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On mixed boundary value problems for functions analytic in a simply-connected domain

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1. Formulation of problem.

In the preceding papers[1)] we have dealt with a mixed boundary value problem in potential theory. In case the unit circle laid on the z-plane is taken as the basic domain, the previous problem has been formulated as follows: To determine a function $u(z)$ harmonic and bounded in the unit circle $|z|<1$ and satisfying the boundary conditions

$$\begin{aligned} u(e^{i\varphi}) &= U_j(\varphi) \quad \text{for} \quad a_j<\varphi<b_j, \\ \frac{\partial u(e^{i\varphi})}{\partial \nu} &= V_j(\varphi) \quad \text{for} \quad b_j<\varphi<a_{j+1} \end{aligned} \qquad (j=1,\cdots,m),$$

a_{m+1} being supposed to be coincident with $a_1+2\pi$ and $\partial/\partial\nu\equiv\partial/\partial\nu_\varphi$ denoting the differentiation along the inward normal at $e^{i\varphi}$. The prescribed boundary functions $U_j(\varphi)$ and $V_j(\varphi)$ are supposed, for instance, continuous and bounded over their respective intervals of definition.

It has been shown that the solution of the problem is surely existent and uniquely determined and further that it can be represented by the integral formula

$$u(z)=\frac{1}{2\pi}\sum_{j=1}^{m}\left\{\int_{a_j}^{b_j} U_j(\varphi)\frac{\partial}{\partial\nu}\lg|\Phi(e^{i\varphi},z)|\,d\varphi \right.$$
$$\left. -\int_{b_j}^{a_{j+1}} V_j(\varphi)\lg|\Phi(e^{i\varphi},z)|\,d\varphi\right\}.$$

Here, $\Phi(\zeta,z)$ denotes the function mapping $|\zeta|<1$ schlicht and conformally onto the exterior of the unit circle cut along m radial slits starting orthogonally at points on the unit circumference in such a way that the arcs $|\zeta|=1$, $a_j<\arg\zeta\leq b_j$ $(j=1,\cdots,m)$ correspond as the whole to the unit circumference, while the arcs $|\zeta|=1$, $b_j<\arg\zeta<a_{j+1}$ $(j=1,\cdots,m)$ correspond to the radial slits, and it is further normalized at a preassigned parameter point $\zeta=z$ such as $(\zeta-z)\Phi(\zeta,z)\rightarrow 1$ for $\zeta\rightarrow z$.

In the present Note we shall deal with another mixed boundary value problem on analytic functions, which is closely related to the above-mentioned one on harmonic functions and has been once, together with the latter, discussed by A. Signorini.[2] *The problem is to determine a function*

$$f(z)=u(z)+i\tilde{u}(z)$$

analytic and bounded in the unit circle $|z|<1$ *and satisfying the boundary conditions*

$$\begin{array}{lll} u(e^{i\varphi})=U_j(\varphi) & \text{for} & a_j<\varphi<b_j, \\ \tilde{u}(e^{i\varphi})=\tilde{U}_j(\varphi) & \text{for} & b_j<\varphi<a_{j+1} \end{array} \qquad (j=1,\cdots,m),$$

a_{m+1} *coinciding here again with* $a_1+2\pi$ *and the boundary functions* $U_j(\varphi)$ *and* $\tilde{U}_j(\varphi)$ *being supposed, for instance, continuous and bounded in their respective intervals of definition;* $u(z)$ *and* $\tilde{u}(z)$ *represent, of course, the real and imaginary parts of* $f(z)$.

The close relation between the present problem and the former one will readily be suggested by a heuristic consideration. Namely, if, in particular, the solution $f(z)$ of the present problem remains regular even on the boundary $|z|=1$, then a Cauchy-Riemann equation implies

$$\frac{\partial u(e^{i\varphi})}{\partial\nu}=-\frac{\partial\tilde{u}(e^{i\varphi})}{\partial\varphi}=-\tilde{U}_j'(\varphi) \text{ for } b_j<\varphi<a_{j+1} \quad (j=1,\cdots,m).$$

The harmonic function $u(z)=\Re f(z)$ is, therefore, the solution of the former problem in which the boundary functions $V_j(\varphi)$ $(j=1,\cdots,m)$ are replaced by the $-\tilde{U}_j'(\varphi)$, respectively. Conversely, denoting by $\tilde{u}(z)$ a harmonic function conjugate to the solution $u(z)$ of the former problem, the function defined by $f(z)=u(z)+i\tilde{u}(z)$ is then the solution of the present problem in which the boundary functions $\tilde{U}_j(\varphi)$ $(j=1,\cdots,m)$ are replaced by the $\tilde{U}_j(b_j)-\int_{b_j}^{\varphi}V_j(\psi)d\psi$, respectively, the $\tilde{U}_j(b_j)$ $(j=1,\cdots,m)$ being the constants to be determined suitably.

On the other hand, in connection with the last-mentioned fact, we are now in position to emphasize that there exists an essential difference between two problems. In fact, the boundary functions $U_j(\varphi)$ and $V_j(\varphi)$ $(j=1,\cdots,m)$ of the former problem can be prescribed quite independently, i.e., with no functional restriction among them, while the boundary functions $U_j(\varphi)$ and $\tilde{U}_j(\varphi)$ $(j=1,\cdots,m)$ of the present problem must be subjected to certain relations, provided m is greater than 1, in order that the existence of a solution is assured. In fact, as already pointed out by A. Signorini and also shown in the following lines, certain $m-1$ definite relations are necessary and sufficient for the existence of a solution.

Although A. Signorini has once discussed the present problem in detail, it seems his results are not yet of completely explicit nature. Accordingly, it will not be quite useless to attempt here again deriving an explicit integral formula for the solution more concretely in terms of the familiar canonical mapping functions.

The simpler cases of the former problem where there exist merely one or two pairs of the boundary arcs bearing alternately the boundary values of the desired function itself and of its normal derivative have been treated in the preceding paper in particular details. The corresponding cases of the present problem will be also illustrated in details by expressing the kernels contained in the general formula concretely in terms of elementary or elliptic functions.

We shall confine ourselves throughout the present Note to the case where the prescribed boundary functions are continuous and bounded in their respective intervals of definition. However, our integral representation of the solution, which will be derived in the following lines under these assumptions, defines surely a definite analytic function regular in $|z|<1$, provided the boundary functions are merely supposed integrable and of order $o(|\varphi-c|^{-1/2})$ at every junction $\varphi=c$ of the adjacent arcs. Consequently, the converse problem will arise successively. Though we shall omit the precise discussion of the last problem, it will readily be verified that the function thus defined satisfies, in general, the boundary conditions almost everywhere and further that it satisfies them surely at every continuity point of the boundary functions.

Finally, it would especially be noticed that *our problem remains invariant under any conformal mapping*. Namely, if the basic domain

$|z|<1$ is mapped, by means of an analytic function $z=z(\hat{z})$, onto a domain $\hat{D}$ laid on the $\hat{z}$-plane, then the solution $f(z)$ of the original problem is transferred into the function $\hat{f}(\hat{z})\equiv f\bigl(z(\hat{z})\bigr)$ analytic and bounded in $\hat{D}$ and satisfying the boundary conditions

$$\begin{aligned} &\Re\hat{f}(\hat{z})=U_j(\varphi) \quad \text{for} \quad z(\hat{z})=e^{i\varphi}, \quad a_j<\varphi<b_j, \\ &\Im\hat{f}(\hat{z})=\tilde{U}_j(\varphi) \quad \text{for} \quad z(\hat{z})=e^{i\varphi}, \quad b_j<\varphi<a_{j+1} \end{aligned} \qquad (j=1,\cdots,m).$$

Thus, it depends on a mere convenience to restrict the basic domain to the unit circle, and the result covers the case of any simply connected basic domains too.

2. Construction of solution.

It may previously be noted that the *unicity assertion for the solution* of our present problem is quite evident, provided once its existence is established. In fact, we have only to remember that, in general, a function $f_0(z)=u_0(z)+i\tilde{u}_0(z)$ analytic and bounded in $|z|<1$ and satisfying the boundary conditions

$$\begin{aligned} &u_0(e^{i\varphi})=0 \quad \text{for} \quad a_j<\varphi<b_j, \\ &\tilde{u}_0(e^{i\varphi})=0 \quad \text{for} \quad b_j<\varphi<a_{j+1} \end{aligned} \qquad (j=1,\cdots,m)$$

must reduce to the constant 0, a fact which is quite obvious.

For a later use, we notice further that the apparently weaker boundary conditions

$$\begin{aligned} &u_0(e^{i\varphi})=\varepsilon_j \quad \text{for} \quad a_j<\varphi<b_j, \\ &\tilde{u}_0(e^{i\varphi})=0 \quad \text{for} \quad b_j<\varphi<a_{j+1} \end{aligned} \qquad (j=1,\cdots,m),$$

the ε_j being constant, imply also the constancy of the function $f_0(z)$, a fact which can be readily verified, based on the assertion that any point set laid on m straight lines parallel to the imaginary axis or on the real axis alone cannot be the boundary of a bounded domain.

We first propose to construct an integral representation for the solution under the assumption that it exists at any rate. The conditions for its existence will be spontaneously obtained during our following procedure.

Let now $\omega^{(1)}(z, \varphi)$ denote the harmonic measure of the boundary part consisting of (both banks of) the arcs

$$|\zeta|=1, \quad a_j \leqq \arg\zeta \leqq b_j \ (j=1,\cdots,\kappa-1) \text{ and } a_\kappa \leqq \arg\zeta \leqq \varphi,$$

for any parametric value φ with $a_\kappa < \varphi \leqq b_\kappa$, with respect to the fixed m-ply connected slit domain laid on the ζ-plane and bounded by m circular slits

$$|\zeta|=1, \qquad a_j \leqq \varphi \leqq b_j \qquad (j=1,\cdots,m).$$

Let $\tilde{\omega}^{(1)}(z, \varphi)$ be a harmonic function conjugate to $\omega^{(1)}(z, \varphi)$; for instance, we may put

$$\tilde{\omega}^{(1)}(z, \varphi) = \int_0^z \left(\frac{\partial \omega^{(1)}(z, \varphi)}{\partial x} dy - \frac{\partial \omega^{(1)}(z, \varphi)}{\partial y} dx \right), \qquad z=x+iy.$$

Consider then an analytic function defined by

$$f^{(1)}(z) \equiv u^{(1)}(z) + i\tilde{u}^{(1)}(z) = \sum_{j=1}^m \int_{a_j}^{b_j} U_j(\varphi) d\left(\omega^{(1)}(z, \varphi) + i\tilde{\omega}^{(1)}(z, \varphi)\right).$$

It is regular and bounded in $|z|<1$ and satisfies the boundary conditions

$$\left.\begin{aligned} &u^{(1)}(e^{i\varphi}) = U_j(\varphi) \quad &&\text{for} \quad a_j < \varphi < b_j, \\ &\frac{\partial u^{(1)}(e^{i\varphi})}{\partial \nu} = 0 &&\text{for} \quad b_j < \varphi < a_{j+1} \end{aligned}\right\} \qquad (j=1,\cdots,m).$$

The latter relations may be verified, for instance, by remembering the symmetry property of the harmonic measure $\omega^{(1)}$, i. e.

$$\omega^{(1)}(1/\bar{z}, \psi) \equiv \omega^{(1)}(z, \psi),$$

an identity which implies immediately

$$\frac{\partial \omega^{(1)}(e^{i\varphi}, \psi)}{\partial \nu} = 0 \quad \text{for} \quad b_j < \varphi < a_{j+1} \qquad (j=1,\cdots,m).$$

These relations further yield that $\tilde{u}^{(1)}(e^{i\varphi})$ remains constant for every value of φ with $b_j < \varphi < a_{j+1}$. Consequently, we put

$$\tilde{u}^{(1)}(e^{i\varphi}) = \alpha_j \quad \text{for} \quad b_j < \varphi < a_{j+1} \quad (j=1,\cdots,m).$$

Next, let $\omega^{(2)}(z, \varphi)$ denote the harmonic measure of the boundary part consisting of (both banks of) the arcs

$$|\zeta|=1, \quad b_j \leq \arg\zeta \leq a_{j+1} \ (j=1,\cdots,\kappa-1) \text{ and } b_\kappa \leq \arg\zeta \leq \varphi ,$$

for any parametric value φ with $b_\kappa < \varphi \leq a_{\kappa+1}$, with respect to the fixed m-ply connected slit domain laid on the ζ-plane bounded by m circular slits

$$|\zeta|=1, \quad b_j \leq \arg\zeta \leq a_{j+1} \quad (j=1,\cdots,m).$$

Let $\tilde{\omega}^{(2)}(z, \varphi)$ be a harmonic function conjugate to $\omega^{(2)}(z, \varphi)$; it is defined, for instance, by

$$\tilde{\omega}^{(2)}(z, \varphi)=\int_0^z \left(\frac{\partial \omega^{(2)}(z, \varphi)}{\partial x}\, dy - \frac{\partial \omega^{(2)}(z, \varphi)}{\partial y}\, dx \right), \qquad z=x+iy .$$

Consider then an analytic function defined by

$$\begin{aligned} f^{(2)}(z) &= \tilde{u}^{(2)}(z)-iu^{(2)}(z) \\ &= \sum_{j=1}^{m} \int_{b_j}^{a_{j+1}} \left(\tilde{U}_j(\varphi)-\alpha_j\right) d\left(\omega^{(2)}(z, \varphi)+i\tilde{\omega}^{(2)}(z, \varphi)\right) . \end{aligned}$$

It is regular and bounded in $|z|<1$ and satisfies the boundary conditions

$$\left.\begin{aligned} &\tilde{u}^{(2)}(e^{i\varphi})=\tilde{U}_j(\varphi)-\alpha_j && \text{for} \quad b_j<\varphi<a_{j+1} , \\ &\frac{\partial \tilde{u}^{(2)}(e^{i\varphi})}{\partial \nu}=0 && \text{for} \quad a_j<\varphi<b_j \end{aligned}\right\} \quad (j=1,\cdots,m).$$

Thus, $u^{(2)}(e^{i\varphi})$ remaining constant for every value of φ with $a_j<\varphi<b_j$, we put

$$u^{(2)}(e^{i\varphi})=-\beta_j \quad \text{for} \quad a_j<\varphi<b_j \qquad (j=1,\cdots,m).$$

We then consider the analytic function defined by

$$f(z)=u(z)+i\tilde{u}(z)=f^{(1)}(z)+if^{(2)}(z)+\beta_1 .$$

Its real and imaginary parts being then given by

$$u(z)=u^{(1)}(z)+u^{(2)}(z)+\beta_1 ,$$

$$\tilde{u}(z)=\tilde{u}^{(1)}(z)+\tilde{u}^{(2)}(z) ,$$

respectively, they satisfy the boundary conditions

$$u(e^{i\varphi})=U_j(\varphi)-\beta_j+\beta_1 \quad \text{for} \quad a_j<\varphi<b_j, \qquad (j=1,\cdots,m).$$
$$\tilde{u}(e^{i\varphi})=\tilde{U}_j(\varphi) \quad \text{for} \quad b_j<\varphi<a_{j+1}$$

In view of the unicity assertion announced above, it is readily concluded that *a necessary and sufficient condition for the existence of a solution is expressed by* $m-1$ *relations*

$$\beta_1=\beta_2=\cdots=\beta_m;$$

the solution is then uniquely determined.

In the following lines we confine ourselves merely to the case where the last-mentioned condition for existence is surely fulfilled. The solution is then represented in its full form by

$$f(z)=\sum_{j=1}^{m}\left\{\int_{a_j}^{b_j}U_j(\varphi)\,d\left(\omega^{(1)}(z,\varphi)+i\tilde{\omega}^{(1)}(z,\varphi)\right)\right.$$
$$\left.+i\int_{b_j}^{a_{j+1}}\left(\tilde{U}_j(\varphi)-\alpha_j\right)d\left(\omega^{(2)}(z,\varphi)+i\tilde{\omega}^{(2)}(z,\varphi)\right)\right\}+\beta_1,$$

the constants α_j and β_1 being given by

$$\alpha_j=\sum_{\kappa=1}^{m}\int_{a_\kappa}^{b_\kappa}U_\kappa(\varphi)\,d\tilde{\omega}^{(1)}(e^{i(b_j+0)},\varphi),$$

$$\beta_j=\sum_{\kappa=1}^{m}\int_{b_\kappa}^{a_{\kappa+1}}\left(\tilde{U}_\kappa(\varphi)-\alpha_\kappa\right)d\tilde{\omega}^{(2)}(e^{i(a_j+0)},\varphi).$$

We make here, by the way, a preparation, in order that we shall bring later the solution just obtained into another clear form. Consider now the problem to determine a function

$$g(z)=v(z)+i\tilde{v}(z)$$

analytic and bounded in $|z|<1$ and satisfying the boundary conditions

$$v(e^{i\varphi})=\tilde{U}_j(\varphi) \quad \text{for} \quad b_j<\varphi<a_{j+1}, \qquad (j=1,\cdots,m).$$
$$\tilde{v}(e^{i\varphi})=-U_j(\varphi) \quad \text{for} \quad a_j<\varphi<b_j$$

Again in virtue of the unicity assertion, the solution is evidently given by

$$g(z)=-if(z).$$

Hence, applying the above result to $g(z)$, we may write the solution of the original problem also in the form

$$f(z)=\sum_{j=1}^{m}\left\{\int_{a_j}^{b_j}\big(U_j(\varphi)+\gamma_j\big)\,d\big(\omega^{(1)}(z,\varphi)+i\tilde{\omega}^{(1)}(z,\varphi)\big)\right.$$
$$\left.+i\int_{b_j}^{a_{j+1}}\tilde{U}_j(\varphi)\,d\big(\omega^{(2)}(z,\varphi)+i\tilde{\omega}^{(2)}(z,\varphi)\big)\right\}+i\delta_1,$$

the constants γ_j and δ_1 being given by

$$\gamma_j=\sum_{\kappa=1}^{m}\int_{b_\kappa}^{a_{\kappa+1}}\tilde{U}_\kappa(\varphi)\,d\tilde{\omega}^{(2)}(e^{i(a_j+0)},\varphi),$$

$$\delta_j=\sum_{\kappa=1}^{m}\int_{a_\kappa}^{b_\kappa}\big(-U_\kappa(\varphi)-\gamma_\kappa\big)\,d\tilde{\omega}^{(1)}(e^{i(b_j+0)},\varphi).$$

A necessary and sufficient condition for the existence of a solution may also be expressed in the form

$$\delta_1=\delta_2=\cdots=\delta_m.$$

Comparison of both representations derived above for $f(z)$ implies now immediately the relations

$$\sum_{j=1}^{m}\gamma_j\int_{a_j}^{b_j}d\omega^{(1)}(z,\varphi)=\sum_{j=1}^{m}\alpha_j\int_{b_j}^{a_{j+1}}d\tilde{\omega}^{(2)}(z,\varphi)+\beta_1,$$

$$\sum_{j=1}^{m}\gamma_j\int_{a_j}^{b_j}d\tilde{\omega}^{(1)}(z,\varphi)+\delta_1=-\sum_{j=1}^{m}\alpha_j\int_{b_j}^{a_{j+1}}d\omega^{(2)}(z,\varphi).$$

Thus, the solution is also representable in the form

$$f(z)=\sum_{j=1}^{m}\left\{\int_{a_j}^{b_j}U_j(\varphi)\,d\big(\omega^{(1)}(z,\varphi)+i\tilde{\omega}^{(1)}(z,\varphi)\big)+\gamma_j\int_{a_j}^{b_j}d\omega^{(1)}(z,\varphi)\right.$$
$$\left.+i\int_{b_j}^{a_{j+1}}\tilde{U}_j(\varphi)\,d\big(\omega^{(2)}(z,\varphi)+i\tilde{\omega}^{(2)}(z,\varphi)\big)-i\alpha_j\int_{b_j}^{a_{j+1}}d\omega^{(2)}(z,\varphi)\right\}.$$

3. Another expression for the solution.

We now propose to bring the expression of the solution derived just above into another more clear form in terms of the functions mapping the basic domain onto certain canonical domains. The main task is to establish the connections of the harmonic measures and their harmonic conjugates availed above with such canonical maps.

At the beginning of the present paper we have introduced a canonical mapping function $\Phi(\zeta, z)$. Together with it, we further introduce another function $\Psi(\zeta, z)$ mapping $|\zeta|<1$ schlicht and conformally onto the exterior of the unit circle cut along m radial slits starting orthogonally at points on the unit circumference in such a way that the arcs $|\zeta|=1,\ b_j<\arg\zeta\leqq a_{j+1}$ $(j=1,\cdots,m)$ correspond as the whole to the unit circumference, while the arcs $|\zeta|=1,\ a_j<\arg\zeta<b_j$ $(j=1,\cdots,m)$ correspond to the radial slits, and it is moreover normalized at a parameter point $\zeta=z$ such as $(\zeta-z)\Psi(\zeta,z)\to 1$ for $\zeta\to z$.

Both functions $\Phi(\zeta, z)$ and $\Psi(\zeta, z)$ can be prolonged analytically, by means of the inversion principle, beyond the arcs $|\zeta|=1,\ a_j<\arg\zeta<b_j$ and $|\zeta|=1,\ b_j<\arg\zeta<a_{j+1}$, respectively, the defining equations for prolongation being, of course, given by

$$\Phi(1/\bar{\zeta}, z)=1/\overline{\Phi(\zeta, z)}, \qquad \Psi(1/\bar{\zeta}, z)=1/\overline{\Psi(\zeta, z)}.$$

Consequently, *the function $\Phi(\zeta, z)$ or $\Psi(\zeta, z)$ may also be characterized as the one which maps the whole plane cut along m circular slits $|\zeta|=1,\ b_j\leqq\arg\zeta\leqq a_{j+1}$ $(j=1,\cdots,m)$ or along m circular slits $|\zeta|=1,$ $a_j\leqq\arg\zeta\leqq b_j$ $(j=1,\cdots,m)$, respectively, onto the whole plane cut along m radial slits centred at the origin in such a manner that the points $\zeta=z$ and $\zeta=1/\bar{z}$ correspond, in either case, to the point at infinity and the origin, respectively, and further that the same normalization at $\zeta=z$, as stated above, is preassigned.* It will also be evident that the functions thus prolonged satisfy the further functional equations

$$\overline{\Phi(1/\bar{\zeta}, 1/\bar{z})}=-z^2\Phi(\zeta, z), \qquad \overline{\Psi(1/\bar{\zeta}, 1/\bar{z})}=-z^2\Psi(\zeta, z).$$

We are now in position to state the **main theorem** of the present Note:

The solution of our mixed boundary value problem is represented by the integral formula

$$f(z)=\frac{1}{2\pi}\sum_{j=1}^{m}\left\{\int_{a_j}^{b_j}U_j(\varphi)\,d\left(-\arg\Phi(e^{i\varphi},z)+i\lg|\Psi(e^{i\varphi},z)|\right)\right.$$
$$\left.+i\int_{b_j}^{a_{j+1}}\tilde{U}_j(\varphi)\,d\left(-\arg\Psi(e^{i\varphi},z)+i\lg|\Phi(e^{i\varphi},z)|\right)\right\},$$

provided the condition for the existence of the solution is fulfilled.

We begin with considering the harmonic measure $\omega^{(1)}(z,\varphi)$. It satisfies, as a function of $z=re^{i\theta}$ harmonic in $|z|<1$, the boundary conditions

$$\omega^{(1)}(e^{i\theta},\varphi)=\begin{cases}1 & \text{for } a_j<\theta<b_j\ (j=1,\cdots 1,\kappa-1) \text{ and } a_\kappa<\theta<\varphi,\\ 0 & \text{for } \varphi<\theta<b_\kappa \text{ and } a_j<\theta<b_j\ (j=\kappa+1,\cdots,m),\end{cases}$$

$$\frac{\partial\omega^{(1)}(e^{i\theta},\varphi)}{\partial\nu}=0 \quad \text{for } b_j<\theta<a_{j+1}\ (j=1,\cdots,m)$$

for any value of φ with $a_\kappa<\varphi<b_\kappa$, $\partial/\partial\nu=\partial/\partial\nu_0$ denoting the differentiation along the inward normal at $e^{i\theta}$. Hence, applying the formula referred to at the beginning of the present paper which remains valid for the boundary functions with a finite number of jumps, we get, for $a_\kappa<\varphi<b_\kappa$,

$$\omega^{(1)}(z,\varphi)=\frac{1}{2\pi}\sum_{j=1}^{\kappa-1}\int_{a_j}^{b_j}\frac{\partial}{\partial\nu}\lg|\Phi(e^{i\theta},z)|\,d\theta+\frac{1}{2\pi}\int_{a_\kappa}^{\varphi}\frac{\partial}{\partial\nu}\lg|\Phi(e^{i\theta},z)|\,d\theta$$
$$=-\frac{1}{2\pi}\sum_{j=1}^{\kappa-1}\int_{a_j}^{b_j}d\arg\Phi(e^{i\theta},z)-\frac{1}{2\pi}\int_{a_\kappa}^{\varphi}d\arg\Phi(e^{i\theta},z).$$

In view of an evident relation $\Phi(e^{ib_j},z)=\Phi(e^{ia_{j+1}},z)$, it further becomes

$$\omega^{(1)}(z,\varphi)=-\frac{1}{2\pi}\arg\frac{\Phi(e^{i\varphi},z)}{\Phi(e^{ia_1},z)},$$

a relation which connects $\omega^{(1)}(z,\varphi)$ with $\Phi(e^{i\varphi},z)$ and is independent of κ.

Quite similarly, we obtain a corresponding connection

$$\omega^{(2)}(z,\varphi)=-\frac{1}{2\pi}\arg\frac{\Psi(e^{i\varphi},z)}{\Psi(e^{ib_1},z)}.$$

Our next step is to investigate a harmonic conjugate of the harmonic measure. For that purpose, we consider, for a while, any analytic function regular in the closed unit circle. Let it be $f^*(z)=u^*(z)+i\tilde{u}^*(z)$, and put

$$\begin{aligned} u^*(e^{i\varphi}) &= U_j^*(\varphi) \quad \text{for} \quad a_j<\varphi<b_j, \\ \tilde{u}^*(e^{i\varphi}) &= \tilde{U}_j^*(\varphi) \quad \text{for} \quad b_j<\varphi<a_{j+1} \end{aligned} \qquad (j=1,\cdots,m).$$

Since $f^*(z)$ is supposed regular even along the unit circumference, there holds a Cauchy-Riemann relation

$$\frac{\partial u^*(e^{i\varphi})}{\partial \nu} = -\frac{\partial \tilde{u}^*(e^{i\varphi})}{\partial \varphi} = -\tilde{U}_j^{*\prime}(\varphi) \quad \text{for } b_j<\varphi<a_{j+1} \qquad (j=1,\cdots,m).$$

The integral formula for the solution of the former mixed boundary value problem implies thus

$$\begin{aligned} u^*(z) &= \frac{1}{2\pi}\sum_{j=1}^{m}\left\{\int_{a_j}^{b_j} U_j^*(\varphi)\frac{\partial}{\partial\nu}\lg|\Phi(e^{i\varphi},z)|\,d\varphi \right. \\ &\qquad \left. +\int_{b_j}^{a_{j+1}}\tilde{U}_j^{*\prime}(\varphi)\lg|\Phi(e^{i\varphi},z)|\,d\varphi\right\} \\ &= -\frac{1}{2\pi}\sum_{j=1}^{m}\left\{\int_{a_j}^{b_j} U_j^*(\varphi)\,d\arg\Phi(e^{i\varphi},z)+\int_{b_j}^{a_{j+1}}\tilde{U}_j^*(\varphi)\,d\lg|\Phi(e^{i\varphi},z)|\right\}. \end{aligned}$$

On the other hand, applying the general formula derived in the last section, we have

$$\begin{aligned} u^*(z) &= \Re f^*(z) \\ &= \sum_{j=1}^{m}\int_{a_j}^{b_j}\left(U_j^*(\varphi)+\gamma_j^*\right)d\omega^{(1)}(z,\varphi)-\int_{b_j}^{a_{j+1}}\tilde{U}_j^*(\varphi)\,d\tilde{\omega}^{(2)}(z,\varphi)\Big\}, \end{aligned}$$

the constants γ_j^* being defined by

$$\gamma_j^* = \sum_{\kappa=1}^{m}\int_{b_\kappa}^{a_{\kappa+1}}\tilde{U}_\kappa^*(\varphi)\,d\tilde{\omega}^{(2)}(e^{i(a_j+0)},\varphi).$$

By virtue of the relation already established between $\omega^{(1)}$ and Φ, the last equation for $u^*(z)$ becomes

$$u^*(z)=-\sum_{j=1}^{m}\left\{\frac{1}{2\pi}\int_{a_j}^{b_j}U_j^*(\varphi)\,d\arg\Phi(e^{i\varphi},z)+\int_{b_j}^{a_{j+1}}\tilde{U}_j^*(\varphi)\,d\tilde{\omega}^{(2)}(z,\varphi)\right\}$$

$$-\sum_{j=1}^{m}\frac{1}{2\pi}\arg\frac{\Phi(e^{ib_j},z)}{\Phi(e^{ia_j},z)}\sum_{\kappa=1}^{m}\int_{b_\kappa}^{a_{\kappa+1}}\tilde{U}_\kappa^*(\varphi)\,d\tilde{\omega}^{(2)}(e^{i(a_j+0)},\varphi)$$

$$=-\frac{1}{2\pi}\sum_{j=1}^{m}\left\{\int_{a_j}^{b_j}U_j^*(\varphi)\,d\arg\Phi(e^{i\varphi},z)+\int_{b_j}^{a_{j+1}}\tilde{U}_j^*(\varphi)\,d\Big(2\pi\tilde{\omega}^{(2)}(z,\varphi)\right.$$

$$\left.+\sum_{\kappa=1}^{m}\tilde{\omega}^{(2)}(e^{i(a_\kappa+0)},\varphi)\arg\frac{\Phi(e^{ib_\kappa},z)}{\Phi(e^{ia_\kappa},z)}\Big)\right\}.$$

Comparing this expression with the previous one for $u^*(z)$, we obtain

$$0=\sum_{j=1}^{m}\int_{b_j}^{a_{j+1}}\tilde{U}_j^*(\varphi)\,d\Big(\tilde{\omega}^{(2)}(z,\varphi)$$

$$+\frac{1}{2\pi}\sum_{\kappa=1}^{m}\tilde{\omega}^{(2)}(e^{i(a_\kappa+0)},\varphi)\arg\frac{\Phi(e^{ib_\kappa},z)}{\Phi(e^{ia_\kappa},z)}-\frac{1}{2\pi}\lg|\Phi(e^{i\varphi},z)|\Big)$$

$$=\sum_{j=1}^{m}\left\{\Big[\tilde{U}_j^*(\varphi)\Big(\tilde{\omega}^{(2)}(z,\varphi)+\frac{1}{2\pi}\sum_{\kappa=1}^{m}\tilde{\omega}^{(2)}(e^{i(a_\kappa+0)},\varphi)\arg\frac{\Phi(e^{ib_\kappa},z)}{\Phi(e^{ia_\kappa},z)}\right.$$

$$-\frac{1}{2\pi}\lg|\Phi(e^{i\varphi},z)|\Big)\Big]_{\varphi=b_j}^{a_{j+1}}-\int_{b_j}^{a_{j+1}}\Big(\tilde{\omega}^{(2)}(z,\varphi)$$

$$\left.-\frac{1}{2\pi}\sum_{\kappa=1}^{m}\tilde{\omega}^{(2)}(e^{i(a_\kappa+0)},\varphi)\arg\frac{\Phi(e^{ib_\kappa},z)}{\Phi(e^{ia_\kappa},z)}-\frac{1}{2\pi}\lg|\Phi(e^{i\varphi},z)|\Big)d\tilde{U}_j^*(\varphi)\right\}.$$

Since $\tilde{U}_j^*(\varphi)$ may be chosen here as a boundary function along $a_j<\varphi<b_j$ of any function $u^*(z)$ regular harmonic in $|z|\leq 1$, the fundamental lemma in the calculus of variations in its slightly extended form implies thus the identity

$$\tilde{\omega}^{(2)}(z,\varphi)+\frac{1}{2\pi}\sum_{\kappa=1}^{m}\tilde{\omega}^{(2)}(e^{i(a_\kappa+0)},\varphi)\arg\frac{\Phi(e^{ib_\kappa},z)}{\Phi(e^{ia_\kappa},z)}=\frac{1}{2\pi}\lg|\Phi(e^{i\varphi},z)|.$$

Quite similarly, we conclude a corresponding relation

$$\tilde{\omega}^{(1)}(z,\varphi)+\frac{1}{2\pi}\sum_{\kappa=1}^{m}\tilde{\omega}^{(1)}(e^{i(b_\kappa+0)},\varphi)\arg\frac{\Psi(e^{ia_{\kappa+1}},z)}{\Psi(e^{ib_\kappa},z)}=\frac{1}{2\pi}\lg|\Psi(e^{i\varphi},z)|.$$

The differentials with respect to φ of the last two equations, after multiplied by $\tilde{U}_j(\varphi)$ and $U_j(\varphi)$, respectively, integrated over b_j to a_{j+1} and a_j to b_j, respectively, and then added with respect to j, imply the relations

$$\sum_{j=1}^{m}\int_{b_j}^{a_{j+1}}\tilde{U}_j(\varphi)\,d\tilde{\omega}^{(2)}(z,\varphi)-\sum_{\kappa=1}^{m}\gamma_\kappa\int_{a_\kappa}^{b_\kappa}d\omega^{(1)}(z,\varphi)$$
$$=\frac{1}{2\pi}\sum_{j=1}^{m}\int_{b_j}^{a_{j+1}}\tilde{U}_j(\varphi)\,d\lg|\Phi(e^{i\varphi},z)|,$$

$$\sum_{j=1}^{m}\int_{a_j}^{b_j}U_j(\varphi)\,d\tilde{\omega}^{(1)}(z,\varphi)-\sum_{\kappa=1}^{m}\alpha_\kappa\int_{b_\kappa}^{a_{\kappa+1}}d\omega^{(2)}(z,\varphi)$$
$$=\frac{1}{2\pi}\sum_{j=1}^{m}\int_{a_j}^{b_j}U_j(\varphi)\,d\lg|\Psi(e^{i\varphi},z)|.$$

Substituting these relations, together with those obtained above between $\omega^{(1)}$ and Φ and between $\omega^{(2)}$ and Ψ, into the formula at the end of the last section, we reach really the desired formula to be proved.

4. Case of a single pair of arcs.

If there exists merely a single pair of boundary arcs bearing respectively the prescribed boundary values of the real and imaginary parts of an analytic function to be determined, the kernels contained in the integral representation of the solution can be explicitly expressed within the range of elementary functions. Let the proposed problem be formulated as follows: *To determine a function $f(z)=u(z)+i\tilde{u}(z)$ analytic and bounded in $|z|<1$ and satisfying the boundary conditions*

$$u(e^{i\varphi})=U(\varphi)\quad for\quad a<\varphi<b,$$
$$\tilde{u}(e^{i\varphi})=\tilde{U}(\varphi)\quad for\quad b<\varphi<a+2\pi,$$

the functions $U(\varphi)$ and $\tilde{U}(\varphi)$ being supposed continuous and bounded in their respective intervals of definition.

We first remember the formula on the solution of the corresponding mixed boundary value problem

$$\Delta u(z)=0 \qquad \text{in} \qquad |z|<1,$$

$$u(e^{i\varphi})=U(\varphi) \qquad \text{for} \qquad a<\varphi<b,$$

$$\frac{\partial u(e^{i\varphi})}{\partial \nu}=V(\varphi) \qquad \text{for} \qquad b<\varphi<a+2\pi.$$

In our previous papers[3)] it has been solved in two apparently different but mutually equivalent ways. We prefer here, for instance, the formula of the form

$$u(z)=\Re\Bigg\{\frac{1}{2\pi}\int_a^b U(\varphi)\frac{e^{i(2\varphi-a-b)/4}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}}d\varphi$$

$$-\frac{1}{\pi}\int_b^{a+2\pi}V(\varphi)$$

$$\times\lg\frac{\left(e^{i(b-a)/8}\left(\sin\frac{\varphi-b}{2}\right)^{1/2}\sqrt{z-e^{ia}}+e^{-i(b-a)/8}\left(\sin\frac{\varphi-a}{2}\right)^{1/2}\sqrt{z-e^{ib}}\right)^2}{\sin\frac{b-a}{2}(z-e^{i\varphi})}d\varphi\Bigg\},$$

the square roots $\sqrt{z-e^{ia}}$ and $\sqrt{z-e^{ib}}$ denoting here and below the branch which attains the values $ie^{ia/2}$ and $ie^{ib/2}$, respectively at the origin.

We consider accordingly an analytic function defined by

$$f^{(1)}(z)\equiv u^{(1)}(z)+i\tilde{u}^{(1)}(z)$$

$$=\frac{1}{2\pi}\int_a^b U(\varphi)\frac{e^{i(2\varphi-a-b)/4}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}}d\varphi.$$

It is evidently regular and bounded in $|z|<1$ and further satisfies the boundary conditions

$$u^{(1)}(e^{i\varphi})=U(\varphi) \quad \text{for} \quad a<\varphi<b,$$

$$\tilde{u}^{(1)}(e^{i\varphi})=0 \qquad \text{for} \quad b<\varphi<a+2\pi.$$

In fact, the first relation is immediate from the definition of $f^{(1)}(z)$, while the second will be derived by actual computation; namely, we get, for $b<\varphi<a+2\pi$,

$$v^{(1)}(e^{i\varphi})=\Im\frac{1}{2\pi}\int_a^b U(\psi)\frac{e^{i(2\psi-a-b)/4}}{\left(\sin\frac{\psi-a}{2}\sin\frac{b-\psi}{2}\right)^{1/2}}$$

$$\frac{ie^{i(\varphi+a)/4}\left(\sin\frac{\varphi-a}{2}\right)^{1/2}ie^{i(\varphi+b)/4}\left(\sin\frac{\varphi-b}{2}\right)^{1/2}}{e^{i(\psi+\varphi)/2}\sin\frac{\varphi-\psi}{2}}d\psi$$

$$=\Im\frac{1}{2\pi}\int_a^b U(\psi)\left(\frac{\sin\frac{\varphi-a}{2}\sin\frac{\varphi-b}{2}}{\sin\frac{\psi-a}{2}\sin\frac{b-\psi}{2}}\right)^{1/2}\frac{1}{\sin\frac{\varphi-\psi}{2}}d\psi=0\,.$$

We next consider an analytic function $f^{(2)}(z)$ obtained by replacing a, b, $U(\varphi)$ in the expression of $f^{(1)}(z)$ by b, $a+2\pi$, $\tilde{U}(\varphi)$, respectively. Since $\sqrt{z-e^{i(a+2\pi)}}$ must then be identified with $-\sqrt{z-e^{ia}}$, we accordingly get

$$f^{(2)}(z)=v^{(2)}(z)-iu^{(2)}(z)$$

$$=\frac{i}{2\pi}\int_b^{a+2\pi}\tilde{U}(\varphi)\frac{e^{i(2\varphi-a-b)/4}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{\varphi-b}{2}\right)^{1/2}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}}d\varphi\,,$$

the square roots denoting, of course, again the same branch as above. It is also regular and bounded in $|z|<1$ and further, as verified similarly as above, satisfies the boundary conditions

$$v^{(2)}(e^{i\varphi})=\tilde{U}(\varphi)\quad\text{for}\quad b<\varphi<a+2\pi\,,$$

$$u^{(2)}(e^{i\varphi})=0\qquad\text{for}\quad a<\varphi<b\,.$$

Thus, *the analytic function defined by*

$$f(z)=f^{(1)}(z)+if^{(2)}(z)=u(z)+iv(z)$$

$$=\frac{1}{2\pi}\left\{\int_a^b U(\varphi)\frac{e^{i(2\varphi-a-b)/4}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}}d\varphi\right.$$

$$\left.-\int_b^{a+2\pi}\tilde{U}(\varphi)\frac{e^{i(2\varphi-a-b)/4}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{\varphi-b}{2}\right)^{1/2}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}}d\varphi\right\},$$

the square root $\sqrt{z-e^{ic}}$ denoting the branch which attains the value $ie^{ic/2}$ at the origin, is regular and bounded in $|z|<1$ and satisfies the boundary conditions

$$u(e^{i\varphi})=U(\varphi) \quad \textit{for} \quad a<\varphi<b,$$

$$\tilde{u}(e^{i\varphi})=\tilde{U}(\varphi) \quad \textit{for} \quad b<\varphi<a+2\pi,$$

and hence solves the mixed boundary value problem in consideration.

By the way, it may be noted that, by means of the identities

$$2i\frac{\partial}{\partial\varphi}$$

$$\times \lg\frac{\left(e^{i(b-a)/8}\left(\sin\frac{b-\varphi}{2}\right)^{1/2}\sqrt{z-e^{ia}}-ie^{-i(b-a)/8}\left(\sin\frac{\varphi-a}{2}\right)^{1/2}\sqrt{z-e^{ib}}\right)^2}{z-e^{i\varphi}}$$

$$=\frac{e^{i(2\varphi-a-b)/4}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}}$$

and

$$2\frac{\partial}{\partial\varphi}$$

$$\times \lg\frac{\left(e^{i(b-a)/8}\left(\sin\frac{\varphi-b}{2}\right)^{1/2}\sqrt{z-e^{ia}}+e^{-i(b-a)/8}\left(\sin\frac{\varphi-a}{2}\right)^{1/2}\sqrt{z-e^{ib}}\right)^2}{z-e^{i\varphi}}$$

$$=\frac{e^{i(2\varphi-a-b)/4}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{\varphi-b}{2}\right)^{1/2}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}},$$

the formula for the solution just obtained can be written also in the form

$$f(z)=\frac{1}{\pi}\left\{i\int_a^b U(\varphi)\right.$$

$$\times d\lg\frac{\left(e^{i(b-a)/8}\left(\sin\frac{b-\varphi}{2}\right)^{1/2}\sqrt{z-e^{ia}}-ie^{-i(b-a)/8}\left(\sin\frac{\varphi-a}{2}\right)^{1/2}\sqrt{z-e^{ib}}\right)^2}{z-e^{i\varphi}}$$

$$-\int_b^{a+2\pi}\tilde{U}(\varphi)$$

$$\left.\times d\lg\frac{\left(e^{i(b-a)/8}\left(\sin\frac{\varphi-b}{2}\right)^{1/2}\sqrt{z-e^{ia}}+e^{-i(b-a)/8}\left(\sin\frac{\varphi-a}{2}\right)^{1/2}\sqrt{z-e^{ib}}\right)^2}{z-e^{i\varphi}}\right\}.$$

Finally, an attention should be called to the fact that in this simplest case there exists always a unique solution without any functional restriction on the prescribed boundary functions $U(\varphi)$ and $\tilde{U}(\varphi)$.

5. Case of two pairs of arcs.

In case there exist two pairs of boundary arcs bearing alternately the prescribed boundary values of the real and imaginary parts of an analytic function to be determined, the kernels contained in the integral representation of the solution are explicitly expressible within the range of elliptic functions. The proposed original problem is to determine a function $f(z)=u(z)+i\tilde{u}(z)$ analytic and bounded in $|z|<1$ and satisfying the boundary conditions

$$u(e^{i\varphi})=U_1(\varphi) \text{ for } a_1<\varphi<b_1, \qquad u(e^{i\varphi})=U_2(\varphi) \text{ for } a_2<\varphi<b_2,$$

$$\tilde{u}(e^{i\varphi})=\tilde{U}_1(\varphi) \text{ for } b_1<\varphi<a_2, \qquad \tilde{u}(e^{i\varphi})=\tilde{U}_2(\varphi) \text{ for } b_2<\varphi<a_1+2\pi,$$

the boundary functions $U_1(\varphi)$, $U_2(\varphi)$, $\tilde{U}_1(\varphi)$ and $\tilde{U}_2(\varphi)$ being supposed continuous and bounded in their respective intervals of definition.

Now, based on the conformal invariance of our problem, we may choose any simply connected domain instead of the unit circle. Similarly as in the preceding paper[3)], we again prefer here, for convenience sake, a rectangle of the form

$$\lg q < \Re \hat{z} < 0, \qquad 0 < \Im \hat{z} < \pi,$$

laid on the $\hat{z}$-plane, as the basic domain, q being a positive constant less than unity. As well-known, the unit circle $|z|<1$ can be mapped onto a rectangle of this form in such a manner that the points e^{ia_1}, e^{ib_1}, e^{ia_2} and e^{ib_2} on $|z|=1$ correspond to the vertices 0, $i\pi$, $\lg q + i\pi$ and $\lg q$, respectively. Since the ratio of the length of adjacent sides of the rectangle is a conformal invariant called its *modulus*, the number q is uniquely determined by the assigned correspondence. Moreover, the mapping function is really defined by the equations

$$\hat{z} = i\int_{\infty}^{X} \frac{dX}{\sqrt{4(X-e_1)(X-e_2)(X-e_3)}},$$

$$(X, e_1, e_2, e_3) = (z, e^{ib_1}, e^{ia_2}, e^{ib_2}),$$

where the triple of the real constants e_1, e_2 and e_3 with $e_1 > e_2 > e_3$ is subjected to the conditions

$$e_1+e_2+e_3=0, \qquad \frac{e_1-e_3}{e_1-e_2} = \left(\sin\frac{b_2-b_1}{2}\sin\frac{a_2-a_1}{2}\right) \Big/ \left(\sin\frac{b_2-a_1}{2}\sin\frac{a_2-b_1}{2}\right)$$

and a remaining freedom of a common factor for the triple is to be determined in such a way that the primitive periods of the elliptic function $X = \wp(-i\hat{z})$ are

$$2\omega_1 = 2\pi, \qquad 2\omega_3 = -2i\lg q.$$

The number q is then defined by the equation

$$\left(\sin\frac{b_2-a_1}{2}\sin\frac{a_2-b_1}{2}\right) \Big/ \left(\sin\frac{b_2-b_1}{2}\sin\frac{a_2-a_1}{2}\right) = \frac{e_1-e_2}{e_1-e_3} = \prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^8,$$

which is also equivalent to the equation

$$\lg q = -K'/K,$$

where K and K' denote, as usual, the quantities given by

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \qquad K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}}$$

with

$$k'^2 = 1 - k^2 = \frac{e_1 - e_2}{e_1 - e_3}.$$

According to the remark just stated, we now choose, for the sake of mere convenience, a rectangle as the basic domain, which is laid on the z-plane——for brevity sake, we again write z instead of $\hat{z}$——. Let it be accordingly expressed by

$$\lg q < \Re z < 0, \qquad 0 < \Im z < \pi.$$

The problem is then to determine an explicit expression for a function $f(z) = u(z) + i\tilde{u}(z)$ *analytic and bounded in the rectangle and satisfying the boundary conditions*

$$u(it) = M(t) \quad \textit{and} \quad u(\lg q + it) = N(t) \quad \textit{for} \quad 0 < t < \pi,$$

$$\tilde{u}(s) = \tilde{M}(s) \quad \textit{and} \quad \tilde{u}(s + i\pi) = \tilde{N}(s) \quad \textit{for} \quad \lg q < s < 0,$$

As a consequence of the general discussion, the condition for the existence of a solution must be given by a single functional relation among the prescribed boundary functions. Although the relation will be derived naturally during the following arguments, it can also be previously formulated in quite a brief manner. We now commence with the statement on the condition for the existence of a solution:

A necessary and sufficient condition for the existence of a solution of our mixed boundary value problem is given by

$$\int_0^\pi \big(M(t) - N(t)\big)\, dt + \int_{\lg q}^0 \big(\tilde{M}(s) - \tilde{N}(s)\big)\, ds = 0.$$

The necessity of the condition is quite evident. In fact, by considering the contour integral of $f(z)$ extended along the whole boundary of the basic rectangle, we get

$$0 = \int f(z)\, dz = \int_0^\pi \big(M(t) - N(t)\big)\, dt + \int_{\lg q}^0 \big(\tilde{M}(s) - \tilde{N}(s)\big)\, ds.$$

The sufficiency proof may proceed as follows. Let $g(z) = \tilde{v}(z) - iv(z)$ be any function analytic and bounded in the rectangle, continuous on its closure and further satisfying the conditions

$$\tilde{v}(s) = \tilde{M}(s) \quad \text{and} \quad \tilde{v}(s + i\pi) = \tilde{N}(s) \quad \text{for} \quad \lg q < s < 0.$$

Such a function can be constructed in various ways. Now, the function $w=e^z$ maps the basic rectangle onto the upper half of the annulus

$$q<|w|<1, \qquad \Im w>0.$$

We then solve a Dirichlet problem for the annulus $q<|w|<1$ obtained by duplicating the semi-annulus, of which the boundary conditions are given in the form

$$\left.\begin{aligned} u^*(e^{i\psi}) &= u^*(e^{-i\psi}) = M(\psi)-v(i\psi), \\ u^*(qe^{i\psi}) &= u^*(qe^{-i\psi}) = N(\psi)-v(\lg q+i\psi) \end{aligned}\right\} \quad \text{for } 0<\psi<\pi.$$

Let $u^*(w)$ be its solution and let $\tilde{u}^*(w)$ denote a branch of any harmonic function conjugate to $u^*(w)$. The symmetry character of the boundary conditions implies immediately the relation

$$\frac{\partial \tilde{u}^*(w)}{\partial \Re w} = -\frac{\partial u^*(w)}{\partial \Im w} = 0 \qquad \text{for} \quad q<|w|<1, \quad \Im w=0.$$

Hence $\tilde{u}^*(w)$ must remain constant along each boundary segment of the upper semi-annulus; we put

$$\tilde{u}^*(w) = c_\pm \qquad \text{for} \quad q<|w|<1, \quad \Re w \gtrless 0, \quad \Im w=0.$$

Consider then the function defined by

$$f(z) \equiv u(z)+i\tilde{u}(z) = u^*(e^z)+i\left(\tilde{u}^*(e^z)-c_+\right)+v(z)+i\tilde{v}(z).$$

It is analytic and bounded in the original rectangle and satisfies the boundary conditions

$$\left.\begin{aligned} u(it) &= u^*(e^{it})+v(it) &&= M(t), \\ u(\lg q+it) &= u^*(qe^{it})+v(\lg q+it) &&= N(t) \end{aligned}\right\} \quad \text{for } 0<t<\pi;$$

$$\left.\begin{aligned} \tilde{u}(s) &= \tilde{u}^*(e^s)-c_+ +\tilde{v}(s) &&= \tilde{M}(s), \\ \tilde{u}(s+i\pi) &= \tilde{u}^*(-e^s)-c_+ +\tilde{v}(s+i\pi) &&= c_- - c_+ + \tilde{N}(s) \end{aligned}\right\} \quad \text{for } \lg q<s<0.$$

By taking into account the necessity condition satisfied by $f(z)$, we have

$$0 = \oint f(z)\,dz = -\lg q \cdot (c_- - c_+).$$

Hence, the function $f(z)$ satisfying entirely the boundary conditions, it is just a solution of the problem, what completes the desired proof.

Once the existence of a solution having been ensured, its uniqueness is a matter of course.

By the way, it may be noticed that *the condition in consideration ensures the one-valuedness of the function $\tilde{u}^*(w)$ in the whole annulus $q<|w|<1$.* In fact, the relation $c_-=c_+$ implies that the function $f^*(w)=u^*(w)+i\tilde{u}^*(w)$ analytic in the upper semi-annulus is analytically prolongable, based on the functional equation

$$f^*(\overline{w})=\overline{f^*(w)}+2ic_+,$$

beyond the boundary segments on the real axis into the lower semi-annulus.

Moreover, the connection of the condition in consideration with the so-called *monodromy condition*[4)] will also be evident. Namely, in view of the former together with itself applied to $ig(z)$ defined above, we get

$$\begin{aligned}
&\int_{-\pi}^{\pi} u^*(e^{i\psi})\,d\psi-\int_{-\pi}^{\pi} u^*(qe^{i\psi})\,d\psi\\
&=2\int_0^{\pi}\bigl(M(\psi)-v(i\psi)\bigr)\,d\psi-2\int_0^{\pi}\bigl(N(\psi)-v\bigl(\lg q+i\psi\bigr)\bigr)\,d\psi\\
&=2\Bigl\{\int_0^{\pi}\bigl(M(\psi)-N(\psi)\bigr)\,d\psi+\int_{\lg q}^{0}\bigl(\tilde{M}(s)-\tilde{N}(s)\bigr)\,ds\Bigr\}\\
&-2\Bigl\{\int_0^{\pi}\bigl(v(i\psi)-v(\lg q+i\psi)\bigr)\,d\psi+\int_{\lg q}^{0}\bigl(\tilde{v}(s)-\tilde{v}(s+i\pi)\bigr)\,ds\Bigr\}=0.
\end{aligned}$$

The last relation is nothing but the monodromy condition ensuring again that the harmonic function $\tilde{u}^*(w)$ conjugate to $u^*(w)$ is one-valued throughout the whole annulus.

We now turn our attention to the main discourse. As shown in the previous paper, the corresponding mixed boundary value problem

$$\begin{aligned}
&\Delta u(z)=0 \quad &&\text{in}\quad \lg q<\Re z<0,\ 0<\Im z<\pi,\\
&u(it)=M(t)\quad\text{and}\quad u(\lg q+it)=N(t) &&\text{for}\quad 0<t<\pi,\\
&\frac{\partial u(s)}{\partial \nu}=P(s)\quad\text{and}\quad \frac{\partial u(s+i\pi)}{\partial\nu}=Q(s) &&\text{for}\quad \lg q<s<0
\end{aligned}$$

is solved by the formula

$$u(z)=\Re\left\{\frac{1}{\pi i}\int_0^\pi\left(\frac{2\eta_3 z}{\lg q}\left(M(t)-N(t)\right)\right.\right.$$

$$\left.+M(t)\left(\zeta(iz+t)+\zeta(iz-t)\right)-N(t)\left(\zeta_3(iz+t)+\zeta_3(iz-t)\right)\right)dt$$

$$+\frac{1}{\pi}\int_{\lg q}^{0}\left(\frac{2\eta_3 z}{i\lg q}s\left(P(s)+Q(s)\right)\right.$$

$$\left.\left.+P(s)\lg\frac{\sigma(iz-is)}{\sigma(iz+is)}+Q(s)\lg\frac{\sigma_1(iz-is)}{\sigma_1(iz+is)}\right)ds\right\},$$

the notations from the Weierstrassian theory of elliptic functions referring here to those with the primitive periods

$$2\omega_1=2\pi, \qquad 2\omega_3=-2i\lg q.$$

Accordingly, we first consider an analytic function defined by

$$f^{(1)}(z)=u^{(1)}(z)+i\tilde{u}(z)=\frac{1}{\pi i}\int_0^\pi\left(\frac{2\eta_3 z}{\lg q}\left(M(t)-N(t)\right)\right.$$

$$\left.+M(t)\left(\zeta(iz+t)+\zeta(iz-t)\right)-N(t)\left(\zeta_3(iz+t)+\zeta_3(iz-t)\right)\right)dt.$$

It is regular and bounded in the basic rectangle and satisfies the boundary conditions

$$u^{(1)}(it)=M(t) \text{ and } u^{(1)}(\lg q+it)=N(t) \qquad \text{for } 0<t<\pi,$$

$$\tilde{u}^{(1)}(s)=0 \qquad \text{and } \tilde{u}^{(1)}(s+i\pi)=-\frac{1}{\lg q}\int_0^\pi\left(M(t)-N(t)\right)dt \text{ for } \lg q<s<0.$$

In fact, the former is an immediate consequence of the definition, while the latter will be verified by actual computation. Namely, we get, for $\lg q<s<0$,

$$\tilde{u}^{(1)}(s)=\Im\left\{\frac{1}{\pi i}\int_0^\pi\left(\frac{2\eta_3 s}{\lg q}\left(M(t)-N(t)\right)+M(t)\left(\zeta(is+t)+\zeta(is-t)\right)\right.\right.$$

$$\left.\left.-N(t)\left(\zeta_3(is+t)+\zeta_3(is-t)\right)\right)dt\right\}=0,$$

$$
\begin{aligned}
\tilde{u}^{(1)}(s+i\pi) = \Im \Big\{ \frac{1}{\pi i} \int_0^\pi \Big(& \frac{2\eta_3(s+i\pi)}{\lg q} \big(M(t)-N(t)\big) \\
& + M(t)\big(\zeta(is-\pi+t)+\zeta(is-\pi-t)\big) \\
& - N(t)\big(\zeta_3(is-\pi+t)-\zeta_3(is-\pi-t)\big)\Big)\, dt \Big\} \\
= \frac{1}{\pi} \Big(\frac{2\eta_3\pi}{i\lg q} & + 2\eta_1\Big) \int_0^\pi \big(M(t)-N(t)\big)\, dt \\
= - \frac{1}{\lg q} \int_0^\pi & \big(M(t)-N(t)\big)\, dt\,.
\end{aligned}
$$

Here we have taken into account the facts that, since the primitive periods $2\omega_1=2\pi$ and $2\omega_3=-2i\lg q$ are real and purely imaginary, respectively, and hence the quantities η_1 and $i\eta_3$ are both real and further the relations

$$
\overline{\zeta(w)} = \zeta(\overline{w})\,, \qquad \zeta_3(w) = \zeta_3(\overline{w})
$$

are valid identically, together with the fundamental properties

$$
\zeta(w+\pi) = \zeta(w) + \eta_1\,, \qquad \zeta_3(w+\pi) = \zeta_3(w) + \eta_1
$$

as well as the Legendre's identity

$$
\pi i/2 = \eta_1\omega_3 - \eta_3\omega_1 = -i\eta_1 \lg q - \eta_3\pi\,.
$$

We next consider a function $\hat{f}^{(2)}(\hat{z})$ analogous to $f^{(1)}(z)$, which is obtained from the latter by replacing z, t, q, $M(t)$ and $N(t)$ respectively by

$$
\hat{z} = \frac{\pi}{-\lg q}(iz - i\lg q)\,, \quad \hat{s} = \frac{\pi}{-\lg q}(s-\lg q)\,, \quad \hat{q} = \exp\frac{\pi^2}{\lg q}\,,
$$

$$
\tilde{M}\Big(\frac{-\lg q}{\pi}(\hat{s}-\pi)\Big) \quad \text{and} \quad \tilde{N}\Big(\frac{-\lg q}{\pi}(\hat{s}-\pi)\Big),
$$

i. e.

$$
\begin{aligned}
\hat{f}^{(2)}(\hat{z}) &= \tilde{\hat{u}}^{(2)}(\hat{z}) - i\hat{u}^{(2)}(\hat{z}) \\
&= \frac{1}{\pi i} \int_0^\pi \Big(\frac{2\hat{\eta}_3\hat{z}}{\lg \hat{q}} \Big(\tilde{M}\Big(\frac{-\lg q}{\pi}(\hat{s}-\pi)\Big) - \tilde{N}\Big(\frac{-\lg q}{\pi}(\hat{s}-\pi)\Big)\Big)
\end{aligned}
$$

$$+\tilde{M}\Big(\frac{-\lg q}{\pi}(\hat{s}-\pi)\Big)\Big(\hat{\zeta}(i\hat{z}+\hat{s})+\hat{\zeta}(i\hat{z}-\hat{s})\Big)$$

$$-\tilde{N}\Big(\frac{-\lg q}{\pi}(\hat{s}-\pi)\Big)\Big(\hat{\zeta}_3(i\hat{z}+\hat{s})+\hat{\zeta}_3(i\hat{z}-\hat{s})\Big)\Big)\,d\hat{s}\,,$$

the notations from the Weierstrassian theory of elliptic functions, marked by ^, now referring to those with the primitive periods

$$2\hat{\omega}_1=2\pi\,, \qquad 2\hat{\omega}_3=-2i\lg\hat{q}\,.$$

By making use of the identities

$$\hat{\omega}_1\hat{\zeta}(\hat{\omega}_1 Z)=\omega_3\zeta(\omega_3 Z)\,, \qquad \hat{\omega}_1\hat{\zeta}_3(\hat{\omega}_1 Z)=\omega_3\zeta_1(\omega_3 Z)\,,$$

$$\hat{\omega}_1\hat{\eta}_1=\omega_3\eta_3\,, \qquad \hat{\omega}_1\hat{\eta}_3=-\omega_3\eta_1\,; \qquad \hat{\omega}_3/\hat{\omega}_1=-\omega_1/\omega_3=\pi/i\lg q\,,$$

the functions depending on the primitive periods $2\hat{\omega}_1=2\pi$ and $2\hat{\omega}_3=-2i\lg\hat{q}$ can be replaced by those on $2\omega_1=2\pi$ and $2\omega_3=-2i\lg q$. Thus, returning to the original variable

$$z=\frac{-\lg q}{i\pi}(\hat{z}-i\pi)\,,$$

after substituting the new integration variable by means of $s=(-i\lg q/\pi)(\hat{s}-\pi)$, we get

$$f^{(2)}(z)\equiv\hat{f}^{(2)}(\hat{z})\equiv\tilde{u}^{(2)}(z)-iu^{(2)}(z)$$

$$=-\frac{1}{\pi}\int_{\lg q}^{0}\Big(\frac{2\eta_1 z+\pi}{\pi i}\big(\tilde{M}(s)-\tilde{N}(s)\big)$$

$$+\tilde{M}(s)\big(\zeta(iz-is)+\zeta(iz+is)\big)-\tilde{N}(s)\big(\zeta_1(iz-is)+\zeta_1(iz+is)\big)\Big)\,ds\,.$$

The function $f^{(2)}(z)$ thus defined is regular and bounded in the original rectangle and satisfies the boundary conditions

$$\tilde{u}^{(2)}(s)=\tilde{M}(s) \quad\text{and}\quad \tilde{u}^{(2)}(s+i\pi)=\tilde{N}(s) \quad\text{for } \lg q<s<0,$$

$$u^{(2)}(it)=-\frac{1}{\pi}\int_{\lg q}^{0}\big(\tilde{M}(s)-\tilde{N}(s)\big)\,ds \quad\text{and}\quad u^{(2)}(\lg q+it)=0 \quad\text{for } 0<t<\pi.$$

In fact, the former is evident from the manner of constructing $f^{(2)}(z)$

and the latter will be verified by actual computation; or else one may also remark the relation

$$u^{(2)}(it)=\frac{1}{\lg q}\int_0^{\pi}\left(\tilde{M}\left(\frac{-\lg q}{\pi}(\hat{s}-\pi)\right)-\tilde{N}\left(\frac{-\lg q}{\pi}(\hat{s}-\pi)\right)\right)d\hat{s}$$

$$=-\frac{1}{\pi}\int_{\lg q}^{0}\left(\tilde{M}(s)-\tilde{N}(s)\right)ds \qquad \text{for } 0<t<\pi,$$

which follows immediately from our previous consideration on $\tilde{u}^{(1)}(s+i\pi)$ with $\lg q<s<0$.

We finally define an analytic function by

$$f(z)\equiv u(z)+i\tilde{u}(z)$$

$$=f^{(1)}(z)+if^{(2)}(z)+\frac{\lg q-z}{\pi\lg q}\int_{\lg q}^{0}\left(\tilde{M}(s)-\tilde{N}(s)\right)ds.$$

It is regular and bounded in the basic rectangle and satisfies the boundary conditions

$$u(it)=u^{(1)}(it)+u^{(2)}(it)+\frac{1}{\pi}\int_{\lg q}^{0}\left(\tilde{M}(s)-\tilde{N}(s)\right)ds$$

$$=M(t)-\frac{1}{\pi}\int_{\lg q}^{0}\left(\tilde{M}(s)-\tilde{N}(s)\right)ds+\frac{1}{\pi}\int_{\lg q}^{0}\left(\tilde{M}(s)-\tilde{N}(s)\right)ds=M(t),$$

$$u(\lg q+it)=u^{(1)}(\lg q+it)+u^{(2)}(\lg q+it)=N(t) \qquad \text{for } 0<t<\pi;$$

$$\tilde{u}(s)=\tilde{u}^{(1)}(s)+\tilde{u}^{(2)}(s)=\tilde{M}(s),$$

$$\tilde{u}(s+i\pi)=\tilde{u}^{(1)}(s+i\pi)+\tilde{u}^{(2)}(s+i\pi)-\frac{1}{\lg q}\int_{\lg q}^{0}\left(\tilde{M}(s)-\tilde{N}(s)\right)ds$$

$$=-\frac{1}{\lg q}\int_0^{\pi}\left(M(t)-N(t)\right)dt+\tilde{N}(s)$$

$$-\frac{1}{\lg q}\int_{\lg q}^{0}\left(\tilde{M}(s)-\tilde{N}(s)\right)ds=\tilde{N}(s) \qquad \text{for } \lg q<s<0;$$

in the last equation the condition for the existence of the solution is taken into account. Thus, the function $f(z)$ solves surely the mixed boundary value problem in consideration.

By remembering again the condition for the existence of the solution and also by making use of the Legendre's identity

$$2\eta_1/\pi + 1/\lg q = 2\eta_3 i/\lg q ,$$

the final expression of the solution in its fully explicit form becomes

$$f(z) = \frac{1}{\pi i} \int_0^\pi \Big(M(t) \big(\zeta(iz+t) + \zeta(iz-t) \big) - N(t) \big(\zeta_3(iz+t) + \zeta_3(iz-t) \big) \Big) dt$$

$$+ \frac{1}{\pi i} \int_{\lg q}^0 \Big(\tilde{M}(s) \big(\zeta(iz-is) + \zeta(iz+is) \big) - \tilde{N}(s) \big(\zeta_1(iz-is) + \zeta_1(iz+is) \big) \Big) ds .$$

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[2] A. Signorini, Sopra un problema al contorno nella teoria delle funzioni di variabile complessa. Ann. di Mat. (3) 25 (1916), 253 273.

[3] Cf. Y. Komatu and I. Hong, loc. cit.[1]

[4] For instance, cf. Y. Komatu, Ein alternierendes Approximationsverfahren für konforme Abbildung von einem Ringgebiete auf einen Kreisring. Proc. Imp. Acad. Tokyo 21 (1945), 146 155.