Journal of the Mathematical Society of Japan Vol. 5, Nos. 3~4, November, 1953.

On the regularity of homeomorphisms of E^n.

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(Received Feb. 16, 1953)

Introduction. Let X be a compact metric space and h a homeomorphism of X onto itself. The homeomorphism h has been called by B. v. Kerékjártó [3][1] *regular* at $p \in X$, if h satisfies the following condition: for each $\epsilon > 0$ there exists $\delta > 0$ such that for each x with $d(p, x) < \delta$ and for each integer m

$$d\left(h^m(p), h^m(x)\right) < \epsilon .$$

One of the purpose of this paper is to prove the following

THEOREM 1. *Let X be a compact metric space and h a homeomorphism of X onto itself. Assume that X and h have the following property: there exist two distinct points a and b such that*

(i) *for each point $x \in X-b$ the sequence $\{h^m(x)\}$ converges to a and*

(ii) *for each point $x \in X-a$ the sequence $\{h^{-m}(x)\}$ converges to b, where $m=1, 2, 3, \cdots$.*

Then h is regular at every point of X except for a and b.

As a corollary of Theorem 1 we have the following

THEOREM 2. *Let h be a homeomorphism of the n-dimensional sphere S^n onto itself satisfying the same condition as that of Theorem 1. Then h is regular at every point of S^n except for a and b.*

Now let S^n be the n-dimensional sphere in the $(n+1)$-dimensional Euclidean space E^{n+1} and let P be a point of S^n. Let $p(x)$ be the stereographic projection of $S^n - P$ from P onto the n-dimensional Euclidean space E^n tangent at the antipode O of P, where we assume that O is the origin of E^n. Let h be a homeomorphism of E^n onto itself. Put $\bar{h}(x) = p^{-1}hp(x)$ where $x \in S^n - P$ and put $\bar{h}(P) = P$. Then we have a homeomorphism $\bar{h}$ of S^n onto itself. B. v. Kerékjártó [3] called a

1) The numbers in the brackets refer to the references at the end of this paper.

homeomorphism h of E^n onto itself *regular* at $p \in E^n$, if $\bar{h}$ is regular at $p^{-1}(p)$. By Theorem 2 we have immediately the following

THEOREM 3. *Let h be a homeomorphism of E^n onto itself satisfying the following conditions:*

(*i*) *for each $x \in E^n$ the sequence $\{h^m(x)\}$ converges to the origin O,*

(*ii*) *for each $x \in E^n$ except for O the sequence $\{h^{-m}(x)\}$ converges to the point at infinity ∞, where $m=1, 2, 3, \cdots$.*

Then h is regular at every point of E^n except for O.

If $n=2$, in virtue of a theorem of Kerékjártó [3], we have immediately the following

THEOREM 4. *Let h be a homeomorphism of the plane onto itself satisfying the same conditions as that of Theorem 3. If h is sense-preserving, then h is topologically equivalent to the transformation*

$$x' = \frac{1}{2} x, \qquad y' = \frac{1}{2} y,$$

and if h is sense-reversing, then h is topologically equivalent to the transformation

$$x' = \frac{1}{2} x, \qquad y' = -\frac{1}{2} y,$$

in Cartesian coordinates.

Since Theorem 2 follows immediately from Theorem 1, Theorem 3 immediately from Theorem 2, and Theorem 4 immediately from Theorem 3, we shall prove in this paper Theorem 1 only. To this purpose a notion of *bulging sequences* will be introduced in § 1. Then in § 2 Theorem 1 will be proved. In § 3 we shall give another application of bulging sequences in relation to the works of A. S. Besicovitch [1] [2].

§ 1. Bulging sequences.

Let A be a subset of a separable metric space X and let f be a continuous mapping of X into itself. A sequence $\{f^n(A)\}$ will be said to be a *bulging sequence*, if for each natural number n

$$f^n(A) - \bigcup_{i=0}^{n-1} f^i(A) \neq 0 .$$

LEMMA 1. *Let A be compact. If $\bigcup_{n=0}^{\infty} f^n(A)$ is not compact, then $\{f^n(A)\}$ is a bulging sequence.*

PROOF. Suppose on the contrary that $\{f^n(A)\}$ is not a bulging sequence and that there exists a natural number m such that

$$f^m(A) \subset A \cup f(A) \cup \cdots \cup f^{m-1}(A) .$$

Then it is easy to see that for each natural number i

$$f^{m+i}(A) \subset A \cup f(A) \cup \cdots \cup f^{m-1}(A) .$$

Therefore we have

$$(*) \qquad \bigcup_{n=0}^{\infty} f^n(A) = A \cup f(A) \cup \cdots \cup f^{m-1}(A) .$$

Since a continuous image of a compactum is compact and since a finite sum of compacta is also compact, the right hand side of (*) is compact, which is a contradiction.

LEMMA 2. *Let $\{f^n(A)\}$ be a bulging sequence and let*

$$C_n = A \cap f^{-n}\left(f^n(A) - \bigcup_{i=0}^{n-1} f^i(A)\right)$$

for every natural number n. Then $C_n \neq 0$ and $C_n \supset C_{n+1}$.

PROOF. First we prove that $C_n \neq 0$. Since $\{f^n(A)\}$ is a bulging sequence, there exists a point $p \in f^n(A) - \bigcup_{i=0}^{n-1} f^i(A)$. Then there exists a point $q \in A$ such that $f^n(q) = p$ and then $q \in A \cap f^{-n}\left(f^n(A) - \bigcup_{i=0}^{n-1} f^i(A)\right) = C_n$. Therefore $C_n \neq 0$.

Now we prove that $C_n \supset C_{n+1}$. Let x be a point of C_{n+1} and suppose that $x \notin C_n$. Then there exists an $m < n$ such that $f^n(x) \in f^m(A)$. Therefore $f^{n+1}(x) \in f^{m+1}(A)$, which contradicts $x \in C_{n+1}$.

LEMMA 3. *Let A be compact and let $\{f^n(A)\}$ be a bulging sequence. Then there exists a point $p \in A$ such that for each natural number n*

$$f^n(p) \cap \mathrm{Int}\,(A) = 0 .$$

PROOF. Let C_m be the same as in Lemma 2. Take $x_m \in C_m$. Since A is compact, there exists a subsequence $\{x_{m_i}\}$ which converges to a point $p \in A$. Then $\{f^n(x_{m_i})\}$ converges to $f^n(p)$ for every n. If $m_i > n$, then $f(x_{m_i}) \in f^n(C_{m_i}) \subset f^n(C_n)$ by Lemma 2. Since $f^n(C_n) \cap A = 0$ by the definition of C_n, $f^n(x_{m_i}) \cap A = 0$ for every $m_i > n$. Then we have $f^n(p) \cap \mathrm{Int}\,(A) = 0$ for every n, and the proof is complete.

§ 2. Proof of Theorem 1.

In § 2 we suppose that X is a non-degenerated compactum. Take two distinct points a and b of X and let φ be a continuous real-valued function on X such that

$$\begin{cases} -\frac{1}{2}\pi \leqq \varphi(x) \leqq \frac{1}{2}\pi & \text{for each } x \in X, \\ \varphi(x) = \frac{1}{2}\pi & \text{if and only if } x=a, \\ \varphi(x) = -\frac{1}{2}\pi & \text{if and only if } x=b. \end{cases}$$

The existence of such a function is obvious. Put

$$\psi(x) = \tan \varphi(x).$$

For each real number r put

$$A(r) = \{x \mid \psi(x) \geqq r\} \smile a, \qquad \text{and}$$

$$B(r) = \{x \mid \psi(x) \leqq r\} \smile b.$$

It is easy to see that

(i) $A(r)$ and $B(r)$ are compact,

(ii) if $r > r'$, then $\overline{A(r)} \subset A(r')$ and $B(r) \supset \overline{B(r')}$,

(iii) if r tends to $+\infty$, then $A(r)$ converges to a, and

(iv) if r tends to $-\infty$, then $B(r)$ converges to b.

Now we prove the following

LEMMA 4. *Let f be a continuous mapping of X into itself such that for each $x \in X - b$ the sequence $\{f^n(x)\}$ converges to a. Then $\bigcup_{n=0}^{\infty} f^n(A(r))$ is compact for every r.*

PROOF. Suppose on the contrary that $\bigcup_{n=0}^{\infty} f^n(A(r))$ is not compact. Then by Lemma 1 $\{f^n(A(r))\}$ is a bulging sequence. Therefore by Lemma 3 there exists a point $p \in A(r)$ such that for each n

$$f^n(p) \frown \operatorname{Int}(A(r)) = 0.$$

Then $\{f^n(p)\}$ does not converge to a, which is a contradiction.

Hereafter in § 2 we assume that a homeomorphism h of X onto itself satisfies the condition of Theorem 1. Then we have the following

LEMMA 5. *For each r the sequence $\{h^n(A(r))\}$ converges to a.*

PROOF. Since $\bigcup_{n=0}^{\infty}(A(r))$ is compact by Lemma 4, there exists a real number r_0 such that $\bigcup_{n=0}^{\infty} h^n(A(r)) \subset A(r_0)$. Take $x_n \in h^n(A(r))$. It is easy to see that if we prove that the sequence $\{x_n\}$ converges to a, then the proof of Lemma 5 is complete.

Since $x_n \in A(r_0)$, the set $\bigcup_{n=0}^{\infty} x_n$ has a limit point. Now we suppose that $\bigcup_{n=0}^{\infty} x_n$ has a limit point $p \in A(r_0)$ different from a. Then there exists a subsequence $\{x_{n_i}\}$ which converges to p. Then $\{h^{-m}(x_{n_i})\}$ converges to $h^{-m}(p)$ for every natural number m. Now put $y_{n_i}=h^{-n_i}(x_{n_i})$, then $y_{n_i} \in A(r)$. If $n_i > m$, then

$$h^{-m}(x_{n_i})=h^{-m}h^{n_i}(y_{n_i})=h^{n_i-m}(y_{n_i}) \in h^{n_i-m}(A(r)) \subset A(r_0)\,.$$

Therefore $h^{-m}(p) \in A(r_0)$ for every m. Then $\{h^{-m}(p)\}$ does not converge to b, which is a contradiction.

Similarly we have the following

LEMMA 6. *For each r the sequence $\{h^{-n}(B(r))\}$ converges to b.*

PROOF OF THEOREM 1. Let $p \in X-a-b$ and let ε be a given positive real number. Then there exist real numbers r_1 and r_2 such that

$$p \in \operatorname{Int}(A(r_1)) \quad \text{and} \quad p \in \operatorname{Int}(B(r_2))\,,$$

respectively. Put

$$U_1=\left\{x \mid d(a, x) < \frac{1}{2}\varepsilon\right\} \qquad \text{and}$$

$$U_2=\left\{x \mid d(b, x) < \frac{1}{2}\varepsilon\right\}.$$

By Lemma 5 and Lemma 6, there exist natural numbers n_1 and n_2 such that $h^n(A(r_1)) \subset U_1$ for every $n > n_1$ and that $h^{-n}(B(r_2)) \subset U_2$ for every $n > n_2$, respectively. Now let V_1 and V_2 be neighbourhoods of p such that $\delta(h^n(V_1)) < \varepsilon$ for every $0 \leqq n \leqq n_1$ and that $\delta(h^{-n}(V_2)) < \varepsilon$ for every $0 \leqq n \leqq n_2$, respectively. Take $\delta > 0$ such that

$$\{x \mid d(p, x) < \delta\} \subset V_1 \cap V_2 \cap \operatorname{Int}(A(r_1)) \cap \operatorname{Int}(B(r_2))\,.$$

Then it is easy to see that for each $x \in X$ with $d(p, x) < \delta$ and for each integer m

$$d\big(h^m(p), h^m(x)\big) < \varepsilon .$$

Therefore h is regular at every point of X except for a and b, and the proof is complete.

§ 3. Another application of bulging sequences.

Let X be a separable metric space and let f be a continuous mapping of X into itself. For each point $x \in X$ the set $\bigcup_{n=1}^{\infty} f^n(x)$ will be said to be a *positive half-orbit* of x. Let $P(f)$ be the set of points whose positive half-orbits are everywhere dense in X and put $Q(f) = X - P(f)$. It is easy to see that if $P(f) \neq 0$ then $P(f)$ is everywhere dense in X. Now we prove the following

THEOREM 5. *Let X be a locally compact, non compact, separable, metric space and let f be a continuous mapping of X into itself. Then $Q(f)$ is everywhere dense in X.*

PROOF. Suppose on the contrary that $Q(f)$ is not everywhere dense in X. Then there exist a point p and a neighbourhood U of p such that $Q(f) \frown U = 0$ $\big($i.e. $U \subset P(f)\big)$. Since X is locally compact, there exists a neighbourhood V of p with $\overline{V} \subset U$ such that $\overline{V}$ is compact.

Now we prove that $\{f^n(\overline{V})\}$ is a bulging sequence. In fact, if $\{f^n(\overline{V})\}$ is not a bulging sequence, then the set $W = \bigcup_{n=0}^{\infty} f^n(\overline{V})$ is compact by Lemma 1. Since $\overline{V} \subset U \subset P(f)$, $W = \overline{W} = X$ is compact, which is a contradiction. Therefore $\{f^n(\overline{V})\}$ is a bulging sequence.

Then by Lemma 3 there exists a point $q \in \overline{V}$ such that $f^n(q) \bar{\in} V$ for every natural number n. Therefore $q \in Q(f)$. Since $q \in \overline{V} \subset U$, we have $q \in P(f)$, which is also a contradiction, and the proof is complete.

COROLLARY. *Let f be a continuous mapping of E^n into itself. Then $Q(f)$, i.e. the set of points whose positive half-orbits are not everywhere dense in E^n, is everywhere dense in E^n.*

REMARK 1. A. S. Besicovitch [1] has shown that there exists a homeomorphism of the plane onto itself such that there exists a point whose positive half-orbit by this homeomorphism is everywhere dense on the plane. His statement that by this homeomorphism the positive half-orbit of every point of the plane except for the origin is everywhere dense on the plane is erroneous, as he has shown in his recent paper [2]. The fault of his assertion can also be seen by the above Corollary.

REMARK 2. If h is a homeomorphism of E^n onto itself, then the set $Q(f)$ will be seen to be an F_σ without difficulty.

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