Journal of the Mathematical Society of Japan Vol. 6, No. 1, March, 1954

On the normal form of cohomology groups.

By Tadao TANNAKA

(Received July 15, 1953)

In a previous note[1] I have obtained an extension of the well-known Hilbert's norm theorem concerning cyclic fields to the case of abelian extensions, but only for the case of p-adic number fields as ground fields. The present investigation started at its first stage to fill up this lack, but the problem was quite difficult to overwhelm, and the results we have obtained have an entirely different meaning.

Instead of the factor sets treated in 1), we take up generally the cochain of degree n, with finite abelian operator domain.

For the case of $n=2$, O. Schreier has obtained a normal form for the factor sets, in his famous research on group extension.[2] A part of this result is generalized by Lyndon,[3] and we shall reproduce the results by another approach. We then give a more precise normal form, which is a direct generalization of the fact, that the Galois-cohomology group of dimension 2 by a cyclic field is isomorphic to the norm residue class group.

The main principle which governs this paper lies in the concept of splitting groups, which have slightly weakend property compared with Artin's splitting group.

Several parts of our investigation were also obtained by S. Takahashi independently and appeared in a recent number of Tohoku Mathematical Journal.[4]

§1. Throughout this paper we treat the cocycles with values in an abelian group Ω and with an abelian operator domain G. We call a cocycles to be splitting or bounding, if it is a bounding cocycle. Then we have the following (weakend) analogy of Artin's splitting group, which is known as the reduction theorem.

THEOREM 1. *To each cocycle f, we find an extension Ω^* of Ω, in which f splits.*

PROOF. For the sake of brevity, we shall assume that the dimension n of f is 3. We have by the assumption

(1) $$\frac{f(\beta,\gamma,\delta)f(\alpha,\beta\gamma,\delta)f(\alpha,\beta,\gamma)^{\delta}}{f(\alpha\beta,\gamma,\delta)f(\alpha,\beta,\gamma\delta)}=1\,.$$

or

(2) $$f(\alpha,\beta,\gamma)^{\delta}=\left\{\frac{f(\beta,\gamma,\delta)f(\alpha,\beta\gamma,\delta)}{f(\alpha\beta,\gamma,\delta)f^{\gamma}(\alpha,\beta,\delta)}\right\}^{-1}$$

if we put

(3) $$f^{\gamma}(\alpha,\beta,\delta)=f(\alpha,\beta,\gamma\delta)\,.$$

Now we observe the group Ω^* of 1-cochain, that is, the group whose elements are functions with parameter δ in G and values in Ω. We identify the element c in Ω to the function c^{δ} in Ω^*. We also make G to an operator domain by the convention

(4) $$g^{\gamma}(\delta)=g(\gamma\delta) \qquad (\gamma\in G)\,.$$

By these conventions, the equation (2) becomes the following equality, which asserts that f splits in Ω^*, as was required:

(5) $$f(\alpha,\beta,\gamma)=\frac{c(\beta,\gamma)\,c(\alpha,\beta\gamma)}{c(\alpha\beta,\gamma)\,c(\alpha,\beta)^{\gamma}}=(\delta c)(\alpha,\beta,\gamma)$$

$$\begin{pmatrix} c(\alpha,\beta)=f(\alpha,\beta,\delta)^{-1}\in\Omega^* \\ \delta\text{: coboundary operator}\end{pmatrix}.$$

§ 2. Now, as well known, one can find in every cohomology class a cocycle which takes the value 1 whenever at least one variable reduces to 1. We shall call every cocycle or cochain with this property normalized. Furthermore we shall call a cocycle f weakly normal, if it satisfies the following relation

(6) $$\begin{aligned} f(\alpha_1\beta_1,\alpha_2\beta_2,\cdots,\alpha_n\beta_n)=&f(\alpha_1,\alpha_2,\cdots,\alpha_n)^{\beta_1\beta_2\beta_3\cdots\beta_n}\\ &\cdot f(\beta_1,\alpha_2,\alpha_3,\cdots,\alpha_n)^{\beta_2\beta_3\cdots\beta_n}\\ &\cdot f(\beta_1,\beta_2,\alpha_3,\cdots,\alpha_n)^{\beta_3\cdots\beta_n}\\ &\cdots\cdots\cdots\\ &\cdots\cdots\cdots\\ &\cdot f(\beta_1,\beta_2,\beta_3,\cdots,\beta_n)\end{aligned}$$

for $\alpha_i\in A$, $\beta_i\in B$.

We now proceed to the second step and prove the

THEOREM 2. *If $G=A\times B$, then there exists a normalized, weakly normal cocycle f in every cohomology class.*

PROOF. The theorem is obvious if $n=0$. We shall prove the theorem, by mathematical induction on n, and suppose we have already completed the proof for smaller values of n. By theorem 1 the given cocycle f splits in some extension Ω^* of Ω:

$$f=\delta c. \tag{7}$$

It follows then c is a $(n-1)$-cocycle with values on Ω^*/Ω, for $\delta c=f\equiv 1$ (mod. Ω). By the assumption of induction we have an $(n-1)$-cochain $c'(\gamma_1,\cdots,\gamma_{n-1})$ such that

$$\begin{cases} c'(\gamma_1,\cdots,\gamma_{n-1})\equiv c(\gamma_1,\cdots\gamma_{n-1})\,(\delta c)\,(\gamma_1,\cdots,\gamma_{n-1}) \\ \qquad\qquad\qquad\qquad\qquad\qquad (\text{mod. } \Omega)\,, \\ c'(\alpha_1\beta_1,\cdots,\alpha_{n-1}\beta_{n-1})\equiv c'(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})^{\beta_1\cdots\beta_{n-1}} \\ \qquad\qquad\qquad\qquad c'(\beta_1,\alpha_2,\cdots,\alpha_{n-1})^{\beta_2\cdots\beta_{n-1}} \\ \qquad\qquad\qquad\qquad\qquad \cdots\cdots\cdots(\text{mod. } \Omega)\,, \end{cases} \tag{8}$$

so that

$$\begin{aligned} c'(\gamma_1,\cdots,\gamma_{n-1})=c(\gamma_1,\cdots,\gamma_{n-1})\,(\delta\bar{c})\,(\gamma_1,\cdots,\gamma_{n-1}) \\ b(\gamma_1,\cdots,\gamma_{n-1}) \\ \big(b(\gamma_1,\cdots,\gamma_{n-1})\in\Omega\big)\,. \end{aligned}$$

Now the n-cocycle $f'=\delta c'$ satisfies the relation

$$f'=\delta c_{\prime}=\delta c\cdot\delta\delta\bar{c}\cdot\delta b=\delta c\cdot\delta b=f\cdot\delta b\sim f\,(\text{in } \Omega)\,. \tag{9}$$

On the other hand if we put

$$\begin{aligned} c''(\alpha_1\beta_1,\cdots,\alpha_{n-1}\beta_{n-1})=c'(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})^{\beta_1\cdots\beta_{n-1}} \\ c'(\beta_1,\alpha_2,\cdots,\alpha_{n-1})^{\beta_2\cdots\beta_{n-1}} \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ c'(\beta_1,\beta_2,\cdots,\beta_{n-1})\,, \end{aligned}$$

we have

$$c''(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}) = c'(\alpha_1, \alpha_2, \cdots, \alpha_{n-1}),$$
$$c''(\beta_1, \alpha_2, \cdots, \alpha_{n-1}) = c'(\beta_1, \alpha_2, \cdots, \alpha_{n-1}),$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots ,$$

so that

$$c''(\alpha_1\beta_1, \alpha_2\beta_2, \cdots, \alpha_{n-1}\beta_{n-1}) = c''(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})^{\beta_1\beta_2\cdots\beta_{n-1}} c''(\beta_1, \alpha_2, \cdots, \alpha_{n-1})^{\beta_2\cdots\beta_{n-1}} \cdots\cdots , \tag{10}$$

and moreover by (8)

$$c''(\gamma_1, \gamma_2, \cdots, \gamma_{n-1}) \equiv c'(\gamma_1, \gamma_2, \cdots, \gamma_{n-1}) \qquad (\text{mod. } \mathfrak{Q}),$$

and hence

$$f'' = \delta c'' \sim f' = \delta c' \qquad (\text{in } \mathfrak{Q}).$$

By our assumption of the induction, $c'(\gamma_1, \cdots, \gamma_n)$ mod. $\mathfrak{Q}$ may be supposed to be normalized, so that, as is easily seen from the construction of f'', f'' may be taken as normalized, and as f'' is weakly normal, as will be shown in the next §, this proves our assertion.

§ 3. In the preceding paragraph we have proved except the point of which we shall complete the proof in this paragraph, that every cohomology class contains a normalized weakly normal cocycle. More precisely we can suppose

$$f = \delta c, \tag{11}$$

where c is a normalized weakly normal cochain, that is

$$c(\alpha_1\beta_1, \alpha_2\beta_2, \cdots, \alpha_{n-1}\beta_{n-1}) = c(\alpha_1, \alpha_2, \cdots, \alpha_{n-1})^{\beta_1\beta_2\cdots\beta_{n-1}} c(\beta_1, \alpha_2, \cdots, \alpha_{n-1})^{\beta_2\cdots\beta_{n-1}} \cdots\cdots \tag{12}$$

We have left thereby the following result unproved.

THEOREM 3. *If we put*

$$f(\gamma_1, \gamma_2, \cdots, \gamma_n) = (\delta c)(\gamma_1, \gamma_2, \cdots, \gamma_n),$$

where c is a normalized cochain satisfying the relation given in (12), *then f is normalized weakly normal.*

PROOF. We prove this by induction on n.

We first prove that the following identity holds, when we regard δ as a parameter, as was done in § 1:

$$c(\alpha_1\beta_1,\cdots,\alpha_{n-2}\beta_{n-2},\delta)=c^{\beta_1\cdots\beta_{n-2}}(\alpha_1,\alpha_2,\cdots,\alpha_{n-2},\delta) \tag{13}$$
$$c^{\beta_2\cdots\beta_{n-2}}(\beta_1,\alpha_2,\cdots,\alpha_{n-2},\delta)$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$c(\beta_1,\cdots,\beta_{n-2},\delta)\,.$$

Indeed the left-hand side of (13) is

$$c(\alpha_1,\alpha_2,\cdots,\alpha_{n-2},\alpha)^{\beta_1\beta_2\cdots\beta_{n-2}\beta}$$
$$c(\beta_1,\alpha_2,\cdots,\alpha_{n-2},\alpha)^{\beta_2\beta_3\cdots\beta_{n-2}\beta}$$
$$\cdots\cdots\cdots\cdots$$
$$c(\beta_1,\beta_2,\cdots,\beta_{n-2},\beta)\,,$$

where $\alpha\beta=\delta$, and the right-hand side of (13) is, by the definition of the symbolic power $f^{\gamma}(\delta)=f(\gamma\delta)$,

$$c(\alpha_1,\alpha_2,\cdots,\alpha_{n-2},\beta_1\cdots\beta_{n-2}\delta)$$
$$c(\beta_1,\alpha_2,\cdots,\alpha_{n-2},\beta_2\cdots\beta_{n-2}\delta)$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$c(\beta_1,\beta_2,\cdots,\beta_{n-2},\delta)$$
$$=c(\alpha_1,\alpha_2,\cdots,\alpha_{n-2},\alpha)^{\beta_1\cdots\beta_{n-2}\beta}$$
$$c(\beta_1,\alpha_2,\cdots,\alpha_{n-2},\alpha)^{\beta_2\cdots\beta_{n-2}\beta}$$
$$\cdots\cdots\cdots\cdots\cdots$$
$$c(\beta_1,\beta_2,\cdots,\beta_{n-2},\alpha)^{\beta}$$
$$c(\beta_1,\beta_2,\cdots,\beta_{n-2},\beta)\,,$$

which proves the required identity.

To prove our theorem, put $\gamma_i=\alpha_i\beta_i$ and $\delta=\alpha\beta$, then by the definition $f=\delta c$, we have

$$f(\gamma_1,\gamma_2,\cdots,\gamma_{n-1},\delta)=c(\gamma_2,\cdots,\gamma_{n-1},\delta)\,c(\gamma_1\gamma_2,\cdots,\gamma_{n-1},\delta)^{-1}\cdots$$

$$c(\gamma_1,\cdots,\gamma_{n-2},\gamma_{n-1}\delta)^{\pm 1}\,c(\gamma_1,\cdots,\gamma_{n-1})^{\mp\delta}$$
$$=(\delta_{n-1}c)(\gamma_1,\cdots,\gamma_{n-1},\delta)\,c(\gamma_1,\cdots,\gamma_{n-1})^{\mp\delta}$$

or $\delta_{n-1}c=f\cdot c^{\pm\delta}$. The formula (13) proved above shows that the cochain c, constituted as a function with values in Ω^*, is weakly normal; hence by the assumption of induction, we have for $f'=\delta_{n-1}c$ (this is a function of δ),

$$\begin{aligned}
f'(\gamma_1,\cdots,\gamma_{n-1})&=f'(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})^{\beta_1\cdots\beta_{n-1}}\\
&\qquad f'(\beta_1,\alpha_2,\cdots,\alpha_{n-1})^{\beta_2\cdots\beta_{n-1}}\\
&\qquad \cdots\cdots\cdots\cdots\\
&=\left(f(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})\cdot c(\alpha_1,\alpha_2,\cdots,\alpha_{n-1})^{\pm\delta}\right)^{\beta_1\cdots\beta_{n-1}}\\
&\quad\left(f(\beta_1,\alpha_2,\cdots,\alpha_{n-1})\cdot c(\beta_1,\alpha_2,\cdots,\alpha_{n-1})^{\pm\delta}\right)^{\beta_2\cdots\beta_{n-1}}\\
&\quad\cdots\cdots\cdots\cdots\\
&\quad\left(f(\beta_1,\beta_2,\cdots,\beta_{n-1})\cdot c(\beta_1,\beta_2,\cdots,\beta_{n-1})^{\pm\delta}\right)\\
&=f(\alpha_1,\alpha_2,\cdots,\alpha_{n-1},\beta_1\cdots\beta_{n-1}\delta)\\
&\quad f(\beta_1,\alpha_2,\cdots,\alpha_{n-1},\beta_2\cdots\beta_{n-1}\delta)\\
&\quad\cdots\cdots\cdots\cdots\\
&\quad f(\beta_1,\beta_2,\cdots,\beta_{n-1},\delta)\\
&\quad c(\alpha_1\beta_1,\cdots,\alpha_{n-1}\beta_{n-1})^{\pm\delta},
\end{aligned}$$

so that we have the equality

$$\begin{aligned}
&f(\gamma_1,\gamma_2,\cdots,\gamma_{n-1},\delta)\\
&=f(\alpha_1,\alpha_2,\cdots,\alpha_{n-1},\beta_1\cdots\beta_{n-1}\delta)\\
&\quad f(\beta_1,\alpha_2,\cdots,\alpha_{n-1},\beta_2\cdots\beta_{n-1}\delta)\\
&\quad\cdots\cdots\cdots\cdots\\
&\quad f(\beta_1,\beta_2,\cdots,\beta_{n-1},\delta).
\end{aligned}$$

Our final goal is to prove the equality of this expression and

$$\begin{aligned}
&f(\alpha_1,\alpha_2,\cdots,\alpha_{n-1},\alpha)^{\beta_1\cdots\beta_{n-1}\beta}\\
&f(\beta_1,\alpha_2,\cdots,\alpha_{n-1},\alpha)^{\beta_2\cdots\beta_{n-1}\beta}\\
&\cdots\cdots\cdots\cdots
\end{aligned}$$

$$f(\beta_1,\beta_2,\cdots,\beta_{n-1},\beta).$$

This will truely hold, if we can prove generally

$$\begin{aligned}&f(\beta_1,\cdots,\beta_{k-1},\alpha_k,\cdots,\alpha_{n-1},\alpha\beta)\\&=f(\beta_1,\cdots,\beta_{k-1},\alpha_k,\cdots,\alpha_{n-1},\alpha)^\beta \qquad (k<n),\end{aligned}$$

and

$$\begin{aligned}&f(\beta_1,\cdots,\beta_{n-1},\alpha\beta)\\&=f(\beta_1,\cdots,\beta_{n-1},\alpha)^\beta\\&\quad f(\beta_1,\cdots,\beta_{n-1},\beta),\end{aligned}$$

but this will follow immediately from the very definition of f, the first one for instance

$$\begin{aligned}&f(\beta_1,\cdots,\beta_{k-1},\alpha_k,\cdots,\alpha_{n-1},\alpha\beta)\\&=c(\beta_2,\cdots,\beta_{k-1},\alpha_k,\cdots,\alpha_{n-1},\alpha\beta)\\&\quad c(\beta_1\beta_2,\cdots,\beta_{k-1},\alpha_k,\cdots,\alpha_{n-1},\alpha\beta)^{-1}\\&\quad \cdots\cdots\cdots\cdots\\&=c(\beta_2,\cdots,\beta_{k-1},\alpha_k,\cdots,\alpha_{n-1},\alpha)^\beta\\&\quad c(\beta_1\beta_2,\cdots,\beta_{k-1},\alpha_k,\cdots,\alpha_{n-1},\alpha)^{-\beta}\\&\quad \cdots\cdots\cdots\cdots\\&=f(\beta_1,\cdots,\beta_{k-1},\alpha_k,\cdots,\alpha_{n-1},\alpha)^\beta,\end{aligned}$$

and our theorem is now proved.

§ 4. The results of the preceding paragraphs can be generalized to the case of direct product $G=A\times B\times C$ or groups with more than 3 direct factors, the proof of which is almost same as for the case $G=A\times B$. We have for instance

THEOREM 4. *For the case of $G=A\times B\times C$ the theorem 2 remains true, if we understand by weakly normal cochains, those which satisfy the relation*

$$\begin{aligned}f(\alpha_1\beta_1\gamma_1,\cdots,\alpha_n\beta_n\gamma_n)&=f(\alpha_1,\alpha_2,\cdots,\alpha_n)^{(\beta_1\gamma_1)\cdots(\beta_n\gamma_n)}\\&\quad f(\beta_1,\alpha_2,\cdots,\alpha_n)^{\gamma_1(\beta_2\gamma_2)\cdots(\beta_n\gamma_n)}\\(14)\qquad&\quad \cdots\cdots\cdots\cdots\end{aligned}$$

$$f(\gamma_1,\cdots,\beta_k,\cdots,\alpha_l,\cdots)^{\gamma_k\cdots(\beta_l\gamma_l)\cdots} \quad ,$$

$$\cdots\cdots\cdots\cdots$$

$$f(\gamma_1,\gamma_2,\cdots,\gamma_n)$$

where α, β, γ *denote the elements of* A, B *and* C *respectively.*

§5. Before going further we shall explain the concept of the "strict normality" for the case of 2-cocycles. This will be done most rapidly, if we utilize the extension group of Ω by G. This is the group which contains Ω as normal subgroup and with representants $u_\sigma(\sigma\in G)$, which satisfy

$$u_\sigma u_\tau = u_{\sigma\tau} f(\sigma,\tau)\,,$$
$$u_\sigma^{-1} z u_\sigma = z^\sigma \qquad (z\in\Omega)\,.$$

If we choose another system of representants v_σ, defined by

$$v_{\alpha\beta} = u_\alpha u_\beta \qquad (G=A\times B,\ \alpha\in A,\ \beta\in B)$$

then we see easily that the factor set corresponding to v_σ is weakly normal. If we put further

$$v_{\sigma_1^{x_1}\cdots\sigma_r^{x_r}} = u_{\sigma_1^{x_1}\cdots\sigma_r^{x_r}}$$

where $G=\{\sigma_1\}\times\cdots\times\{\sigma_r\}$ is the canonical decomposition of G into the cyclic groups of orders $e_1,\cdots,e_r$, and $0\leqq x_i < e_i$, we obtain a more specialized normal cocycle, which we call "strictly normal". This result will be generalized in the next lines. One should notice that the abovementioned method, employing the group extension, is not applicable for the case of higher dimensions $n\geqq 3$.

§6. From now on we assume that the cocycles in consideration are always normalized. We shall modify also for convenience, the definition of the "weak normality," i.e. a cocycle f will be called weakly normal in the sequel if we have

$$f(\alpha_1\beta_1\gamma_1,\cdots,\alpha_n\beta_n\gamma_n)=f(\alpha_1,\alpha_2,\cdots,\alpha_n) \tag{15}$$

$$f(\alpha_1,\alpha_2,\cdots,\alpha_{n-1},\beta_n)^{\alpha_n}$$

$$\cdots\cdots\cdots\cdots$$

$$f(\alpha_1,\cdots,\beta_k,\cdots,\gamma_l\cdots)^{\alpha_k\cdots(\alpha_l\beta_l)\cdots}$$

$$\cdots\cdots\cdots\cdots$$

instead of (14), which means only that the order $\{\sigma_1\}, \{\sigma_2\}, \cdots, \{\sigma_r\}$ is from now on to be inverted to the order $\{\sigma_r\}, \cdots, \{\sigma_2\}, \{\sigma_1\}$.

We use following abbreviations:

$$[\sigma^x] = 1 + \sigma + \sigma^2 + \cdots + \sigma^{x-1},$$
$$N_i = 1 + \sigma_i + \sigma_i^2 + \cdots + \sigma_i^{e_i - 1},$$
$$\Delta_i = 1 - \sigma_i$$
$$[x]_i = \left[\frac{x}{e_i}\right] \qquad ([\;]\text{: Gauss' symbol}),$$

and

$$(16) \qquad \begin{aligned} &[x_1, x_2, \cdots, x_{r_1}; y_1, y_2, \cdots, y_{r_2}; \cdots] \\ &= [x_1 + x_2]_1 [x_3 + x_4]_1 \cdots [y_1 + y_2]_2 [y_3 + y_4]_2 \cdots (r_1 + r_2 + \cdots = n). \end{aligned}$$

In the last notation we make the convention, if for instance r_1 is odd number, the same symbol means

$$= [x_1 + x_2]_1 [x_3 + x_4]_1 \cdots [\sigma_1^{x_{r_1}}] [y_1 + y_2]_2 [\;]_2 \cdots .$$

An n-cochain f is said to have the standard form (or to be strictly normal), if it is weakly normal with respect to the decomposition $G = \{\sigma_1\} \times \{\sigma_2\} \times \cdots$, and moreover satisfies the conditions for $0 \leqq x_i < e_1$, $0 \leqq y_j < e_2 \cdots$

$$(17) \qquad \begin{aligned} &f(\sigma_1^{x_1}, \sigma_1^{x_2}, \cdots; \sigma_2^{y_1}, \sigma_2^{y_2}, \cdots) \\ &= f(\sigma_1, N_1, \sigma_1, N_1, \cdots; \sigma_2, N_2, \sigma_2, \cdots)^{[x_1, x_2, \cdots; y_1, y_2 \cdots]} \end{aligned}$$

or by a simple example

$$f(\sigma_1^{x_1}, \sigma_2^{x_2}, \sigma_1^{x_3}; \sigma_2^{y_1}, \sigma_2^{y_2}) = f(\sigma_1, N_1, \sigma_1; \sigma_2, N_2)^{[x_1, x_2, x_3; y_1, y_2]},$$

with

$$\begin{aligned} &[x_1, x_2, x_3; y_1, y_2] = [x_1 + x_2]_1 [\sigma_1^{x_3}] [y_1 + y_2]_2 \\ &= \left[\frac{x_1 + x_2}{e_1}\right] (1 + \sigma_1 + \ldots + \sigma_1^{x_3 - 1}) \left[\frac{y_1 + y_2}{e_2}\right]. \end{aligned}$$

By $f(\sigma_1, N_1, \sigma_1; \sigma_2, N_2)$ we mean in obvious manner

$$\prod_{i,j} f(\sigma_1, \sigma_1^i, \sigma_1; \sigma_2, \sigma_2^j) \qquad (0 \leqq i < e_1, 0 \leqq j < e_2).$$

Now we can state the followng theorem:

THEOREM 5. *Every cohomology class contains a cocycle with standard form.*

This is precisely the result announced in the preceding paragraph. We now proceed to the proof, by mathematical induction on n, and assume the theorem for smaller dimension than n.

As in the proof of the weak normality, we may agree to set

$$f(\gamma_1, \gamma_2, \cdots, \gamma_n) = (\delta\, c)\,(\gamma_1, \cdots, \gamma_n)\,,$$

$$c \equiv c' \cdot \delta \bar{c} \ (\text{mod. } \Omega)\,,$$

where c' is strictly normal with respect to the coefficient group Ω^*/Ω.

If we put

$$c = c' \cdot \delta \bar{c} \cdot b \quad \big(b\text{: } (n-1)\text{-cochain with values in } \Omega\big)\,,$$

then we have

$$f = \delta\, c' \cdot \delta b \sim \delta\, c' \qquad (\text{in } \Omega)\,.$$

Starting from the values $c''(\sigma_1, N_1, \sigma_1, \cdots; \sigma_2, N_2, \sigma_2, \cdots) = c'(\sigma_1, N_1, \sigma_1, \cdots; \sigma_2, N_2, \sigma_2, \cdots)$ and similar formulas as in (17) and (15) we define an $(n-1)$-cochain c''. Then c'' is strictly normal, and $c'' \sim c'$ (mod. Ω), so that

$$f \sim \delta\, c'' \qquad (\text{in } \Omega)\,.$$

So it remains only to prove the

THEOREM 6. *If c is strictly normal, so is also its coboundary $f = \delta\, c$.*

PROOF. The relation which we must prove is

$$\begin{aligned} &f(\sigma_1^{x_1}, \cdots, \sigma_1^{x_{r_1}}; \sigma_2^{y_1}, \cdots, \sigma_2^{y_{r_2}}, \cdots) \\ (17) \qquad &= f(\sigma_1, N_1; \cdots; \sigma_2, N_2, \cdots; \cdots)^{[x_1, \cdots, x_{r_1}; y_1, \cdots, y_{r_2}; \cdots]} \end{aligned}$$

Both sides of this can be represented in terms of c, and then, by its strict normality, by its "elementary" values $c(\sigma_1, N_1, \cdots; \sigma_2, N_2, \cdots)$. We then compare all the elementary values in both sides. The left-hand side is equal to

$$c(\sigma_1^{x_2}, \cdots, \sigma_1^{x_{r_1}}; \cdots)\, c(\sigma_1^{x_1+x_2}, \sigma_1^{x_3}, \cdots, \sigma_1^{x_{r_1}}; \cdots)^{-1} \cdots$$

$$\cdots c(\sigma_1^{x_1}, \cdots, \sigma_1^{x_{r_1}} \sigma_2^{y_1}, \cdots)^{\pm 1}\, c(\sigma_1^{x_1}, \cdots, \sigma_1^{x_{r_1}}, \sigma_2^{y_1+y_2}, \cdots)^{\mp 1} \cdots.$$

In the following lines we restrict our consideration to the cases $f(\sigma_1^a, \sigma_1^b; \sigma_2^c)$ and $f(\sigma_1^a, \sigma_1^b, \sigma_1^c; \sigma_2^d)$, for we can see easily that these cover essentially the general case, as will be seen by the representation (16).

To make the printing easier we shall also use the additive manner of writing for the group Ω^*, so we have

$$\begin{aligned} f(\sigma_1^a, \sigma_1^b; \sigma_2^c) &= c(\sigma_1^b, \sigma_2^c) - c(\sigma_1^{a+b}, \sigma_2^c) + c(\sigma_1^a, \sigma_1^b\sigma_2^c) - \sigma_2^c\, c(\sigma_1^a, \sigma_1^b) \\ &= [b; c]\, c(\sigma_1, \sigma_2) - [(a+b)_1; c]\, c(\sigma_1, \sigma_2) \\ &\qquad + \{c(\sigma_1^a, \sigma_1^b) + \sigma_1^b\, c(\sigma_1^a, \sigma_2^c)\} - \sigma_2^c\, c(\sigma_1^a, \sigma_1^b) \\ &= \{[b; c] - [(a+b)_1; c] + \sigma_1^b[a; c]\} c(\sigma_1, \sigma_2) \\ &\qquad + \{[a, b] - [a, b]\, \sigma_2^c\} c(\sigma_1, N_1)\,, \end{aligned}$$

where $(a+b)_1$ means the least non-negative residue of $a+b$ mod. e_1.

In a similar way we have

$$\begin{aligned} [a, b; c] f(\sigma_1, N_1; \sigma_2) &= [a, b; c]\, \{c(N_1, \sigma_2) - c(\sigma_1 N_1, \sigma_2) + c(\sigma_1, N_1\sigma_2) \\ &\qquad\qquad - \sigma_2\, c(\sigma_1, N_1)\} \\ &= [a, b; c]\, \{N_1 c(\sigma_1, \sigma_2) + c(\sigma_1, N_1) - c(\sigma_1, N_1)\}\,, \end{aligned}$$

so that it remains only to prove

$$[b; c] - [(a+b)_1; c] + \sigma_1^b[a; c] = [a, b; c] N_1$$

and

$$[a, b]\,(1 - \sigma_2^c) = [a, b; c] \times (1 - \sigma_2)\,.$$

Both of these identities are proved by a simple formal computation.

For the case of $f(\sigma_1^a, \sigma_1^b, \sigma_1^c; \sigma_2^d)$, we have first

$$\begin{aligned} & f(\sigma_1^a, \sigma_1^b, \sigma_1^c; \sigma_2^d) = c(\sigma_1^b, \sigma_1^c; \sigma_2^d) - c(\sigma_1^{a+b}, c_1^c, \sigma_2^d) \\ & + c(\sigma_1^a, \sigma_1^{b+c}, \sigma_2^d) - c(\sigma_1^a, \sigma_1^b, \sigma_1^c\sigma_2^d) + \sigma_2^d\, c(\sigma_1^a, \sigma_1^b, \sigma_1^c) \\ & = \{[b, c; d] - [(a+b)_1, c; d] + [a, (b+c)_1; d] \\ & - [a, b; d]\sigma_1^c\} c(\sigma_1, N_1, \sigma_2) + (\sigma_2^d - 1)\,[a, b, c]\, c(\sigma_1, N_1, \sigma_1)\,, \end{aligned}$$

and on the other hand

$$\begin{aligned} & [a, b, c; d] f(\sigma_1, N_1, \sigma_1; \sigma_2) = [a, b, c; d]\{c(N_1, \sigma_1, \sigma_2) - c(\sigma_1 N_1, \sigma_1, \sigma_2) \\ & + c(\sigma_1, N_1\sigma_1, \sigma_2) - c(\sigma_1, N_1, \sigma_1\sigma_2) + \sigma_2\, c(\sigma_1, N_1, \sigma_1)\} \\ & = [a, b, c; d]\, \Delta_1\, c(\sigma_1, N_1, \sigma_2) - [a, b, c; d]\, \Delta_2\, c(\sigma_1, N_1, \sigma_1)\,, \end{aligned}$$

and the equality in question will be proved by means of the simple identity

$$[b, c]-[(a+b)_1, c]+[a, (b+c)_1]-[a, b]=0$$

or

$$\left[\frac{b+c}{e_1}\right]-\left[\frac{(a+b)_1+c}{e_1}\right]+\left[\frac{a+(b+c)_1}{e_1}\right]-\left[\frac{a+b}{e_1}\right]=0$$

$$(0 \leqq a, b, c < e_1).$$

The last identity is proved without difficulty, if we distinguish all possible cases $b+c \leqq e_1$ or $b+c > e_1$ and so on, accordingly also the proof of our theorem is completed.

§ 7. In the preceding paragraphs we have proved the existence of strictly normal cocycles in each cohomology class. From this fact there arises naturally the question how we can characterize the "elementary" values $f(\sigma_1, N_1, \sigma_1, \cdots \sigma_2, N_2, \sigma_2, \cdots)$ of a strictly normal cocycle f, or how we can deduce the properties of cocycles from that elementary values. We consider now this problems.

THEOREM 7. *If f is a strictly normal cocycle, then for every set of natural numbers $i_1 < i_2 < \cdots < i_k$ and $r_1, r_2, \cdots, r_k$ with $r_1+r_2+\cdots+r_k = n+1$ we have*

$$(18) \qquad (-1)^{r_1}\varepsilon(i_1, r_1) f(\underbrace{\sigma_{i_1}, N_{i_1}, \cdots}_{r_1-1};\ \underbrace{\sigma_{i_2}, N_{i_2}, \cdots}_{r_2};\ \cdots)$$

$$+(-1)^{r_1+r_2}\varepsilon(i_1, r_2) f(\underbrace{\sigma_{i_1}, N_{i_1}, \cdots}_{r_1};\ \underbrace{\sigma_{i_2}, N_{i_2}, \cdots}_{r_2-1};\ \cdots)$$

$$\cdots\cdots\cdots\cdots$$

$$+(-1)^{r_1+r_2\cdots+r_k}\varepsilon(i_k, r_k) f(\underbrace{\sigma_{i_1}, N_{i_1}, \cdots}_{r_1};\underbrace{\quad}_{r_2};\ \underbrace{;\quad}_{r_k-1})=0.$$

Thereby we have set,

$$\varepsilon(i, r)=\begin{cases} N_i & \text{(for } r \text{ even)} \\ -\Delta_i & \text{(for } r \text{ odd).} \end{cases}$$

PROOF. We confine ourselves to give the proof by a special example:

$$
\begin{aligned}
0&=(\delta f)(\sigma_1, N_1, \sigma_1, \sigma_2, N_2)\\
&=(\delta f)(\sigma_1-1, N_1, \sigma_1-1, \sigma_2-1, N_2)\\
&=(\delta f)(\varDelta_1', N_1, \varDelta_1', \varDelta_2', N_2) \qquad\qquad (\varDelta_i'=\sigma_i-1)\\
&=(1-1)f(N_1, \varDelta_1', \varDelta_2', N_2)-f(\varDelta_1' N_1, \varDelta_1', \varDelta_2', N_2)\\
&\quad+f(\varDelta_1', N_1\varDelta_1', \varDelta_2', N_2)-f(\varDelta_1', N_1, \varDelta_1'\varDelta_2', N_2)\\
&\quad+f(\varDelta_1', N_1, \varDelta_1', \varDelta_2' N_2)-N_2 f(\varDelta_1', N_1, \varDelta_1', \varDelta_2')\\
&=-f(\varDelta_1', N_1, \varDelta_1'\varDelta_2', N_2)-N_2 f(\varDelta_1', N_1, \varDelta_1', \varDelta_2')\\
&=-\varDelta_1' f(\varDelta_1', N_1, \varDelta_2', N_2)-N_2 f(\varDelta_1', N_1, \varDelta_1', \varDelta_2')\\
&=-\varDelta_1' f(\sigma_1, N_1, \sigma_2, N_2)-N_2 f(\sigma_1, N_1, \sigma_1, \sigma_2)\,, \qquad \text{q. e. d.}
\end{aligned}
$$

The factor $(1-1)$ in the first term means, the second factor occurs once with positive sign and once with negative sign:

$$
\begin{aligned}
(\delta f)(\varDelta_1', N_1, \cdots)&=(\delta f)(\sigma_1, N_1, \cdots)-(\delta f)(1, N_1, \cdots)\\
&=\{f(N_1, \varDelta_1', \cdots)-\cdots\}\\
&\quad-\{f(N_1, \varDelta_1', \cdots)-\cdots\}\,.
\end{aligned}
$$

It will be often convenient, to use the abbreviations, such as $f(\sigma_1, N_1, \sigma_1, \sigma_2, N_2)=f(1,1,1,2,2)$, for which then holds the following theorem, which is the converse to theorem 7.

THEOREM 8. *If a system* $f(i_1, i_1, \cdots, i_2, i_2 \cdots)$ $(i_1<i_2<\cdots)$ *of elements in* Ω *satisfies the same condition* (18) *as in the theorem 7, then these elements induce a strictly normal cocycle.*

PROOF. We can associate to $f(i_1, i_1, \cdots i_2, i_2, \cdots)$ a strictly normal cocycle by the formulas similar to (17) and (15). Then as we have proved earlier, δf is also strictly normal, so that if the elementary values of δf vanish (and this condition is nothing other than (17) itself) then also δf vanishes identically. q. e. d.

§ 8. If $n=2$, all possible $r_1, r_2, \cdots, r_k$ with $r_1+r_2+\cdots+r_k=n+1=3$ are

$$3;\,1,2;\,2,1 \text{ and } 1,1,1$$

so that (18) reduces to

$$
(19)\qquad \begin{cases} \Delta_i f(i,i)=0\,, \\ \Delta_i f(j,j)-N_j f(i,j)=0\,, \\ N_i f(i,j)+\Delta_j f(i,i)=0\,, \\ \Delta_i f(j,k)-\Delta_j f(i,j)+\Delta_k f(i,j)=0\,, \qquad (i<j<k)\,. \end{cases}
$$

These are precisely those formulas to be found in Schreier's extension theory.

Similar formulas for $n=3$ are

$$
(20)\qquad \begin{cases} N_i f(i,i,i)=0 \\ \Delta_i f(j,j,j)-\Delta_j f(i,j,j)=0 \\ N_i f(i,j,j)+N_j f(i,i,j)=0 \\ \Delta_i f(i,i,j)-\Delta_j f(i,i,i)=0 \\ \Delta_i f(j,k,k)-\Delta_j f(i,k,k)+N_k f(i,j,k)=0 \\ \Delta_i f(j,j,k)-N_j f(i,j,k)-\Delta_k f(i,j,j)=0 \\ N_i f(i,j,k)+\Delta_j f(i,i,k)-\Delta_k f(i,i,j)=0 \\ \Delta_i f(j,k,l)-\Delta_j f(i,k,l)+\Delta_k f(i,j,l)-\Delta_l f(i,j,k)=0 \\ \qquad\qquad (i<j<k<l)\,. \end{cases}
$$

§ 9. We now prove two theorems, which are easily deduced from our preceding investigations.

THEOREM 9. *If K/k is an abelian field with the Galois group $G=\{\sigma_1\}\times\{\sigma_2\}$, then every element of K with relative norm 1 can be represented in the form $f(\sigma_1,\sigma_2)/f(\sigma_2,\sigma_1)$ with a suitable cocycle f.*

PROOF. By the relations (19) of Schreier we have

$$
(21)\qquad \begin{cases} \Delta_i f(i,i)=0 & (i=1,2) \\ \Delta_1 f(2,2)-N_2 f(1,2)=0 \\ N_1 f(1,2)+\Delta_2 f(1,1)=0\,, \end{cases}
$$

the last relation in (19) being meaningless under our assumption.

Let now A be an element of K with norm 1. Then the norm $N_1(A)$ can be represented in the form, additively written, $-\Delta_2 B_1$ with $\Delta_1 B=0$ by Hilbert's norm theorem, and by the same reason we have a representation of the form

$$
N_2(A)=\Delta_1 B_2 \qquad \text{with} \qquad \Delta_2 B_2=0\,.
$$

If we put

$$B_1=f(1,1), \qquad B_2=f(2,2), \qquad A=f(1,2),$$

we have our theorem for

$$A=f(1,2)-f(2,1) \qquad \bigl(f(2,1)=0\bigr).$$

THEOREM 10. *Under the same assumption as in the preceding theorem, we have the isomorphism*

$$H_3(K^*, G)\cong N_1(K_1^*)\cap N_2(K_2^*)/N(K^*).$$

Thereby K_i $(i=1,2)$ denote the fields corresponding to the subgroups $\{\sigma_i\}$ of G, and K^*, K_1^* and K_2^* mean multiplicative groups of non-zero elements in K, K_1 and K_2 respectively.

PROOF. The formulas (20) reduce to

$$\text{(22)} \qquad \begin{cases} N_i f(i,i,i)=0 & (i=1,2) \\ \Delta_1 f(1,1,1)-\Delta_2 f(1,2,2)=0 \\ N_1 f(1,2,2)+N_2 f(1,1,2)=0 \\ \Delta_1 f(1,1,2)-\Delta_2 f(1,1,1)=0. \end{cases}$$

By virtue of Hilbert's norm theorem we have

$$\text{(23)} \qquad f(i,i,i)=\Delta_i g_i \qquad (i=1,2).$$

If we put $g(i,i)=g_i$ and $g(1,2)=0$ and construct the coboundary of the strictly normal cochain generated by these values, we have

$$(\delta g)(\sigma_1, N_1, \sigma_1)=\Delta_1 g(1,1)$$
$$(\delta g)(\sigma_1, N_1, \sigma_2)=\Delta_2 g(1,1)+N_1 g(1,2)=\Delta_2 g(1,1)$$
$$(\delta g)(\sigma_1, \sigma_2, N_2)=\Delta_1 g(2,2)-N_2 g(1,2)=\Delta_1 g(2,2)$$
$$(\delta g)(\sigma_2, N_2, \sigma_2)=\Delta_2 g(2,2),$$

so that $f'=f-\delta g$ satisfies the conditions

$$f'(i,i,i)=0 \qquad (i=1,2),$$
$$\Delta_2 f'(1,2,2)=0 \qquad \bigl(\text{i. e. } f'(1,2,2)\in K_2\bigr),$$
$$N_1 f'(1,2,2)+N_2 f'(1,1,2)=0,$$
$$\Delta_1 f'(1,1,2)=0 \qquad \bigl(\text{i. e. } f'(1,1,2)\in K_1\bigr),$$

We now make correspond to the cohomology class containing f, the class

$$N_1 f'(1,2,2) \left(=-N_2 f'(1,1,2)\right) \quad \text{mod. } N(K^*)$$

in the factor group $N_1(K_1^*) \frown N_2(K_2^*)/N(K^*)$. It remains only to prove the isomorphism property of this mapping. Let us assume

$$N_1 f'(1,2,2) = NA \qquad (A \in K).$$

Then we have by Hilbert's norm theorem

$$f'(1,2,2) = \Delta_1 B_2 + N_2 A,$$

and likewise from the equality

$$-N_2 f'(1,1,2) = NA$$

we have

$$f'(1,1,2) = \Delta_2 B_1 - N_1 A.$$

If we put

$$g'(1,2) = -A, \qquad g'(1,1) = B_1, \qquad g'(2,2) = B_2,$$

then we have

$$f \sim f' = \delta g',$$

and this proves our assertion.

Mathematical Institute,
Tohoku University, Sendai

Bibliography

[1] T. Tannaka, Some remarks concerning p-adic number field, Jour. of Math. Soc. of Japan, 3 (1951), pp. 252-257.

[2] O. Schreier, Über die Erweiterung von Gruppen I, Monatsh. für Math. u. Phys., **34** (1926), pp. 165-180.

[3] R. C. Lyndon, The cohomology theory of group extensions, Duke Math. Journal, **15** (1948), pp. 271-292.

[4] S. Takahashi, Cohomology groups of finite abelian groups, Tohoku Math. Journ., **4** (1952), pp. 294-302.