Journal of the Mathematical Society of Japan Vol. 5, Nos. 3~4, November, 1953.

An imbedding theorem on finite covering surfaces of the Riemann sphere.

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(Received April 10, 1953)

Let Δ be a finite covering surface of the Riemann sphere Σ, i. e. a covering surface consisting of a finite number of closed triangles. Then, it was proved in [1] that *Δ can be imbedded in a closed covering surface of Σ*. According to the construction described therein, however, the genus of the resulting closed surface is in general higher than that of Δ. In § 1 of the present paper, we shall prove:

A finite covering surface Δ of Σ can be imbedded in a closed covering surface of the same genus.

In § 2, an analogous theorem concerning analytic differentials is stated and proved. Finally, in § 3, an application is made to the theory of open Riemann surfaces.

1. It is a trivial fact that, as an abstract Riemann surface, Δ can be imbedded in a closed Riemann surface of the same genus. In this connection, our theorem may be formulated in the following form:

THEOREM 1. *Let D be a subregion of a Riemann surface F, such that the closure $\bar{D}$ of D is compact and that the boundary of D consists of a finite number of Jordan curves. Let $f(p)$ be a function defined and analytic on $\bar{D}$ (poles being admitted). Then, $\bar{D}$ can be imbedded in a closed Riemann surface D^* of the same genus as D, in such a manner that $f(p)$ can be continued to a function defined and analytic on D^*.*

PROOF. We assume that $f(p) \neq$ const., since, otherwise, any closed prolongation of D of the same genus has the required property. Further, we may assume that $f(p)$ is defined and analytic throughout F. If the values of $f(p)$ are represented by points on the Riemann sphere Σ, F is mapped by $f(p)$ onto a covering surface Φ of Σ. The image

Δ of $\overline{D}$ on Φ is a finite covering surface of Σ.

Let C denote one of the connected components of the boundary of D, and Γ be the image of C on Φ. While deforming C and extending D slightly, we may assume:

i) Γ is piece-wise smooth and passes no branch-point of Φ;

ii) the projection of Γ on Σ has only a finite number $n(\Gamma)$ of double points and no multiple points of higher order;

iii) at each double point, the two branches of the projection of Γ are smooth and intersect each other making a positive angle.

We shall first prove, by induction with respect to $n(\Gamma)$, that Γ can be imbedded in a closed covering surface Ψ of genus zero of Σ, in such a manner that Γ passes no branch-point of Ψ.

If $n(\Gamma)=0$, then Γ can be imbedded in a replica of Σ itself. Suppose that the imbedding is possible for any Γ with $n(\Gamma)<k$, and that $n(\Gamma)=k$ for Γ in question. Let z_0 be one of the k double points of the projection of Γ. Dividing Γ into two arcs by the two points lying above z_0, and identifying the two end-points of each arc respectively, we obtain two closed curves Γ_1 and Γ_2, such that $n(\Gamma_1)$, $n(\Gamma_2)$ $<k$. By the assumption, Γ_1 and Γ_2 can be imbedded in closed covering surfaces Ψ_1 and Ψ_2 of genus zero respectively. Let P_1 and P_2 denote the points on Γ_1 and Γ_2, which have the common projection z_0. Then, for a sufficiently small disc K about z_0, Ψ_1 and Ψ_2 contain schlicht discs K_1, K_2 about P_1, P_2 with the common projection K. Let s be a segment in K which intersects the projections of Γ_1 and Γ_2 at z_0 and has no other point in common with them. While cutting Ψ_1 and Ψ_2 along segments in K_1, K_2 with the projection s respectively, and joining them cross-wise along these slits, we obtain a closed covering surface Ψ. Since Γ_1 and Γ_2 are imbedded in Ψ_1 and Ψ_2 respectively, Γ is imbedded in Ψ. Further, Ψ_1 and Ψ_2 being of genus zero, Ψ is of genus zero. Evidently Γ passes no branch-point of Ψ. Thus, the possibility of the mentioned imbedding is proved.

Since Ψ is of genus zero and Γ is a simple closed curve on Ψ, Γ divides Ψ into two parts, one of which, Ψ', lies to the right of Γ taken in positive direction with respect to Δ. We join Ψ' to Δ along the boundary curve Γ to obtain a finite covering surface Δ' containing Δ.

With natural local parameters, Δ' may be considered as an abstract Riemann surface D', in which D is imbedded. The mapping function of D' onto Δ' is analytic in D' and is equal to $f(p)$ in D and on C. Further, since C is null-homologous in D', D' is of the same genus as D.

Repeating the same construction for each connected component of the boundary of D, we finally obtain a closed Riemann surface D^* containing $\bar{D}$, which is of the same genus as D, and on which $f(p)$ can be continued to a function analytic everywhere, q. e. d.

2. Let F and D have the same meaning as in Theorem 1. We shall further prove:

THEOREM 2. *Let $d\varphi(p)$ be an analytic differential defined on $\bar{D}$ (poles being admitted). Then, $\bar{D}$ can be imbedded in a closed Riemann surface D^* of the same genus as D, in such a manner that $d\varphi(p)$ can be continued to a differential defined and analytic on D^*.*

PROOF. Let C denote one of the connected components of the boundary of D. We may assume that no pole of $d\varphi$ lies on C. Let S be a doubly connected strip region of sufficiently small width on F containing C, such that $d\varphi$ is regular on $\bar{S}$.

Suppose that $\int_C d\varphi=0$. Then, $f(p)=\int^p d\varphi$ is single-valued and analytic on $\bar{S}$. Hence, by Theorem 1, $\bar{S}$ can be imbedded in a closed Riemann surface S^* of genus zero, on which $f(p)$ can be continued to a function analytic everywhere. Then, df is analytic on S^* and $=d\varphi$ on $\bar{S}$.

If $\int_C d\varphi=\alpha\neq 0$, we put $f(p)=\exp\left(\frac{2\pi i}{\alpha}\int^p d\varphi\right)$. Then, we can similarly construct a closed surface S^* of genus zero, on which $(\alpha/2\pi i)(df/f)=d\varphi$ is everywhere analytic.

C divides S^* into two parts, one of which lies to the right of C taken in positive direction with respect to D. We join this part of S^* to D along C. Repeating the same construction for each connected component of the boundary of D, we obtain a closed Riemann surface D^*, which obviously satisfies the requirements.

3. Let F be an open Riemann surface, and G be a subregion of F, whose complement $F-G$ is compact and is bounded by a finite number of Jordan curves. Let $f(p)$ be a given function analytic on the closure $\overline{G}$ of G.

While constructing another Riemann surface containing $\overline{G}$ by means of Theorem 1, we can deduce, in some cases, some properties of $f(p)$ in G from known properties of functions analytic everywhere on the new surface. Following this line, we shall give here a simple proof of the following theorem, a special case of which was stated and proved in [2].

We denote by O_{HB} the class of Riemann surfaces, on which no non-constant bounded harmonic functions exist, and by O_G the class of those, which possess no Green's functions. As is well-known, $O_G \subseteq O_{HB}$.

THEOREM 3. *Suppose that F belongs to the class O_{HB}, and that $z=f(p) \neq$ const. is analytic on $\overline{G}$ and omits in G a set E of values z of positive logarithmic capacity. Then, $f(p)$ takes in G any value z at most a finite number of times N, and, along any continuous curve L: $p=p(t)$ $(0 \leq t < \infty)$ extending itself to the ideal boundary of F,* lim $f(p)$ *exists. All these limiting values form a closed set of logarithmic capacity zero. Further, F belongs really to the class O_G.*

A curve L: $p=p(t)$ on F is said to be extended to the ideal boundary of F, if, for any sequence $t_n \to \infty$ of values of t, the point sequence $\{p(t_n)\}$ has no accumulation point on F. The existence of lim $f(p)$ along any such L may be stated in other words as follows: *at each ideal boundary component of F,* lim $f(p)$ *exists for p tending to it.* As for the precise definition of ideal boundary components (éléments-frontières), cf. [9] or [5].

PROOF. Let D be a subregion of genus zero of G, such that $\overline{D}$ is compact and contains the relative boundary of G. By Theorem 1, $\overline{D}$ can be imbedded in a closed Riemann surface D^* of genus zero, in such a manner that $f(p)$ can be continued to a function analytic everywhere on D^*. Since D^* is of genus zero, each connected component of the boundary of G cuts off from D^* a simply connected region lying outside D. Joining these regions to $\overline{G}$ along the boundary curves, we obtain an open Riemann surface F^*, in which $\overline{G}$ is imbedded and on which $f(p)$ can be continued to a function analytic everywhere. Further, F^*-G being compact, F^* has the same ideal boundary as F

(cf. [4], [7] or [8])).

Since $\in O_{HB}$ is a boundary property ([4], [7], [8]), F^* belongs, together with F, to the class O_{HB}. Since $f(p)$ omits in G the set E of values, the image Φ^* of F^* by $z=f(p)$ covers the set E of positive capacity at most a finite number of times. On the other hand, it is known that a covering surface of the sphere belonging to O_{HB} covers each point of the sphere exactly one and the same (finite or infinite) number of times, except possibly those belonging to a set of capacity zero ([3], [4], [6]). Hence, Φ^* covers each point z, except those belonging to a closed set E^* of capacity zero, exactly a finite number of times N. Such a covering surface Φ^* belongs really to the class O_G ([4]), so that F also.

Suppose that $\lim f(p)$ did not exist along a curve $L: p=p(t)$ extending itself to the ideal boundary, so that the projection of the image of L would not terminate at any point of the sphere. Since E^* is totally disconnected, we could then find a sequence $t_n \to \infty$ and a point $z_0 \notin E^*$, such that $\lim_{n\to\infty} f\big(p(t_n)\big)=z_0$. Since, above a sufficiently small disc on the sphere about z_0, there lie N discs on Φ^*, it follows that the image of $\{p(t_n)\}$ would have an accumulation point on Φ^*, which is impossible by the hypothesis. Hence, $\lim f(p)$ exists along any L. Similarly, we see that the limiting values belong to E^*.

Further, since each point of E^* is an accessible boundary point of the complementary region, we see easily that the set of all the limiting values of $f(p)$ comprises E^*.

Thus, Theorem 3 is proved.

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