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On Neumann's problem for a domain on a closed Riemann surface.

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The Neumann's problem is solved usually by means of integral equations. Recently L. Myrberg[1)] proved simply the existence of the solution of the Neumann's problem for the inside of a unit circle, without use of integral equations. By his method, we shall prove the existence of the solution of the Neumann's problem for a domain on a closed Riemann surface, without use of integral equations.

Let F be a closed Riemann surface spread over the z-plane and D be its sub-domain, whose boundary Γ consists of a finite number of analytic Jordan curves or Jordan arcs $\Gamma=\sum_{i=0}^{n}\Gamma_i$, such that, if Γ_i, Γ_{i+1} meet at a point ζ_i, then they make an inner angle $\alpha_i\pi$ $(0<\alpha_i<2)$ at ζ_i. Let $f(\zeta)$ be a given function on Γ, which is continuous on Γ, except at $\{\zeta_i\}$, where $f(\zeta)$ may be discontinuous, but is bounded on Γ, such that

$$|f(\zeta)|\leqq M \qquad \text{on } \Gamma\,, \tag{1}$$

and satisfies the condition:

$$\int_{\Gamma} f(\zeta)\,|d\zeta|=0\,. \tag{2}$$

Then we shall prove

THEOREM. *There exists a harmonic function $u(z)$ in D, which is continuous in $\overline{D}$, such that*

(i) $$|u(z)|\leqq k_1 M \quad \textit{in} \quad \overline{D}\,,$$

where $k_1=k_1(D)$ is a constant, which depends on D only.

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1) L. Myrberg: Über die vermischte Randwertaufgabe der harmonischen Funktionen. Ann. Acad. Sci. Fenn. Series A, 103 (1951).

(ii) $$\frac{\partial u}{\partial \nu} \to f(\zeta), \qquad \zeta \neq \zeta_i,$$

when z tends to $\zeta \in \Gamma$ along the inner normal ν of at ζ and $\frac{\partial u}{\partial \nu}$ is the derivative of u in the direction ν.

(iii) *Let $D[u]$ be the Dirichlet integral of u on D, then $D[u]$ can be expressed by*

$$D[u] = -\int_\Gamma u \frac{\partial u}{\partial \nu} |d\zeta| = -\int_\Gamma u(\zeta) f(\zeta) |d\zeta|,$$

so that

$$D[u] \leqq k_2 M^2,$$

where $k_2 = k_2(D)$ is a constant, which depends on D only.

PROOF. We may assume that $M=1$, so that

$$|f(\zeta)| \leqq 1 \quad \text{on } \Gamma, \qquad \int_\Gamma f(\zeta)|d\zeta| = 0 \tag{1}$$

and we have to prove that $|u(z)| \leqq k_1(D)$, $D[u] \leqq k_2(D)$.

Let $z=0 \in D$ and $g(z, 0)$ be the Green's function of D with $z=0$ as its pole. We put

$$\varphi(\zeta) = f(\zeta) / \frac{\partial g(\zeta, 0)}{\partial \nu}, \tag{2}$$

where ν is the inner normal, then

$$\int_\Gamma \varphi(\zeta) \frac{\partial g(\zeta, 0)}{\partial \nu} |d\zeta| = 0. \tag{3}$$

Since $g(\zeta, 0)$ is harmonic at $\zeta \neq \zeta_i$ on Γ, $\frac{\partial g(\zeta, 0)}{\partial \nu} > 0$ exists at such points. To investigate the behaviour of $\frac{\partial g(\zeta, 0)}{\partial \nu}$ in a neighbourhood of ζ_i, we map the part $U(\zeta_i)$ of D, contained in $|z-\zeta_i| < \rho$ on a half-disc: $|w| < 1$, $y > 0$ on the $w = x+iy$-plane, such that ζ_i becomes $w=0$ and the part of Γ, which lies in $|z-\zeta_i| \leqq \rho$ becomes $-1 \leqq w \leqq 1$ and put $g(z, 0) = G(w)$, then $G(w)$ is harmonic in $|w| < 1$ and if $\zeta \in \Gamma$ corresponds to ξ,

$$\frac{\partial g(\zeta, 0)}{\partial \nu} |d\zeta| = \left(\frac{\partial G(w)}{\partial \eta}\right)_{\eta=0} d\xi, \qquad (w = \xi + i\eta).$$

Since

$$0<a\leqq\left(\frac{\partial G(w)}{\partial\eta}\right)_{\eta=0}\leqq b \quad \text{for} \quad |\xi|\leqq\delta<1,$$

$$a\left|\frac{\partial\xi}{\partial\zeta}\right|\leqq\frac{\partial g(\zeta,0)}{\partial\nu}\leqq b\left|\frac{d\xi}{d\zeta}\right|.$$

Now $\left|\frac{d\xi}{d\zeta}\right|=\left|\frac{dw}{dz}\right|$. By Kellogg's theorem,[2] we can prove easily that in a neighbourhood of ζ_i,

$$\left.\begin{aligned} A|z-\zeta_i|^{\frac{1-\alpha_i}{\alpha_i}}&\leqq\left|\frac{dw}{dz}\right|\leqq B|z-\zeta_i|^{\frac{1-\alpha_i}{\alpha_i}},\\ A|z-\zeta_i|^{\frac{1}{\alpha_i}}&\leqq|w|\leqq B|z-\zeta_i|^{\frac{1}{\alpha_i}},\end{aligned}\right\} \tag{4}$$

where $A>0$, $B>0$ are constants, so that writing A, B in stead of Aa, Ba, we have

$$A|\zeta-\zeta_i|^{\frac{1-\alpha_i}{\alpha_i}}\leqq\frac{\partial g(\zeta,0)}{\partial\nu}\leqq B|\zeta-\zeta_i|^{\frac{1-\alpha_i}{\alpha_i}}, \tag{5}$$

hence

$$|\varphi(\zeta)|\leqq\frac{1}{A}|\zeta-\zeta_i|^{\frac{\alpha_i-1}{\alpha_i}}. \tag{6}$$

Since similarly

$$A_1|\zeta-\zeta_i|^{\frac{1-\alpha_i}{\alpha_i}}\leqq\frac{\partial g(\zeta,z)}{\partial\nu}\leqq B_1|\zeta-\zeta_i|^{\frac{1-\alpha_i}{\alpha_i}},$$

we have

$$|\varphi(\zeta)|\frac{\partial g(\zeta,z)}{\partial\nu}\leqq\frac{B_1}{A}, \tag{7}$$

so that $\varphi(\zeta)\frac{\partial g(\zeta,z)}{\partial\nu}$ is bounded on Γ, hence we put

$$v(z)=\frac{1}{2\pi}\int_\Gamma\varphi(\zeta)\frac{\partial g(\zeta,z)}{\partial\nu}|d\zeta|, \tag{8}$$

2) S. Warschawski: Über einen Satz von O. D. Kellogg. Göttinger Nachr. 1932. M. Tsuji: The boundary distortion on conformal mapping, (which will appear in this Journal).

then by (3), $v(0)=0$ and $v(z)$ is harmonic in D and, by means of (6), we can prove easily that

$$\lim_{z\to\zeta} v(z)=\varphi(\zeta), \qquad \zeta\neq\zeta_i. \tag{9}$$

Let $h=h(z,0)$ be the conjugate harmonic function of $g(z,0)$ and we denote the niveau curve $h(z,0)=\text{const.}=\alpha$ by L_α and put

$$u(z)=\int_0^z v(t)\,dg(t,0), \tag{10}$$

where we integrate on L_α, then since $v(0)=0$, the integral is finite. We shall prove that $u(z)$ is harmonic in D.

Let z be different from double points $\{a_i\}$ of the niveau curves $h=$const., then at z,

$$\frac{\partial^2 u}{\partial g^2}=\frac{\partial v}{\partial g}, \qquad \frac{\partial^2 u}{\partial h^2}=\int_0^z\frac{\partial^2 v}{\partial h^2}\,dg=-\int_0^z\frac{\partial^2 v}{\partial g^2}\,dg=-\frac{\partial v}{\partial g},$$

so that $\Delta u=\dfrac{\partial^2 u}{\partial g^2}+\dfrac{\partial^2 u}{\partial h^2}=0$, hence $u(z)$ is harmonic at z. Since $u(z)$ is bounded in a neighbourhood of a_i, $u(z)$ is harmonic at a_i, so that $u(z)$ is harmonic in D.

We see from (9), (10), that when $z\to\zeta\neq\zeta_i$ along the inner normal ν of Γ at ζ,

$$\frac{\partial u}{\partial\nu}\to\varphi(\zeta)\frac{\partial g(\zeta,0)}{\partial\nu}=f(\zeta), \qquad \zeta\neq\zeta_i. \tag{11}$$

Hence $u(z)$ is the solution of the Neumann's problem. Next we shall prove that $u(z)$ continuous in $\overline{D}$.

Since $u(z)$ is continuous at $\zeta\neq\zeta_i$ on Γ, we have only to prove that $u(z)$ is continuous at ζ_i. For the sake of brevity, we assume that there is only one ζ_0 on Γ, where Γ_0, Γ_1 meet at an inner angle $\alpha_0\pi$ $(0<\alpha_0<2)$. Let $U(\zeta_0)$ be the part of D, contained in $|z-\zeta_0|<\rho$. We map $U(\zeta_0)$ conformally on $|w|<1$, $y>0$ on the $w=x+iy$-plane, such that ζ_0 becomes $w=0$ and the part of Γ contained in $|z-\zeta_0|\leqq\rho$ becomes $-1\leqq w\leqq 1$. Let the half-discs $|w|\leqq\dfrac{1}{4}$, $y>0$, and $|w|\leqq\dfrac{1}{2}$, $y>0$ be mapped on $U_0(\zeta_0)$, $U_1(\zeta_0)$ respectively and $\Gamma_1(\zeta_0)$ be the part of Γ, which belongs to the boundary of $U_1(\zeta_0)$. Then

$$v(z)=\frac{1}{2\pi}\int_{\Gamma_1(\zeta_0)}\varphi(\zeta)\frac{\partial g(\zeta,z)}{\partial\nu}|d\zeta|+O(1), \quad z\in U_0(\zeta_0), \tag{12}$$

where $O(1)$ is bounded for any $z\in U_0(\zeta_0)$.

Let $z\in U_0(\zeta_0)$, $z_1\in U(\zeta_0)$ correspond to $w=x+iy$, $w_1=x_1+iy_1$ respectively, then $|w|\leqq\frac{1}{4}$, $|w_1|<1$ and put $g(z_1,z)=G(w_1,w)$ and let

$$G(w_1,w)=\log\left|\frac{w_1-\bar{w}}{w_1-w}\right|+\psi(w_1,w). \tag{13}$$

Since $\psi=0$ on $-1\leqq w_1\leqq 1$, $\psi(w_1,w)$ is harmonic in $|w_1|<1$ and we can prove easily that $|\psi(w_1,w)|\leqq\text{const.}=K$ on $|w_1|=\frac{1}{2}$, so that $|\psi(w_1,w)|\leqq K$ in $|w_1|\leqq\frac{1}{2}$, where K is independent of w, $\left(|w|\leqq\frac{1}{4}\right)$. Hence if $\zeta\in\Gamma_1(\zeta_0)$, $(\zeta\neq\zeta_0)$ corresponds to ξ $\left(-\frac{1}{2}\leqq\xi\leqq\frac{1}{2}\right)$, then

$$\begin{aligned}\frac{\partial g(\zeta,z)}{\partial\nu}|d\zeta|&=\left(\frac{\partial}{\partial y_1}\log\left|\frac{w_1-\bar{w}}{w_1-w}\right|_{w_1=\xi}+O(1)\right)d\xi\\&=\left(\frac{2y}{y^2+(x-\xi)^2}+O(1)\right)d\xi.\end{aligned} \tag{14}$$

Since by (6), (4), $\varphi(\zeta)=O(|\zeta-\zeta_0|)^{\frac{\alpha_0-1}{\alpha_0}}$ and $|z-\zeta_0|=O(|w|^{\alpha_0})$, we have $\varphi(\zeta)=O(|\xi|^{\alpha_0-1})$, so that

$$\begin{gathered}\frac{1}{2\pi}\int_{\Gamma_1(\zeta_0)}\varphi(\zeta)\frac{\partial g(\zeta,z)}{\partial\nu}|d\zeta|=O\left(\int_{-\frac{1}{2}}^{\frac{1}{2}}\frac{y\,d\xi}{|\xi|^\beta(y^2+(x-\xi)^2)}\right)+O(1).\\0<\beta=|1-\alpha_0|<1.\end{gathered} \tag{15}$$

If $|\xi|\leqq|x-\xi|$, then

$$\frac{1}{|\xi|^\beta(y^2+(x-\xi)^2)}\leqq\frac{1}{|\xi|^\beta(y^2+\xi^2)}$$

and if $|\xi|\geqq|x-\xi|$, then

$$\frac{1}{|\xi|^\beta(y^2+(x-\xi)^2)}\leqq\frac{1}{|x-\xi|^\beta(y^2+(x-\xi)^2)},$$

so that

$$\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{y\,d\xi}{|\xi|^{\beta}(y^2+(x-\xi)^2)} \leqq 2\int_{-\infty}^{\infty} \frac{y\,d\xi}{|\xi|^{\beta}(y^2+\xi^2)} = O\Big(\frac{1}{y^{\beta}}\Big),$$

hence by (12)

$$v(z)=O\left(\frac{1}{y^{\beta}}\right), \qquad z\in U_0(\zeta_0), \qquad 0<\beta<1. \tag{16}$$

Let $\zeta\in\Gamma$ lies in a small neighbourhood of ζ_0 and z lie on the same niveau curve $h=\alpha$ as ζ and correspond to $w=x+iy$, then if we integrate on L_α,

$$\left|\int_{\zeta}^{z} v(t)\,dg(t,0)\right| = O\Big(\int_0^y \frac{dy}{y^{\beta}}\Big) = O(y^{1-\beta}) < \varepsilon,$$

if z lies in a small neighbourhood of ζ_0. From this, we see that $u(z)$ is continuous at ζ_0. Hence $u(z)$ is continuous in $\overline{D}$. From the proof, we see that $|u(z)|\leqq k_1(D)$ in $\overline{D}$.

Next we shall prove that $D[u]$ can be expressed by

$$D[u] = -\int_{\Gamma} u\frac{\partial u}{\partial \nu}\,|d\zeta|. \tag{17}$$

Let Γ_ρ be the niveau curve $g(z,0)=\text{const.}=\rho>0$ and $\varDelta_\rho$ be the domain, bounded by Γ_ρ, and $D_\rho[u]$ be the Dirichlet integral of u in $\varDelta_\rho$, then

$$D_\rho[u] = -\int_{\Gamma_\rho} u\frac{\partial u}{\partial \nu}\,|dz|. \tag{18}$$

Suppose that as before there is only one ζ_0 on Γ, where Γ_0, Γ_1 meet at an inner angle $\alpha_0\pi$ $(0<\alpha_0<2)$ and let ζ_0 lie on the niveau curve $h(z,0)=0$, and $\Gamma_\rho(\eta)$, $\Gamma(\eta)$ be the part of Γ_ρ, Γ respectively, on which $|h(z,0)|\leqq\eta$, then

$$D_\rho[u] = -\int_{\Gamma_\rho-\Gamma_\rho(\eta)} u\frac{\partial u}{\partial \nu}\,|dz| - \int_{\Gamma_\rho(\eta)} u\frac{\partial u}{\partial \nu}\,|dz|, \tag{19}$$

where

$$\lim_{\rho\to 0}\int_{\Gamma_\rho-\Gamma_\rho(\eta)} u\frac{\partial u}{\partial \nu}\,|dz| = \int_{\Gamma-\Gamma(\eta)} u\frac{\partial u}{\partial \nu}\,|d\zeta|. \tag{20}$$

Now since $u(z)=O(1)$,

$$\int_{\Gamma_\rho(\eta)} u\frac{\partial u}{\partial \nu}|dz| = O\Big(\int_{\Gamma_\rho(\eta)} |v(z)|\,\frac{\partial g(z,0)}{\partial \nu}\,|dz|\Big) = O\Big(\int_{\Gamma_\rho(\eta)} |v(z)|\,dh(z,0)\Big)$$

$$=O\left(\int_{-\eta}^{\eta} dh \int_{\Gamma} |\varphi(\zeta)| \frac{\partial g(\zeta, z)}{\partial \nu} |d\zeta|\right). \tag{21}$$

Let $\eta < \delta < \frac{1}{2}$. If $z \in \Gamma_\rho(\eta)$, $\zeta \in \Gamma - \Gamma(\delta)$, and $z \to \zeta \in \Gamma(\eta)$, then $\frac{\partial g(\zeta, z)}{\partial \nu} \to 0$, hence

$$\int_{\Gamma_\rho(\eta)} u \frac{\partial u}{\partial \nu} |dz| = O\left(\int_{-\eta}^{\eta} dh \int_{\Gamma(\delta)} |\varphi(\zeta)| \frac{\partial g(\zeta, z)}{\partial \nu} |d\zeta|\right) + O(\eta).$$

As before, we map $U(\zeta_0)$ on $|w|<1$, $y>0$ and let $z \in \Gamma_\rho(\eta)$ correspond to $w=x+iy$, then $dh=O(dx)$, so that by (15),

$$\int_{\Gamma_\rho(\eta)} u \frac{\partial u}{\partial \nu} |dz| = O\left(\int_{-\eta}^{\eta} dx \int_{-\delta}^{\delta} \frac{y\, d\xi}{|\xi|^\beta (y^2+(x-\xi)^2)}\right) + O(\eta) =$$

$$O\left(\int_{-\delta}^{\delta} \frac{d\xi}{|\xi|^\beta} \int_{-\eta}^{\eta} \frac{y\, dx}{y^2+(x-\xi)^2}\right) + O(\eta) = O(\delta^{1-\beta}) + O(\eta),\ (0<\beta<1). \tag{22}$$

Since η, δ are arbitrary, we have from (19), (20), (22),

$$D[u] = \lim_{\rho \to 0} D_\rho[u] = -\int_{\Gamma} u \frac{\partial u}{\partial \nu} |d\zeta|. \tag{23}$$

Since $|u| \leqq k_1(D)$, $\left|\frac{\partial u}{\partial \nu}\right| = |f(\zeta)| \leqq 1$, we have $D[u] \leqq k_2(D)$, where $k_2(D)$ is a constant, which depends on D only. Hence our theorem is proved.

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