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On Royden's theorem on a covering surface of a closed Riemann surface.

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Let F be a closed Riemann surface of genus $p\geqq 2$, spread over the z-plane and Φ be its unramified covering surface. Let C_i $(i=1,2,\cdots,p)$ be p disjoint ring cuts of F, such that, if we cut F along $\{C_i\}$, then F becomes a surface of planar character and let C_i' be the conjugate ring cut of C_i, such that C_i' meets C_i at a point and is disjoint to C_j, C_j' $(j=1,2,\cdots,p,\ j\neq i)$. We assume that C_i, C_i' are rectilinear polygons and meet at a positive angle. We denote the both shores of C_i, C_i' by $C_i^+, C_i^-, C_i'^+, C_i'^-$ respectively.

We denote a surface, which is obtained from F by cutting along a certain number of C_i, C_i' by F' in general, then

$$\Phi=\sum_{k=0}^{\infty} F'_k,$$

where F_k' is one F'.

Let Γ_k be the boundary of F_k', which consists of a certain number of $C_i^+, C_i^-, C_i'^+, C_i'^-$, which we denote by $\{\sigma_k^{(i)}\}_{i=1,2,\cdots,\alpha_k}$ so that $\Gamma_k=\sum_i \sigma_k^{(i)}$. Along $\sigma_k^{(i)}$, there connects another F_s' to F_k'.

Then Royden[1] proved the following theorem.

THEOREM. *The necessary and sufficient condition that Φ is of positive boundary is that there exist a contant $m_k^{(i)}$ corresponding to $\sigma_k^{(i)}$, such that if $\sigma_k^{(i)}$ belong to the boundary of another F_s' and $\sigma_k^{(i)}=\sigma_s^{(j)}$, then $m_s^{(j)}=-m_k^{(i)}$ and satisfy the following conditions:*

(i) $\qquad \sum_i m_0^{(i)}\neq 0, \qquad \sum_i m_k^{(i)}=0 \quad (k=1,2,\cdots),$

(ii) $\qquad \sum_{k=0}^{\infty} M_k^2<\infty,$

1) H. L. Royden: Harmonic functions on open Riemann surfaces. Trans. Amer. Math. Soc. 75 (1952).

where $M_k = \operatorname*{Max}_i |m_k^{(i)}|$.

We shall give a simple proof in the following lines.

PROOF. (i) *Necessity.*

Suppose that Φ is of positive boundary. We take a circular disc Δ_0 in F_0' and C_0 be its bounding circle.

Let $u(z)$ be the harmonic measure of the ideal boundary of Φ with respect to $\Phi-\Delta_0$, then since Φ is of positive boundary,

$$u=0 \text{ on } C_0, \qquad \int_{C_0} \frac{\partial u}{\partial \nu} ds > 0, \qquad 0<u<1 \quad \text{ in } \Phi-\Delta_0,$$

$$D_{\Phi-\Delta_0}[u] < \infty,$$

where $D_{\Phi-\Delta_0}[u]$ is the Dirichlet integral of $u(z)$ in $\Phi-\Delta_0$.

Now to F_k', there connects a finite number of F''s: $F_\alpha', F_\beta', \cdots$. Let z be any point of Γ_k, then

$$|\operatorname{grad} u(z)|^2 \leqq \text{const.} \left(D_{F_k'}[u] + \sum_\alpha D_{F_\alpha'}[u]\right),$$

so that

$$\operatorname*{Max}_{z\in\Gamma_k} \left|\frac{\partial u}{\partial \nu}\right|^2 \leqq \text{const.} \left(D_{F_k'}[u] + \sum_\alpha D_{F_\alpha'}[u]\right),$$

where ν is the outer normal of Γ_k with respect to F_k'.

Now we define $m_k^{(i)}$ by

$$\int_{\sigma_k^{(i)}} \frac{\partial u}{\partial \nu} ds = m_k^{(i)}, \tag{1}$$

then

$$\sum_i m_0^{(i)} = \int_{C_0} \frac{\partial u}{\partial \nu} ds > 0, \qquad \sum_i m_k^{(i)} = \int_{\Gamma_k} \frac{\partial u}{\partial \nu} ds = 0 \quad (k \geqq 1). \tag{2}$$

If we put $M_k = \operatorname*{Max}_i |m_k^{(i)}|$, then

$$M_k^2 \leqq \text{const.} \operatorname*{Max}_{z\in\Gamma_k} \left|\frac{\partial u}{\partial \nu}\right|^2 \leqq \text{const.} \left(D_{F_k'}[u] + \sum_\alpha D_{F_\alpha'}[u]\right),$$

so that

$$\sum_{k=0}^{\infty} M_k^2 \leqq \text{const.}\, D_{\Phi-\Delta_0}[u] < \infty. \tag{3}$$

Hencn $\{m_k^{(i)}\}$ satisfy the condition of the theorem.

(ii) *Sufficiency.*

Suppose that there exist constants $m_k^{(i)}$, which satisfy the condition of the theorem. We define $f_k(z)$ on Γ_k by

$$f_k(z)=m_k^{(i)}/|\sigma_k^{(i)}| \quad \text{on} \quad \sigma_k^{(i)} \quad (k=0,1,2,\cdots), \tag{4}$$

where $|\sigma_k^{(i)}|$ is the length of $\sigma_k^{(i)}$, then

$$\int_{\Gamma_k} f_k(z)ds=\sum_i m_k^{(i)}=0 \qquad (k\geqq 1).$$

We solve the Neumann problem for F_k' $(k\geqq 1)$ with the boundary value $f_k(z)$ and let $w_k(z)$ be its solution, such that $w_k(z)$ is harmonic in F_k' and

$$\frac{\partial w_k}{\partial \nu}=f_k(z) \quad \text{on} \quad \Gamma_k. \tag{5}$$

Let $f_k^{(i)}(z)$ $(i\geqq 2)$ be defined on Γ_k by

$$\begin{aligned} f_k^{(i)}(z)&=1/|\sigma_k^{(i)}| \quad \text{on} \quad \sigma_k^{(i)} \ (i\geqq 2), \\ &=-1/|\sigma_k^{(1)}/ \ \text{on} \ \sigma_k^{(1)}, \\ &=0 \qquad\quad \text{on} \ \sigma_k^{(j)} \quad (j\neq i,1), \end{aligned} \tag{6}$$

then $\int_{\Gamma_k} f_k^{(i)}(z)ds=0$. Let $w_k^{(i)}(z)$ be the solution of the Neumann problem for F_k', with the boundary value $f_k^{(i)}(z)$, then since $\sum_i m_k^{(i)}=0$,

$$f_k(z)=\sum_{i=2}^{\alpha_k} m_k^{(i)} f_k^{(i)}(z),$$

so that

$$w_k(z)=\sum_{i=2}^{\alpha_k} m_k^{(i)} w_k^{(i)}(z).$$

Hence

$$\sqrt{D_{F_k'}[w_k]}\leqq \sum_{i=2}^{\alpha_k} |m_k^{(i)}| \sqrt{D_{F_k'}[w_k^{(i)}]}\leqq \text{const.}\, M_k,$$

$$\sum_{k=1}^{\infty} D_{F_k'}[w_k]\leqq \text{const.} \sum_{k=1}^{\infty} M_k^2<\infty. \tag{7}$$

Let $w_0(z)$ be the solution of the mixed boundary value problem for $F_0'-\varDelta_0$, such that $w_0(z)$ is harmonic in $F_0'-\varDelta_0$,

$$w_0=0 \quad \text{on} \quad C_0, \qquad \frac{\partial w_0}{\partial \nu}=f_0(z) \quad \text{on} \quad \Gamma_0. \tag{8}$$

Then

$$\int_{C_0} \frac{\partial w_0}{\nu \partial} ds=\sum_i m_0^{(i)} \neq 0. \tag{9}$$

We define $w(z)$ in $\Phi-\Delta_0$ by putting

$$\begin{aligned} w(z)&=w_0(z) \quad \text{in} \quad F_0'-\Delta_0, \\ &=w_k(z) \quad \text{in} \quad F_k' \qquad (k\geqq 1). \end{aligned} \tag{10}$$

Then by (7), (9),

$$\int_{C_0} \frac{\partial w}{\partial \nu} ds \neq 0, \qquad D_{\Phi-\Delta_0}[w]<\infty. \tag{11}$$

First proof. Let Φ_n be a connected part of Φ, such that

$$\Phi_n=(F_0'-\Delta_0)+\sum_{k=1}^{n} F_k' \tag{12}$$

and $C_0+\gamma_n$ be its boundary. Let $u_n(z)$ be the solution of the mixed boundary value problem for Φ_n, such that $u_n(z)$ is harmonic in Φ_n,

$$u_n=0 \text{ on } C_0, \qquad \frac{\partial u_n}{\partial \nu}=\frac{\partial w}{\partial \nu} \quad \text{on } \gamma_n. \tag{13}$$

Then

$$D_{\Phi_n}[u_n, w]=\int_{\gamma_n} u_n \frac{\partial w}{\partial \nu} ds=\int_{\gamma_n} u_n \frac{\partial u_n}{\partial \nu} ds=D_{\Phi_n}[u_n],$$

so that $D_{\Phi_n}[u_n, w-u_n]=0$, hence

$$D_{\Phi-\Delta_0}[w]\geqq D_{\Phi_n}[w]=D_{\Phi_n}[u_n]+D_{\Phi_n}[w-u_n]\geqq D_{\Phi_n}[u_n], \tag{14}$$

$$\int_{C_0} \frac{\partial u_n}{\partial \nu} ds=\int_{\gamma_n} \frac{\partial u_n}{\partial \nu} ds=\int_{\gamma_n} \frac{\partial w}{\partial \nu} ds=\int_{C_0} \frac{\partial w}{\partial \nu} ds \neq 0. \tag{15}$$

By (14), we can select a partial sequence $u_{n_k}(z)$ from $u_n(z)$, such that $u_{n_k}(z)\to u(z)$ uniformly in the wider sense in $\Phi-\Delta_0$, then $u(z)$ is harmonic in $\Phi-\Delta_0$ and by (13), (14), (15),

$$u_0=0 \quad \text{on} \quad C_0, \qquad \int_{C_0} \frac{\partial u}{\partial \nu} ds \neq 0, \qquad D_{\Phi-\Delta_0}[u]<\infty. \tag{16}$$

Since such a harmonic function $u(z)$ exsits on $\Phi-\Delta_0$, Φ is of positive boundary.

Second proof. (together with A. Mori). Let $u_n(z)$ be the harmonic measure of γ_n with respect to Φ_n, such that $u_n(z)$ is harmonic in Φ_n,

$$u_n=0 \quad \text{on} \quad C_0, \qquad u_n=1 \quad \text{on} \quad \gamma_n.$$

Let $v_n(z)$ be its conjugate harmonic function and put

$$d_n=\int_{C_0} dv_n>0. \tag{17}$$

Let C_ρ be the niveau curve: $u_n(z)=\rho$ $(0\leqq\rho\leqq1)$, then if $w(z)$ is defined by (10),

$$\int_{C_\rho} \frac{\partial w}{\partial \rho} dv_n=\int_{C_\rho} \frac{\partial w}{\partial \nu} ds=\int_{C_0} \frac{\partial w}{\partial \nu} ds=a_0\neq 0,$$

so that

$$a_0^2=\left(\int_{C_\rho} \frac{\partial w}{\partial \rho} dv_n\right)^2\leqq\int_{C_\rho} dv_n\int_{C_\rho}\left(\frac{\partial w}{\partial \rho}\right)^2 dv_n=d_n\int_{C_\rho}\left(\frac{\partial w}{\partial \rho}\right)^2 dv_n,$$

$$a_0^2\leqq d_n\int_0^1 d\rho\int_{C_\rho}\left(\frac{\partial w}{\partial \rho}\right)^2 dv_n\leqq d_n D_{\Phi_n}[w]\leqq d_n D_{\Phi-\Delta_0}[w].$$

Hence d_n does not tend to zero with $n\to\infty$, so that Φ is of positive boundary.

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