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A metrical theorem on the singular set of a linear group of Schottky type.

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Let G be a linear group of Schottky type on the ζ-plane, whose fundamental domain D_0 is bounded by $p\,(\geqq 2)$ pairs of disjoint analytic Jordan curves C_i, C_i' $(i=1,2,\cdots,p)$, where C_i, C_i' are equivalent by G. The equivalents D_ν of D_0 cluster to a non-dense perfect set E, which is called the singular set of G. Myrberg[1] proved that

$$\text{cap. } E>0\,,$$

where cap. E denotes the logarithmic capacity of E, while, in another paper,[2] I have proved that every point of E is a regular point for Dirichlet problem. Since cap. $E>0$, if we map the outside of E on $|w|<1$ conformally, then E is mapped on a set of mesure 2π on $|w|=1$. We shall prove

THEOREM. *Let E_1 be the sub-set of E, which lies in C_1 and every point of which is contained in infinitely many equivalents of C_1. Then*

$$\text{cap. } E_1>0\,,$$

and E_1 is mapped on a set of positive measure on $|w|=1$.

PROOF. Since the proof is the same, we assume that $p=2$.

First we shall prove that cap. $E_1>0$. Let S_1, S_2 be two generators of G, such that $C_1=S_1(C_1')$. $C_2=S_2(C_2')$. If we apply S_1 to D_0, then D_0 becomes D_1, which lies in $C_1=K_1$ and is bounded by K_1 and three other closed curves C_{11}, C_{12}, C_{13}, which are equivalent to C_1 or C_2. Let C_{11} be equivalent to C_1 and we write $C_{11}=K_{11}$. C_{12}, C_{13} are equivalent to C_2. We choose one of them, C_{12}, say. Let D_{12} be the equivalent of

1) P. J. Myrberg: Die Kapazität der singulären Menge der linearen Gruppe. Ann. Fenn. Ser. A 10 (1941).

2). M. Tsuji: On the capacity of general Cantor sets. Journ. of Math. Soc. Japan 5 (1953).

D_0, which lies in C_{12} and is bounded by C_{12} and three other closed curves $C_{121}, C_{122}, C_{123}$, which are equivalent to C_1 or C_2. Let C_{121} be equivalent to C_1 and we write $C_{121}=K_{12}$. Hence, inside of K_1, we have two equivalents K_{11}, K_{12} of C_1.

Similarly we define $K_{1i_1\cdots i_n}$ $(i_1,\cdots,i_n=1,2)$, which are equivalent to C_1, such that if we denote the inside of $K_{1i_1\cdots i_n}$ by $\varDelta_{1i_1\cdots i_n}$, then

$$\varDelta_{1i_1\cdots i_n}\subset\varDelta_{1i_1\cdots i_{n-1}} \qquad (i_n=1,2).$$

We put

$$M=\prod_{n=1}^{\infty}\Big(\sum_{i_1,\cdots,i_n}^{1,2}\varDelta_{1i_1\cdots i_n}\Big). \tag{1}$$

By Koebe's distortion theorem, we can prove easily[3)]

$$\delta(\varDelta_{1i_1\cdots i_n})\geqq a\,\delta(\varDelta_{1i_1\cdots i_{n-1}}) \tag{2}$$

and the mutual distance of

$$\varDelta_{1i_1\cdots i_{n-1},1} \text{ and } \varDelta_{1i_1\cdots i_{n-1},2} \text{ is } \geqq b\ \delta(\varDelta_{1i_1\cdots i_{n-1}}), \tag{3}$$

where $\delta(\varDelta)$ is the diameter of $\varDelta$ and $a>0$, $b>0$ are constants, which are independent of n. Hence cap. $M>0$ by a theorem proved by the author[4)]. Since $M\subset E_1$, we have

$$\text{cap.}\,E_1>0. \tag{4}$$

Next we shall prove that E_1 is mapped on a set of positive measure on $|w|=1$.

If we identify the equivalent points on C_i, C_i', then D_0 becomes a closed Riemann surface F, whose genus is $p=2$. We consider F spread over the z-plane. C_i, C_i' correspond to the both shores γ_i^+, γ_i^- of a ring cut γ_i $(i=1,2)$ of F. If we cut F by γ_1, γ_2, then F becomes a surface F_0, whose boundary consists of $\gamma_1^+, \gamma_1^-, \gamma_2^+, \gamma_2^-$. We write

$$\gamma^{(1)}=\gamma_1^+,\quad \gamma^{(2)}=\gamma_1^-,\quad \gamma^{(3)}=\gamma_2^+,\quad \gamma^{(4)}=\gamma_2^-. \tag{5}$$

In the following, $F_j, F_{ji_1},\cdots$ are the same samples as F_0. Along $\gamma^{(j)}$ $(j=1,2,3,4)$, we connect F_j to F_0. Along three remaining boundary

3), 4) M. Tsuji. l. c. 2).

closed curves of F_j, we connect F_{ji_1} ($i_1=1,2,3$) to F_j. Similarly we define $F_{ji_1\cdots i_n}$ ($j=1,2,3,4$; $i_1,\cdots i_n=1,2,3$) and put

$$F^{(\infty)}=F_0+\sum_j F_j+\sum_{j,i_1} F_{ji_1}+\cdots+\sum_{j,i_1,\cdots,i_n} F_{ji_1\cdots i_n}+\cdots, \tag{6}$$

then $F^{(\infty)}$ is of planar character and is mapped on the outside of E. If $\gamma^{(k)}$ ($k=1,2,3,4$) belongs to the boundary of $F_{ji_1\cdots i_n}$, but does not belong to the boundary of $F_{ji_1\cdots i_{n-1}}$, then we denote it by $\gamma^{(k)}_{ji_1\cdots i_n}$.

Let Φ be a sub-surface of $F^{(\infty)}$, such that

$$\Phi=F_1+\sum_{i_1} F_{1i_1}+\cdots+\sum_{i_1,\cdots,i_n} F_{1i_1\cdots i_n}+\cdots, \tag{7}$$

and put

$$\Phi_n=F_1+\sum_{i_1} F_{1i_1}+\cdots+\sum_{i_1,\cdots,i_n} F_{1i_1\cdots i_n}. \tag{8}$$

Let $\gamma^{(1)}+\Gamma_n$ be the boundary of Φ_n, then Γ_n consists of 3^n closed curves, each of which is γ_1 or γ_2. Let Γ'_n be the sum of those, which are γ_1 and Γ''_n be that of those, which are γ_2, then $\Gamma_n=\Gamma'_n+\Gamma''_n$. Let $u_n(z)$ be the harmonic measure of Γ'_n with respect to Φ_n, such that $u_n(z)$ is harmonic in Φ_n,

$$u_n=1 \text{ on } \Gamma'_n,\ u_n=0 \text{ on } \gamma^{(1)} \text{ and on } \Gamma''_n. \tag{9}$$

We shall prove that $u_n(z)$ does not tend to zero with $n\to\infty$.

Let $v_n(z)$ be the conjugate harmonic function of $u_n(z)$ and put

$$d_n=\int_{\gamma^{(1)}} dv_n>0. \tag{10}$$

We remark that at least one of the boundary curves $\gamma^{(k)}_{1i_1\cdots i_n}$ of $F_{1i_1\cdots i_n}$ belongs to Γ'_n.

Let $F_{1i_1\cdots i_n}$ connect to $F_{1i_1\cdots i_{n-1}}$ along $\gamma=\gamma^{(k)}_{1i_1\cdots i_{n-1}}$. We draw a ring cut γ' in $F_{1i_1\cdots i_{n-1}}$, which lies in a small neighbourhood of γ, such that γ,γ' bound a ring domain Δ in $F_{1i_1\cdots i_{n-1}}$. We add Δ to $F_{1i_1\cdots i_n}$ and put

$$\tilde{F}_{1i_1\cdots i_n}=\Delta+F_{1i_1\cdots i_n}.$$

Let $\omega(z)$ be harmonic in $\tilde{F}_{1i_1\cdots i_n}$, such that

$$\omega=\begin{cases}0 & \text{on } \gamma' \text{ and on } \gamma^{(k)}_{1i_1\cdots i_n}\in\Gamma''_n\,,\\ 1 & \text{on } \gamma^{(k)}_{1i_1\cdots i_n}\in\Gamma'_n\,,\end{cases}\tag{11}$$

then

$$\omega(z)>\alpha>0 \quad\text{on}\quad \gamma\,,$$

where $\alpha>0$ is a constant. Since $u_n(z)>0$ on γ', we have by the maximum principle,

$$u_n(z)\geqq\omega(z)>\alpha>0 \text{ on } \gamma\,.\tag{12}$$

Hence the connected part $\Phi_n(\tau)$ of Φ_n, for which $u_n(z)\leqq\tau$ $(\leqq\alpha)$ and contains $\gamma^{(1)}$ on its boundary does not contain γ, so that if we denote the niveau curve: $u_n(z)=\tau$ $(0\leqq\tau\leqq1)$ by C_τ, then if $\tau\leqq\alpha$,

$$\int_{C_\tau}dv_n=\int_{\gamma^{(1)}}dv_n=d_n\quad(\tau\leqq\alpha)\,.\tag{13}$$

Let $L(\tau)$ be the length of C_τ measured on the z-sphere and $A(\tau)$ be the spherical area of $\Phi_n(\tau)$:

$$L(\tau)=\int_{C_\tau}\frac{|z'|\,dv_n}{1+|z|^2},\qquad z'=\frac{dz}{d\zeta}\,,\ \zeta=u_n+iv_n\,,$$

$$A(\tau)=\int_0^\tau\int_{C_\tau}\left(\frac{|z'|}{1+|z|^2}\right)^2 d\tau\,dv_n\,,\ S(\tau)=A(\tau)/|F|\,,$$

$|F|$ being the spherical area of F, then

$$L(\tau)^2\leqq\int_{C_\tau}dv_n\int_{C_\tau}\left(\frac{|z'|}{1+|z|^2}\right)^2 dv_n\leqq d_n\frac{dA(\tau)}{d\tau}\quad(\tau\leqq\alpha)\,.\tag{14}$$

Now C_τ consists of a finite number $\nu(\tau)$ of disjoint closed curves, each of which is not homotop null, so that each curve has a length $\geqq a>0$, where a is a constant, which depends on F only, hence

$$L(\tau)\geqq a\,\nu(\tau)\,.\tag{15}$$

Let $\rho(\tau)$ be the Euler's characteristic of $\Phi_n(\tau)$, then since $\Phi_n(\tau)$ $(0\leqq\tau\leqq\alpha)$ is of planar character and the boundary of $\Phi_n(\tau)$ consists of C_τ and $\gamma^{(1)}$,

$$\rho(\tau)\leqq\nu(\tau)\leqq L(\tau)/a\,.\tag{16}$$

Now $\Phi_n(\tau)$ is a covering surface of F and $2(p-1)=2$ is the Euler's characteristic of F, so that by Ahlfors' fundamental theorem on covering surfaces[5]:

$$\rho^{+}(\tau) \geqq 2\, S(\tau) - h\, L(\tau)\,, \tag{17}$$

where $h>0$ is a constant, which depends on F only.

Hence from (16), (17), we have

$$A(\tau) \leqq k\, L(\tau)\,, \tag{18}$$

where $k>0$ is a constant, so that from (14),

$$A(\tau)^2 \leqq k^2\, d_n \frac{dA(\tau)}{d\tau} \qquad (\tau \leqq \alpha)\,,$$

hence

$$\frac{\alpha}{2} \leqq k^2\, d_n \int_{\frac{\alpha}{2}}^{\alpha} \frac{dA(\tau)}{A(\tau)^2} \leqq k^2 d_n / A\left(\frac{\alpha}{2}\right), \quad \text{or}$$

$$\frac{\alpha}{2}\, A\left(\frac{\alpha}{2}\right) \leqq k^2 d_n\,.$$

If $u_n(z) \to 0$, then $d_n \to 0$, so that $A\left(\frac{\alpha}{2}\right) \to 0$, which is absurd. Hence $u_n(z)$ does not tend to zero with $n \to \infty$.

Let γ be a ring cut of Φ, which lies in $\Phi - F_1$. If we cut Φ along γ, then Φ breaks up into two parts. We denote that part, which does not contain F_1, by $\Phi[\gamma]$.

With this notation, we put

$$\tilde{\Phi}_n = \Phi_n + \sum_{\gamma_{1i_1\cdots i_n}^{(k)} \in \Gamma'_n} \Phi\, [\gamma_{1i_1\cdots i_n}^{(k)}]\,. \tag{19}$$

Let Λ_n be the compact boundary of $\tilde{\Phi}_n$ and $\tilde{u}_n(z)$ be the harmonic measure of the ideal boundary of $\tilde{\Phi}_n$ with respect to $\tilde{\Phi}_n$, such that $\tilde{u}_n(z)$ is harmonic in $\tilde{\Phi}_n$, $\tilde{u}_n = 0$ on Λ_n, $u_n = 1$ on the ideal boundary of $\tilde{\Phi}_n$, then since cap. $E_1 > 0$, we see easily that $\tilde{u}_n(z) \not\equiv 0$, so that $0 < \tilde{u}_n(z) < 1$ in $\tilde{\Phi}_n$.

We shall prove that $\tilde{u}_n(z)$ does not tend to zero with $n \to \infty$. Let $\gamma = \gamma_{1i_1\cdots i_n}^{(k)} \in \Gamma'_n$. We draw a ring cut γ' in $F_{1i_1\cdots i_n}$, which lies in a small

5) L. Ahlfors: Zur Theorie der Überlagerungsflächen. Acta Math. 65 (1935).

neighbourhood of γ, such that γ, γ' bound a ring domain Δ in $F_{1i_1\cdots i_n}$. Let $\omega(z)$ be the harmonic measure of the ideal boundary of $\Phi[\gamma']$ with respect to $\Phi[\gamma']$, such that $\omega(z)$ is harmonic in $\Phi[\gamma']$, $\omega=0$ on γ', $\omega=1$ on the ideal boundary of $\Phi[\gamma']$, then

$$\omega(z)>\alpha>0 \text{ on } \gamma,$$

where $\alpha>0$ is a constant. Since $\tilde{u}_n(z)>0$ on γ', we have by the maximum principle,

$$\tilde{u}_n(z)\geqq\omega(z)>\alpha>0 \qquad \text{on } \gamma,$$

so that

$$\tilde{u}_n(z)\geqq\alpha\, u_n(z) \qquad \text{on } \gamma,$$

hence by the maximum principle,

$$\tilde{u}_n(z)\geqq\alpha\, u_n(z) \qquad \text{in } \Phi_n. \tag{20}$$

Since $u_n(z)$ does not tend to zero, $\tilde{u}_n(z)$ does not tend to zero with $n\to\infty$, so that

$$\tilde{u}_n(z_0)\geqq\eta>0 \quad (n=1, 2,\cdots), \tag{21}$$

where z_0 is a fixed point of F_1.

We map $\tilde{\Phi}_n$ on $|\tau|<1$ conformally, such that z_0 becomes $\tau=0$ and put $U_n(\tau)=\tilde{u}_n(z)$, then

$$U_n(0)\geqq\eta>(n=1, 2,\cdots). \tag{22}$$

Since the compact boundary of $\tilde{\Phi}_n$ is mapped on a set of arcs on $|\tau|=1$, on which $U_n(\tau)=0$, if we denote the complement of this set by e_n, then

$$me_n\geqq2\pi U_n(0)\geqq2\pi\eta>0 \quad (n=1, 2,\cdots). \tag{23}$$

We map $F^{(\infty)}$ on $|w|<1$ conformally, such that z_0 becomes $w=0$. Then $|\tau|<1$ is mapped on a domains Δ_n in $|w|<1$. Let M_n be the image of e_n on $|w|=1$, then by an extension of Löwner's theorem[6], we have $mM_n\geqq me_n$, so that

$$mM_n\geqq2\pi\eta>0 \qquad (n=1, 2,\cdots).$$

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6) Y. Kawakami: On an extension of Löwner's lemma. Jap. Journ. Math. 17 (1941). M. Tsuji: On an extension of Löwner's theorem. Proc. Imp. Acad. 18 (1942).

Hence if we put $M=\overline{\lim\limits_{n\to\infty}} M_n$, then

$$mM\geqq 2\pi\eta>0. \tag{24}$$

We see easily that M is a sub-set of the image of E_1 on $|w|=1$, hence E_1 is mapped on a set of positive measure on $|w|=1$.

Hence our theorem is proved.

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