq i\leqq n$, $\nabla_{i-1}\cdots\nabla_1\nabla_0\xi^{(i)}$ contains exactly those i-simplexes incident with some simplex in $\xi^{(i)}$, or with some simplex in some component ξ_j with $j<i$. Similarly, if $0\leqq i\leqq n-1$, $\Delta_{i+1}\cdots\Delta_{n-1}\Delta_n\xi_{(i)}$ consists exactly of those i-simplexes which are incident with some simplex in $\xi_{(i)}$, or with some simplex in some component ξ_j with $j>i$. Moreover the expression*

$$\nabla_{i-1}\cdots\nabla_0\xi^{(i)}+\xi_i+\Delta_{i+1}\cdots\Delta_n\xi_{(i)}$$

is reduced (i. e., an i-cochain) *for* $1\leqq i\leqq n-1$. *Similarly*

$$\nabla_{n-1}\cdots\nabla_0\xi^{(n)}+\xi_n \qquad \textit{and} \qquad \xi_0+\Delta_1\cdots\Delta_n\xi_{(0)}$$

are reduced.

PROOF. The first part follows from the fact that any incidence relation between an i-simplex σ'_i and a j-simplex σ_j gives rise to a *connected chain* ([1, p. 11]). The second part follows from Lemma 1, if we note that ∇_i and Δ_j commute for $j-i>1$.

5.—We state here, before beginning further investigations, several numerical notations frequently used in the sequel.

$[x]$ is Gauss' symbol denoting the least integer $\leqq x$.

$c_i=\binom{n}{i}$, $d_i=c_{i+1}/c_i$. There will be no confusion as to n, since we use them for a fixed $FD(n)$.

$A(a_0, \cdots, a_n)$ denotes the number of elements of F with the prescribed length type $(a_0, \cdots, a_n)$.

$m=\left[\frac{n+1}{2}\right]$. Hence c_m is the greatest of the c_i's.

$\varepsilon=0$, or $=1$, according as n is even or odd. Hence $n=2m+\varepsilon$.

LEMMA 3. *Suppose that ξ has the length type $(a_0, \cdots, a_n)$ with $a_0= \cdots =a_{i-1}=0$ and denote by $(a_0', \cdots, a_n')$ the length type of $\nabla_i\xi$. Then*

$$a_0'= \cdots =a_{i-1}'=a_i'=0\,, \qquad a_k'=a_k \qquad (k>i+1)\,,$$

$$a_{i+1}'\geqq a_{i+1}+d_ia_i\,,$$

Similarly if $a_{j+1}= \cdots =a_n=0$, and if the length type of $\Delta_j\xi$ is denoted by $(a_0'', \cdots, a_n'')$, then

$$a_j''=a_{j+1}''= \cdots =a_n''=0\,, \qquad a_k''=a_k \qquad (k<j-1)\,,$$

$$a_{j-1}''\geqq a_{j-1}+\frac{1}{d_{j-1}}a_j\,.$$

PROOF. We need only to prove the first part of the Lemma, and we may consider only the case when ξ has co-degree i, i. e., $a_i>0$. Denote by q the number of $(i+1)$-simplexes in $\nabla_i\xi$. It is the number of $(i+1)$-simplexes incident with ξ_i, and

$$a_{i+1}'=a_{i+1}+q \tag{1}$$

by Lemma 1. Now each of the a_i simplexes in ξ_i contains exactly $n-i$ $(i+1)$-simplexes in $\nabla_i\xi_i$. But no $(i+1)$-simplex is contained in more than $i+1$ i-simplexes in ξ_i, since any $(i+1)$-simplex is contained in exactly $i+1$ i-simplexes *in* F. Comparing numbers of incidences we have:

$$(n-i)a_i\leqq(i+1)q\,, \qquad q\geqq\frac{n-i}{i+1}a_i=d_ia_i\,,$$

which together with (1) proves the Lemma.

LEMMA 4. *Denote by* $(a_0, \cdots, a_n)$ *the length type of* ξ. *Then there are at least*

$$c_i\left(\frac{a_0}{c_0}+\cdots+\frac{a_n}{c_n}\right)$$

i-simplexes incident with some simplex in ξ.

PROOF. Let $1 \leqq i \leqq n-1$ and consider the sequence

$$\xi^{(i)}=\nabla_{-1}\xi^{(i)},\ \nabla_0\xi^{(i)},\ \nabla_1\nabla_0\xi^{(i)}, \cdots, \nabla_{i-1}\cdots\nabla_0\xi^{(i)}$$

of complexes. Then $\nabla_{j-1}\cdots\nabla_0\xi^{(i)}$ has the length type

$$(0, \cdots, 0, a_j^*, a_{j+1}, \cdots, a_{i-1}, 0, \cdots, 0)$$

for $j<i$, with

$$a_j^* \geqq a_j+d_{j-1}a_{j-1}^*, \qquad a_0^*=a_0,$$

and the length type

$$(0, \cdots, 0, a_i^*, 0, \cdots, 0)$$

if $j=i$, where

$$a_i^* \geqq d_{i-1}a_{i-1}^*.$$

It follows that

$$\begin{aligned} a_i^* &\geqq d_{i-1}a_{i-1}^* \geqq d_{i-i}(a_{i-1}+d_{i-2}a_{i-2}^*) \geqq \cdots \\ &\geqq d_{i-1}(a_{i-1}+d_{i-2}(a_{i-2}+\cdots+d_1(a_1+d_0a_0)\cdots)) \\ &= d_{i-1}a_{i-1}+d_{i-1}d_{i-2}a_{i-2}+\cdots+d_{i-1}\cdots d_1a_1+d_{i-1}\cdots d_1d_0a_0 \\ &=c_i\left(\frac{a_0}{c_0}+\frac{a_1}{c_1}+\cdots+\frac{a_{i-1}}{c_{i-1}}\right). \end{aligned}$$

Similarly $\Delta_{i+1}\cdots\Delta_n\xi_{(i)}$ has the length type

$$(0, \cdots, 0, a_i^{**}, 0, \cdots, 0)$$

with

$$\begin{aligned} a_i^{**} &\geqq \frac{a_{i+1}}{d_i}+\frac{a_{i+2}}{d_id_{i+1}}+\cdots+\frac{a_n}{d_i\cdots d_{n-1}} \\ &= c_i\left(\frac{a_{i+1}}{c_{i+1}}+\cdots+\frac{a_n}{c_n}\right). \end{aligned}$$

We know in Lemma 2 that the sum

$$\nabla_{i-1}\cdots\nabla_0\xi^{(i)}+\xi_i+\Delta_{i+1}\cdots\Delta_n\xi_{(i)}$$

is reduced and that this i-cochain consists of i-simplexes incident with some simplex in ξ. Hence there are at least

$$c_i\left(\frac{a_0}{c_0}+\cdots+\frac{a_i}{c_i}+\cdots+\frac{a_n}{c_n}\right)$$

simplexes of that property in all.

The excluded extreme cases $i=n$ and $i=0$ may be treated in quite an analogous way.

6.—An interesting function

$$P(\xi)=\frac{a_0}{c_0}+\cdots+\frac{a_n}{c_n}$$

of a complex in F was found useful in the course of the proof above. It was also proved by the way, that $P(\xi)\leqq 1$ for all complexes. Making use of this function we restate Lemma 4 as

LEMMA 4′. *If ξ has the length type $(a_0, \cdots, a_n)$, then the number of i-simplexes not incident with any simplex in ξ is at most* $[c_i(1-P(\xi))]$.

7.—We are now in a position to give a Lemma usefull for evaluation of $f(n)$

LEMMA 5. *Let* $0', 1', \cdots, n'$ *be a permutation of* $0, 1, \cdots, n$. *Then*

$$A(a_0, \cdots, a_n)\leqq\binom{c_{0'}}{a_{0'}}\binom{\left[c_{1'}\left(1-\frac{a_{0'}}{c_{0'}}\right)\right]}{a_{1'}}\cdots\binom{\left[c_{n'}\left(1-\frac{a_{0'}}{c_{0'}}\cdots\frac{a_{(n-1)'}}{c_{(n-1)'}}\right)\right]}{a_{n'}}$$

PROOF. We dispose to select first $a_{0'}$ $0'$-simplexes, then $a_{1'}$ $1'$-simplexes, and so on, so as to obtain a complex of the length type $(a_0, \cdots, a_n)$. There are obviously $\binom{c_{0'}}{a_{0'}}$ ways of choosing $a_{0'}$ $0'$-simplexes. Suppose we have selected a $0'$-cochain $\xi_{0'}$, containing $a_{0'}$ $0'$-simplexes. We are to select $a_{1'}$ $1'$-simplexes not incident with $\xi_{0'}$. Since by Lemma 4′ there are at most $[c_{1'}(1-P(\xi_{0'}))]=[c_{1'}(1-a_{0'}/c_{0'})]$ such simplexes in all, the number of choices of $\xi_{1'}$, containing $a_{1'}$ $1'$-simplexes not incident with $\xi_{0'}$ is at most

$$\binom{[c_{1'}(1-a_{0'}/c_{0'})]}{a_{1'}}.$$

Now suppose we have selected $\xi_{0'}$ and $\xi_{1'}$ already. Then we are to select a $\xi_{2'}$ containing $a_{2'}$ $2'$-simplexes not incident with $\xi_{0'}+\xi_{1'}$. Since for *any* choice of $\xi_{0'}, \xi_{1'}$,

$$P(\xi_{0'}+\xi_{1'})=\frac{a_{0'}}{c_{0'}}+\frac{a_{1'}}{c_{1'}},$$

this stage of choosing $\xi_{2'}$ is quite similar as that of $\xi_{1'}$ above. The same procedure is feasible at each stage of choosing $\xi_{i'}$, and hence the number of choices of a complex of length type $(a_0, \cdots, a_n)$ does not exceed the right-hand member of Lemma 5.

LEMMA 6. *Let* $0', 1', \cdots, n'$ *be a permutation of* $0, 1, \cdots, n$ *and put* $c_{i'}=c'_i$ $(i=0, 1, \cdots, n)$. *Then* $f(n)$ *does not exceed*

$$((\cdots((1^{1/c'_n}+1)^{c'_n/c'_{n-1}}+1)^{c'_{n-1}}+1)^{c'_{n-1}/c'_{n-2}}\cdots+1)^{c'_1/c'_0}+1)^{c'_0}.$$

PROOF. Lemma 5 shows that $f(n)$ does not exceed the sum of the right-hand side of that Lemma, *extended over all non-negative solution of*

$$a_0/c_0+\cdots+a_n/c_n\leqq 1 \tag{2}$$

(Cf. § 6). Let us evaluate this sum. The summation is made first on $a_{n'}=a'_n$, then on $a_{(n-1)'}=a'_{n-1}$ and so on. Fixing $a_{0'}=a'_0, \cdots, a'_{n-1}$, the sum of the last factor of our summand, extended over a'_n is

$$2^{[c'_n(1-a'_0/c'_0-\cdots-a'_{n-1}/c'_{n-1})]}, \tag{3}$$

which does not exceed

$$2^{c'_n(1-a'_0/c'_0-\cdots-a'_{n-1}/c'_{n-1})}. \tag{4}$$

The next summation on a'_{n-1} of the next-to-the-last factor of our summand, multiplied by (4), yields, after eliminating Gauss' symbol, as was done on (3) to get (4),

$$2^{c'_n(1-a'_0/c'_0-\cdots-a'_{n-1}/c'_{n-2})}(1+2^{-c'_n/c'_{n-1}})^{c'_{n-1}(1-a'_0/c'_0-\cdots-a'_{n-2}/c'_{n-2})}$$
$$=(2^{c'_n/c'_{n-1}}+1)^{c'_{n-1}(1-a'_0/c'_0-\cdots-a'_{n-2}/c'_{n-2})}.$$

Continuing this process we find that $f(n)$ is majorated by the number given in Lemma 6.

8.—It is convenient to make use of the following function

$$F_u(x)=(x^{1/u}+1)^u\,,\qquad u>0\,,\ x>0$$

to express the number obtained above.

LEMMA 6′. $$f(n)<F_{c_{0'}}F_{c_{1'}}\cdots F_{c_{n'}}(1)$$
for any permutation $0',1',\cdots,n'$ *of* $0,1,\cdots,n$.

Note that this function is monotone increasing in x, and that

$$F_u^2(x)=F_uF_u(x)=(x^{1/u}+2)^u\,. \tag{5}$$

It is interesting to find a permutation minimizing the function given in Lemma 6′.

LEMMA 7. *If* $u>v>0$, $x>0$ *then*

$$F_uF_v(x)>F_vF_u(x)\,.$$

It follows that

$$F_{c_{0'}}F_{c_{1'}}\cdots F_{c_{n'}}(1)$$

is minimum if

$$c_{0'}\leqq c_{1'}\leqq\cdots\leqq c_{n'}\,,$$

ex. gr., *if* $0',1',\cdots,n'$ *is the permutation*

$$m,\,m+1,\,m-1,\,m+2,\,m-2,\,\cdots,\,n-1,\,1,\,n,\,0$$

where $m=\left[\dfrac{n+1}{2}\right]$.

PROOF. We prove the first part only. From the identities

$$F_{ut}(x)=(F_t(x^{1/u}))^u\,,\qquad F_{ut}(x^u)=(F_t(x))^u$$

follows that

$$F_uF_v(x)=F_u(F_{v/v}(x^{1/u})^u)=F_1F_{v/u}(x^{1/u}))^u\,,$$

$$F_vF_u(x)=F_v(F_1(x^{1/u})^u)=(F_{v/u}F_1(x^{1/u}))^u\,.$$

Thus our assertion is equivalent to

$$F_tF_1(x)<F_1F_t(x)\qquad\text{for}\qquad 1>t>0\,,\ x>0\,,$$

a special case of the Lemma for $u=1$. This is again equivalent to

$$F_t(x+1)<F_t(x)+1\,,$$

or

$$((x+1)^{1/t}+1)^t < (x^{1/t}+1)^t+1.$$

The last one is nothing but the well-known Minkowski's Inequality (dimension 2, metric $l_{1/t}$). Thus the Lemma was proved.

The minimum found above is

$$F^2_{c_0}F^2_{c_1}\cdots F^2_{c_{m-1}}F_{c_m}(1)$$

or

$$F^2_{c_0}F^2_{c_1}\cdots F^2_{c_{m-1}}F^2_{c_m}(1)$$

according as n is even or odd. By using ε of § 5 and by (5) we have

THEOREM 1. *The order $f(n)$ of $FD(n)$ does not exceed*

$$(\cdots((\varepsilon+2)^{d_{m-2}}+2)^{d_{m-3}}+\cdots+2)^{d_0}+2,$$

where $m=\left[\frac{n+1}{2}\right]$, $n=2m+\varepsilon$, *and* $d_i=c_{i+1}/c_i$.

9.—We now proceed to study asymptotic behaviour of the number presented in Theorem 1. It lies between

$$(b'\sqrt{n})^{c_m} \quad \text{and} \quad (b\sqrt{n})^{c_m}$$

with some absolute constants b' and b. We will, however, prove only the majorating inequality (Theorem 2 below).

Let us write for the moment

(6) $$G_u(x)=x^u+2 \qquad (x>1,\ u>1).$$

Then the number in Theorem 1 is written as

(7) $$G_{d_0}G_{d_1}\cdots G_{d_{m-2}}(\varepsilon+2).$$

Note that all appearing d's are >1. Now it is obvious that

$$G_u(x)<(x+2/u)^u \qquad \text{for} \qquad x>1,\ u>1.$$

Thus (7) is majorated by

$$(\varepsilon+2+2/d_{m-2}+2/d_{m-2}d_{m-3}+\cdots+2/d_{m-2}\cdots d_0)^{d_{m-2}\cdots d_0}$$

$$=(\varepsilon+2+2c_{m-2}/c_{m-1}+2c_{m-3}/c_{m-1}+\cdots+2c_0/c_{m-1})^{c_{m-1}}$$

$$=\left(\varepsilon+\frac{2}{c_{m-1}}\sum_{i=0}^{m-1}c_i\right)c_{m-1}$$

$$<\left(\frac{2}{c_{m-1}}\sum_{i=0}^{m-1}c_i+\frac{1}{2}(\varepsilon+1)c_m\right)^{c_{m-1}}=(2^n/c_{m-1})^{c_{m-1}}.$$

We have thus obtained a very simple

LEMMA 8. $$f(n)<\left(\frac{2^n}{c_{m-1}}\right)^{c_{m-1}}.$$

10.—By Stirling's formula we have

$$c_{m-1}=\sqrt{\frac{2}{\pi}}\,2^n n^{-\frac{1}{2}}(1+O(n^{-1})), \tag{8}$$

and we obtain by Lemma 8

$$f(n)<\left(\sqrt{\frac{\pi}{2}}\,n^{\frac{1}{2}}(1+O(n^{-1}))\right)^{c_{m-1}}.$$

This again together with (7) implies that

$$\log_2 f(n)<\sqrt{\frac{\pi}{2}}\,2^n n^{-\frac{1}{2}}\log_2\sqrt{\frac{n\pi}{2}}\,(1+O(n^{-1})).$$

On the other hand it is almost trivial that

$$2^{c_{m-1}}\leqq f(n). \tag{9}$$

In fact $2^{c_{m-1}}-1$ is the number of non-void $(m-1)$-cochains, and the n-cochain σ_n is never counted in it. Now (9) together with (7) yields

$$\log_2 f(n)\geqq c_{m-1}=\sqrt{\frac{2}{\pi}}\,2^n n^{-\frac{1}{2}}(1+O(n^{-1})).$$

It might hereby be pointed out that Ward's asymptotic relation

$$\log_2\log_2 f(n)\sim n$$

follows from (9) and a more trivial inequality

$$f(n)\leqq 2^{2^n}$$

Thus we have finally proved

THEOREM 2.

$$\sqrt{\frac{2}{\pi}}\,2^n n^{-\frac{1}{2}}(1+O(n^{-1}))<\log_2 f(n)<\sqrt{\frac{2}{\pi}}\,2^n n^{-\frac{1}{2}}\log_2\sqrt{\frac{n\pi}{2}}\,(1+O(n^{-1}))\,.$$

COROLLARY 1. *Let $\delta>0$ be an arbitrary constant. Then*

$$2^n n^{-\frac{1}{2}-\delta}<\log_2 f(n)<2^n n^{-\frac{1}{2}+\delta}\,,$$

if n is sufficiently large.

COROLLARY 2. $$\log_2\log_2 f(n)=n-\frac{1}{2}\log_2 n+O(\log_2\log_2 n)\,.$$

11.—Concluding Remark.—As was observed at the beginning of § 9, we cannot drop the term $O(\log_2\log_2 n)$ in the last formula, if we start from Theorem 1. It is desirable to find a more accurate evaluations for $A(a_0, \cdots, a_n)$ and $f(n)$. It seems likely that only those $A(a_0, \cdots, a_n)$ with

$$\frac{a_0}{c_0}+\cdots+\frac{a_n}{c_n}\quad\text{very near to}\quad\frac{1}{2}$$

make significant contributions to $f(n)$, as is suggested by the Central Limit Theorem in the theory of probability.

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References

[1] Garrett Birkhoff: Lattice Theory, 1949 New York.
[2] Morgan Ward: Abstract 52-5-135, Bull. Amer. Math. Soc., 52 (1946), 423.
[3] Koichi Yamamoto: Note on the order of free distributive lattices, Sci. Rep. Kanazawa Univ., 2 (1953), 5-6.

ERRATA

Symmetrization and univalent functions in an annulus.

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p. 60, l. 12 from bottom: for "R. E. Goodman", read "A. W. Goodman"
p. 66, l. 4 from bottom: for "Goodman, R. E.", read "Goodman, A. W."