Journal of the Mathematical Society of Japan Vol. 5, Nos. 3~4, November, 1953.

On Weierstrass-Stone's theorem.

By Shozo KOSHI

(Received Feb. 12, 1953)

Let Ω be a compact Hausdorff space and $C(\Omega)$ the ring of all real-valued continuous functions on Ω.

We define the norm for an element f of $C(\Omega)$ as

$$||f||=\sup_{x\in\Omega}|f(x)|,$$

then we have a Banach algebra $C(\Omega)$.

Weierstrass-Stone's theorem may be formulated as follows:

Let B be a subring in $C(\Omega)$ which has the following properties:

(1) *if $x_1, x_2, x_1\neq x_2$ are arbitrary elements of Ω, then we can find an element f in B such that $f(x_1)\neq f(x_2)$.*

(2) *B has the unit* 1.

Then B is norm-dense in $C(\Omega)$.

In this theorem, a point of Ω may be considered as a linear functional on $C(\Omega)$. So, we shall consider here generally by what kind of systems of linear functionals on $C(\Omega)$ the set Ω in (1) can be replaced.

DEFINITION. Let $C^*(\Omega)$ be the set of all linear functionals on $C(\Omega)$. A subsystem $\mathfrak{S}$ of $C^*(\Omega)$ is said to satisfy the condition of Weierstrass (shortly W-condition) if $\mathfrak{S}$ satisfies the following condition:

If B is an arbitrary subring of $C(\Omega)$ which contains the unit 1, and if for any two different elements φ, ψ of $\mathfrak{S}$ there exists an f in B such that $\varphi(f)\neq\psi(f)$, then B is norm-dense in $C(\Omega)$.

Clearly the totality of point-functionals satisfies this condition.

LEMMA 1. *Let F be a linear space and φ, ψ two linear functionals on F. If $\varphi(f)=0$ always implies $\psi(f)=0$, then we can find a real number a such that $\psi(f)=a\varphi(f)$.*

PROOF. Let f_0 be an element of F such that $\varphi(f_0)\neq 0$. If we can not find such an element, this lemma follows trivially.

Since $\varphi\{\varphi(f_0)f-\varphi(f)f_0\}=0$ $(f\in F)$, we have by assumption,

$$\psi\{\varphi(f_0)f-\varphi(f)f_0\}=0 \qquad (f\in F),$$

i.e.

$$\varphi(f_0)\psi(f)-\varphi(f)\psi(f_0)=0 \qquad (f\in F).$$

Thus we have $\psi(f)=a\varphi(f)\ (f\in F)$ putting $a=\psi(f_0)/\varphi(f_0)$.

THEOREM 1. *The system $\mathfrak{S}$ satisfies the W-condition, if and only if, for any different points x_1, $x_2\in\Omega$, we can find a real number $p\neq 0$ and φ, $\psi\in\mathfrak{S}$ such that*

$$\mu_{x_1}-\mu_{x_2}=p(\varphi-\psi),$$

where μ_x is the linear functional which corresponds to the point $x\in\Omega$.

PROOF. Sufficiency is a direct consequence of Weierstrass-Stone's theorem. Necessity: Given any two different points $x_1, x_2\in\Omega$, we set $S=\{f\mid(\mu_{x_1}-\mu_{x_2})f=0\}$. Clearly S is a subring with 1 which is not norm dense in $C(\Omega)$. Since $\mathfrak{S}$ satisfies the W-condition, we can find $\varphi, \psi\in\mathfrak{S}$, $\varphi\neq\psi$ such that $(\varphi-\psi)S=0$. By virtue of the above lemma we can find a $p\neq 0$ such that

$$\mu_{x_1}-\mu_{x_2}=p(\varphi-\psi), \qquad \text{q. e. d.}$$

REMARK: Let Ω be a completely regular topological space and $C(\Omega)$ be the set of all bounded, real-valued continuous functions on Ω. Then, by virtue of the theorem of Čech, we can find a compact Hausdorff space $\overline{\Omega}\supset\Omega$ such that $C(\overline{\Omega})\cong C(\Omega)$ where $\cong$ means an isomorphism as Banach algebra. Let $C^*(\Omega)$ be the conjugate space of $C(\Omega)$. Then an element of $\overline{\Omega}$ can be considered as a functional on $C(\Omega)$. A functional ξ is $\pm x$, $x\in\overline{\Omega}$ if and only if ξ is an extreme point of unit sphere of $C^*(\Omega)$.[1)]

In this case, we can consider W-condition for $C^*(\Omega)$ as before. Then we get the following result:

THEOREM 2. *$\mathfrak{S}$ in $C^*(\Omega)$ satisfies the W-condition if and only if for any different extreme points μ_1, μ_2 of unit sphere which satisfy $\mu_1(e)=1$. $\mu_2(e)=1$, (where e is constant 1 in $C(\Omega)$) there exist a real number p and $\varphi,\psi\in\mathfrak{S}$ such that*

$$\mu_1-\mu_2=p(\varphi-\psi).$$

Department of Mathematics, Hokkaido University

1) Arens-Kelley, Characterization of the space of continuous function over a compact Hausdorff space. Trans. Amer. Math. Soc. 62 (1947), pp. 499-508.