On algebraic families of positive divisors
and their associated Varieties
on a projective Variety.[1]

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In spite of their importance, little is known on algebraic families of positive cycles on a projective Variety, except for the fundamental results of Chow-v. d. Waerden on associated-forms of positive cycles.[2] In the case of a Curve, a maximal algebraic family of divisors of a given degree form a *complete* family. But in the case of higher dimensions than 1, the situation is slightly different. A maximal algebraic family of positive divisors of a given degree on a non-singular surface is not determined uniquely in general, and there is a finite number of maximal algebraic families of the given degree, the divisors of which are mutually algebraically equivalent. A non-special linear system of a Curve belongs to the complete algebraic family such that every divisor of the family determines the complete linear system of the same dimension, which is totally contained in the algebraic family. Now the question is, *whether there exists always such a complete algebraic family on algebraic Varieties of higher dimensions*, or more precisely, *how one can obtain such a complete algebraic family from the given maximal algebraic family.*[2'] We shall show that such a family can be obtained always, by adding to the given algebraic family sufficiently large multiples of the hyperplane sections, when the ambient Variety is non-singular (th. 2). Moreover, as we shall show, algebraic families thus obtained generates the Picard Variety of the given Variety, i. e., when

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1)   We shall use the terminologies and conventions in A. Weil's "Foundations of Algebraic Geometry", Amer. Math. Soc. Colloq., vol. 29, 1946 and "Variétés Abéliennes et Courbes Algébrique", Act. Sc. et Ind., no. 1046.

Numbers and letters in brackets refer to Bibliography at the end of this paper.

2)   Cf. [C-W] and [C-2].

2') Cf. G. Albanese, "Intorno ad alcuni concetti e teoremi fondamentali sui sistemi algebrici di curve d'una superficie algebrica." Ann. Mat. pura appl. III. s. vol. 24, 1915,

q is the dimension of the Picard Variety, it consists of ∞^q distinct linear systems. This has a certain contact with the following Severi's theorem (translated in our terminology):

On a surface of irregularity q, every arithmetically effective Curves belongs to a maximal algebraic family consisting of ∞^q distinct linear systems. (cf. [Severi-1])

In order to prove th. 2, we shall show that, on a non-singular projective Variety, the notion of the virtual arithmetic genus of divisors is invariant with respect to algebraic equivalence (th. 1) (cf. [Severi-2]). In the remaining part of this paper, we shall discuss some properties of associated-Varieties of algebraic families of divisors on a normal projective Variety.

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Let V^r be a normal Variety in a projective space defined over a field k_0 which we fix as the *basic field*. All fields we shall consider will be assumed to contain k_0. A *linear equivalence* of divisors on V is defined in the usual manner and is denote by$\sim$. Let X be a positive V-divisor. Considering the coefficients of the associated-form of X as a homogeneous coordinate of a Point in a projective space, we call it the *Chow-Point* of X. *Algebraic equivalence* of divisors are also defined in the usual manner. When a positive integer is given, totalities of Chow-Points of positive V-divisors of that degree distributes in a bunch $\widetilde{\mathfrak{F}}$ in a certain projective space. (cf. [C-W]). Let U be a component of $\widetilde{\mathfrak{F}}$, and $\{X\}$ be the totality of V-divisors whose Chow-Points are on U. We shall say that $\{X\}$ is a *maximal algebraic family*. Let U' be a Subvariety of U and $\{X'\}$ the Subfamily of $\{X\}$ corresponding to U'. We say that U' is the *associated-Variety* of an algebraic family $\{X'\}$. By the *degree* of $\{X'\}$, we mean the degree of a certain divisor X' of $\{X'\}$. This notion is clearly independent of the choice of X'. When X and X' belong to the same maximal algebraic family, $X-X'$ is said to be *algebraically equivalent to zero* and X and X' are said to be *algebraically equivalent*. Denote by $G_a(V)$ the group generated by V-divisors which are algebraically equivalent to zero and $G_l(V)$ the

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group generated by the V-divisors which are linearly equivalent to zero. Assume that, whenever any divisor Y in $G_a(V)$ is given, there are two divisors X and X' in a maximal algebraic family $\{X\}$ in such a way that

$$Y \sim X - X'.$$

Then we say that $\{X\}$ is a maximal *regular* algebraic family. The existence of such a family has been proved in [M-4]-th. 1. We say that a maximal algebraic family $\{X\}$ of positive V-divisors is *complete*, when it holds $l(X) = l(X')$ for any two divisors X and X' in $\{X\}$. When a maximal algebraic family of positive divisors $\{X\}$ is complete, then any algebraic family of positive V-divisors, the divisors of which are algebraically equivalent to a divisor in $\{X\}$, must be a Subfamily of $\{X\}$. That is, the totality of positive V-divisors which are algebraically equivalent to X is a Variety (absolutely irreducible) as a totality of divisors (cf. th. 2). On a non-singular projective Curve, a maximal algebraic family of positive divisors is complete when the degree of it is sufficiently large by the theorem of Riemann-Roch. Let U be the associated-Variety of an algebraic family of positive V-divisors $\{X\}$ and k a field of definition for U. Then we say that $\{X\}$ is *defined over k* or *k is a field of definition* of $\{X\}$. Let X be a divisor in $\{X\}$ such that its Chow-Point is a generic Point of U over k. When that is so, X is said to be a *generic divisor* of $\{X\}$ over k.

Denote by L_s the linear system on V induced by all the positive divisors of degree s on the ambient projective space of V and by C_s a special divisor in L_s. A maximal algebraic family $\{X\}$ of positive V-divisors is said to be *ample* when $X - C_s \sim Y > 0$ for large but fixed value of s, whenever X is in $\{X\}$. Let X be a positive V-divisor, then we denote by $|X|$ the complete linear system determined by X. We say that $|X|$ is defined over a field k, when $|X|$ contains a rational divisor over k. When that is so, there is a rational divisor Z over k on $L^s \times V \left(s = l(X) - 1\right)$ such that $|X|$ is the totality of V-divisors of the form $pr_V[(u \times V) \cdot Z] = Z(u)$, which we show in some details in no. 5, § 3.

Let U and W be two arbitrary Varieties and Z a Subvariety of $U \times W$ such that the projection of Z on U is regular. Following A. Weil (cf. [W-2]-ch. I), we say that Z is the graph of the function g and

when $x \times y$ is a Point of Z such that the projection of Z on U is regular at x, we denote $g(x)=y$. When k is a common field of definition for U, W and Z, we say that g is defined over k. g is said to be the function defined on U with values on W. Let $\mathfrak{F}$ be a bunch on $U \times W$. By $\text{proj}_U \mathfrak{F}$, we denote the "geometric" projection of $\mathfrak{F}$ on U.

$$\S\, 1.$$

1. Throughout this and the following $\S\S$, we always assume that V is non-singular.

LEMMA 1. *Let X be a positive V-divisor. There is a positive integer m_0 such that for every integer $m \geqq m_0$, $|X+C_m|$ has no base Point. The integer m_0 depends only on the degree of X.*

PROOF. There is a positive integer such that

$$X \sim C_t - Y, \qquad Y \succ 0.$$

We can find such an integer by the method of projecting Cones (cf. [v. d. Waerden]) and the projecting cone of X has the same order as X. Hence $t=\deg(X)$ in this case. Let Q be a Point on V. One can choose Y in such a way that every component of it does not contain Q. Again, applying the method of projective cones, one can find an integer such that

$$Y \sim C_s - Z, \qquad Z \succ 0,$$

where Z may be assumed to contain no component containing Q. We have also $s=\deg(Y)=\deg(C_t)-\deg(X)=\deg(V)\cdot\deg(X)-\deg(X)$ by the theorem of Bézout. It follows that

$$X + C_s \sim C_t + Z$$

and from what we have seen above, $X+C_s$ is linearly equivalent to a positive V-divisor not going through Q. Since Q was the preassigned Point, this proves our lemma when we put $m_0=s$, together with the relation $s=\deg(V)\cdot\deg(X)-\deg(X)$.

Now we shall prove the following lemma, which is trivial in the classical case by the theorem of Bertini.

LEMMA 2. *Let X be a V-divisor such that $|X-C_1|$ exists and that $|X|$ has no base Point. Then a generic divisor of $|X+C_1|$ is a nonsingular Variety.* (cf. [Z-1] and [A])

PROOF. Let K be an algebraically closed field of definition for V and $|X|$, V a representative of V and $(x)=(x_0, x_1, \cdots, x_N)$ a generic point of V over K. By our assumption, $|X|$ contains partially C_1 and hence, it cannot be composite with a pencil. This proves that a generic divisor of $|X|$ over K is a Variety by the generalized theorem of Bertini (cf. [Z-2] and [M-1]). Therefore, there is a divisor X_0 in $|X|$ which is a Variety defined over K.

Let $\varphi_0=1$, $\varphi_1, \cdots, \varphi_m$ be a base of the module $L(X_0)$, consisting of functions, which are multiples of X_0; we may assume that they are all defined over K (cf. [W-1], th. 10, ch. VIII). Put $(\varphi_i)=X_i-X_0$. Then by what we have observed above, we may assume that X_i is a Variety and hence

$$(\varphi_i)_0=X_i, \qquad (\varphi_i)_\infty=X_0 \qquad i=1, 2, \cdots, m.$$

We can find a set of functions $\{\varphi_{m+1}, \cdots, \varphi_n\}$ in $L(X_0)$ all defined over K such that $(\varphi_{m+i})=X_{m+i}$ is a Variety, $(\varphi_{m+i})_0=X_{m+i}$, $(\varphi_{m+i})_\infty=X_0$ and that $\bigcap_{i=0}^{n} X_i=\emptyset$.

Consider the linear system generated by the set of functions $\{\varphi_i \tau_j\}$, $0 \leq i \leq n$, $0 \leq j \leq N$ (assume that $x_0=1$) where τ_j is the function defined by $x_j=\tau_j(x)$. We may consider that $(\varphi_i \tau_j)_0=(\varphi_i)_0+(\tau_j)_0$, $(\varphi_i \tau_j)_\infty=(\varphi_i)_\infty+(\tau_j)_\infty$ as X_j is a Variety and $|X-C_1|$ exists. Put $v_i=\varphi_i(x)$ and denote by V' the locus of $(v_i x_j)$ over K. V' determines a projective Variety V' defined over K such that V' is a representive of V'. It is clear that V and V' are birationally equivalent over K by the correspondence induced by $(x) \rightleftarrows (v_i x_j)$. We shall show that the birational correspondence is everywhere biregular. It is regular at every Point on V', which can be easily verified from the structure of V'. Let Q be a Point on V such that it has a representative $Q_\alpha=(a)$ on a representative V_α of V. A generic Point of V_α may be assumed to be $P_\alpha=(1/x_\alpha, \cdots, x_i/x_\alpha, \cdots, x_N/x_\alpha)$. By the choice of X_i, there is an integer β such that $Q \notin X_\beta$. Let $V'_{\alpha\beta}$ be a representative of V', whose generic point over K is $(v_i x_j/v_\beta x_\alpha)$. We have

$(v_i \, x_j/v_\beta \, x_\alpha) = (v_i \, x_j) - (v_\beta \, x_\alpha) = (v_i) - (v_\beta) + (x_j) - (x_\alpha) = X_i - X_\beta + (x_j/x_\alpha)$
and hence $(v_i \, x_j/v_\beta \, x_\alpha) \prec X_\beta + (x_j/x_\alpha)_\infty$. x_j/x_α is in the specialization-ring of Q_α in $K(P_\alpha)$ and $Q \notin X_\beta$, that is, $Q \notin (v_i \, x_j/v_\beta \, x_\alpha)$. Since this holds for every choice of i, j, this proves that the birational correspondence is regular at Q on V, i. e., it is everywhere biregular.

Let $(\lambda) = (\lambda_{ij})$ be a set of independent variables over $K(P_\alpha)$ and consider the divisor $(\sum_{i=0}^{n} \sum_{j=0}^{n} \lambda_{ij} \, v_i \, x_j)_0$ on V. This is transformed to a generic hyperplane section of V' over K. But as we have seen above, V' has no singular Point since it is in everywhere biregular birational correspondence with V (cf. [W-1], th. 17, ch. IV). Therefore a generic hyperplane section has no singular Point (cf. [N]). Hence the divisor considered is a Variety having no singular Point. Thus, we have proved that the complete linear system $|X + C_1|$ contains a non-singular Variety. Now our lemma follows from the following general lemma.

LEMMA 3. *Let Γ be a non-singular Curve and X a divisor on the product $\Gamma \times V$, such that every component of X has the projection Γ on Γ. Let K be a common field of definition for Γ and V over which X is rational, u a generic Point of Γ over K and v a Point of Γ. When $pr_V[(v \times V) \cdot X] = X(v)$ is a non-singular Variety, $X(u)$ is also a non-singular Variety.*

PROOF. Since every component of X has the projection Γ on Γ, the intersection-product $(v \times V) \cdot X$ is defined on $\Gamma \times V$ and is of the form $v \times X(v)$ by [W-1], prop. 16, ch. VII. When v is a generic Point of Γ over K, we have nothing to prove. Hence we may assume that v is algebraic over K. $X(v)$ is a specialization of $X(u)$ over $u \to v$ with reference to K (cf. [M-2] and [S]) and since $X(v)$ is a Variety, $X(u)$ must be a Variety.

Now let x be a multiple Point on $X(u)$, and extend the specialization $\big(u, \, X(u)\big) \to \big(v, \, X(v)\big)$ to a specialization

$$\big(u, \, X(u), \, x\big) \to \big(v, \, X(v), \, x'\big)$$

over K. It is clear that x' is a Point of $X(v)$. Let L'^N be the dual projective space of the ambient space L^N of V^r and $W = L' \times \cdots \times L'$ be the product of $r-1$ factors equal to L'. The hyperplanes passing through x are represented in L' as the Points of the hyperplane $H(x)$

in the usual manner. Then the linear Varieties L^{N-r+1} passing through x are represented as the Points on the product $\underbrace{H(x) \times \cdots \times H(x)}_{r-1}$ on W.

In the same way the linear Varieties of dimension $N-r+1$ passing through x' are represented as the Points on $\underbrace{H(x') \times \cdots \times H(x')}_{r-1}$. More-over, $H(x') \times \cdots \times H(x')$ is a specialization of $H(x) \times \cdots \times H(x)$ over K. This proves that the linear Variety of dimension $N-r+1$ passing through x' is a specialization of a certain linear Variety of dimension $N-r+1$ passing through x over $x \to x'$ with respect to K.

Let L^{N-r+1} be any linear Variety passing through x such that x is a proper intersection of L^{N-r+1} and $X(u)$ on L^N. Then, since x is multiple on $X(u)$, we have

$$i\left(X(u) \cdot L^{N-r+1}, x : L^N\right) \geq 2$$

by [W-1], th. 5, ch. V. One can find L^{N-r+1} in such a way that $X(u) \cdot L^N$ is defined on L^N. In fact, let $(\xi_0, \cdots, \xi_N)$ be a coordinate of x on some representative V of V in L^N and $w_1, \cdots, w_N$ be N indepen-dent variables over $K(u, \xi)$. Assume that $\xi_0 = 1$ for simplicity. Every hyperplane in L^N passing through (ξ) is a specialization of the hyper-plane defined by

$$w_1 X_1 + \cdots + w_N X_N = w_1 \xi_1 + \cdots + w_N \xi_N.$$

Let U be any Variety defined over $K(u, v, \xi)$, containing (ξ). When that hyperplane contains U, the one defined by

$$X_i = \xi_i \qquad (i = 1, 2, \cdots, N)$$

contains U as it is a specialization of the former over K. But this is impossible when dim. $U \geq 1$. This proves that a generic Point of $H(x)$ over $K(u, x)$ determines on L^N a hyperplane L_1, passing through x such that $X(u) \cdot L_1$ is defined on L^N. Let $L_1, \cdots, L_{r-1}$ be hyperplanes on L^N corresponding to $r-1$ independent generic Points of $H(x)$ over $K(u, x)$. Then the above arguments show that $X(u) \cdot L_1 \cdots L_{r-1}$ is de-fined on L^N. The same holds for $X(v)$ and x'. Let L^{*N-r+1} be a linear Variety passing through x', corresponding to a generic Point of $H(x') \times \cdots \times H(x')$ over $K(v, x')$. Then $L^* \cdot X(v)$ is defined on L^N and

$$i\left(X(v) \cdot L^*, x' : L^N\right) = 1$$

since $X(v)$ is non-singular (cf. [W-1]-th. 5, ch, V). As we have re-marked above, L^* is a specialization of a certain L^{N-r+1} passing through x over $(X(u), x) \rightarrow (X(v), x')$ with reference to K. When that is so, $X(u) \cdot L^{N-r+1}$ is defined. Moreover, $X(v) \cdot L^*$ is the uniquely determined specialization of $X(u) \cdot L^{N-r+1}$ over $(X(u), x, L^{N-r+1}) \rightarrow (X(v), x', L^*)$ with reference to K by [S]-th. 2, ch. VI and [M]-th. 2. This proves that

$$i\left(X(u) \cdot L^{N-r+1}, x : L^N\right) = 1$$

This is a contradiction and hence x must be simple on $X(u)$. q. e. d.

Corollary to Lemma 3. *Let X be a positive V-divisor. Then, there is a positive integer m_0 such that for every integer $m \geq m_0$, $|X + C_m|$ exists, and a generic divisor of it over a common field of definition for V and $|X + C_m|$ is a non-singular Variety.*

Now we shall make the following remark. In the proof of lemma 3, we have applied to V an everywhere biregular birational transform-ation, transforming V to a non-singular Variety V' and transforming a certain complete linear system on V to the complete linear system on V' determined by the hyperplane section C_1' of it. Such a birational transformation always exists for a given complete linear system on V, which transforms it to the complete linear system determined by the hyperplane sections, wherever it is sufficiently ample by virtue of lemmas 1 and 2. Now given a Variety W on V and a sufficiently ample complete linear system $|X|$ on V. Let X be a generic divisor of $|X|$ over a common field of definition for V, W and $|X|$. Then the above arguments show that $X \cdot W$ is defined on V by [W-1]-th. 1 ch. V and [M-1], and is a Variety. Therefore we have

Lemma 4. *Let Y be a V-cycle and $|X|$ a sufficiently ample com-plete linear system on V. Let X be a generic divisor of $|X|$ over a common field of definition for V and $|X|$ over which Y is rational. When $Y = \sum a_i W_i$ is the reduced expression for Y, $W_i \cdot X$ is defined on V, is a Varitey and $\sum a_i W_i \cdot X$ is the reduced expression for a X-cycle $Y \cdot X$.*

2. We are now in position to prove the invariance of the notion

of virtual arithmetic genus of divisors on a non-singular projective Variety with respect to algebraic equivalence. For the definition of the virtual arithmetic genus of cycles, see [Z-3]. Denote by $\{A\}$ a maximal regular algebraic family of positive V-divisors, the existence of which is proved in [M-4]-th. 1.

THEOREM 1. *Let X and X' be two V-divisors such that they are algebraically equivalent to each other. Then the virtual arithmetic genus of X and X' are the same.*

PROOF. This theorem is certainly true for the divisors on non-singular projective Curve. We proceed by induction on the dimension of the ambient Variety. Hence we assume that the theorem is proved already for non-singular projective Varieties of dimension $\leq r-1$. The virtual arithmetic genus must satisfy the modular property

$$p_a(D^{r-1}+C_m)=p_a(D)+p_a(C_m)+p_a(D\cdot C_m)$$

whenever $D\cdot C_m$ is defined by [Z-3], part III. Therefore, by our induction assumption, we may assume that X and X' are both positive V-divisors. (cf. [Z-4], Part III). Moreover, we may assume that X and X' belong to one and the same maximal algebraic family of positive V-divisors $\{X\}$.

Let K be a common field of definition for V, $\{A\}$ and $\{X\}$ over which X and X' are rational. By lemma 2, both $|X+C_s|$ and $|X'+C_s|$ contain non-singular Varieties for sufficiently large s, which we denote by Y and Y'. Let A and $\overline{X}$ be independent generic divisors of $\{A\}$ and $\{X\}$ over K and $\{C_s+\overline{X}+A\}$ be a maximal algebraic family of positive V-divisors containing $C_s+\overline{X}+A$ as its divisor. It can be easily seen that it is also a regular family, and is defined over $\overline{K}$. Moreover, it contains C_s+X+A and $C_s+X'+A$ as its divisors and they are linearly equivalent to generic divisors Z, Z' of it over $\overline{K}$ which can be easily verified from the structure of the Picard Variety of V (cf. [M-4], § 4). Therefore

$$C_s+X+A\sim Z, \qquad C_s+X'+A\sim Z'.$$

Since Z and Z' are generic specializations of each other over $\overline{K}$, we have

$$p_a(Z) = p_a(Z').$$

This shows that $p_a(Y+A) = p_a(Y'+A)$. By lemma 2, A may be assumed to be non-singular, and both $|Y+A|$, $|Y'+A|$ contain non-singular Varieties W and W', since $\{A+C_n\}$ is regular for every n as long as $\{A\}$ is regular. Then $p_a(W) = p_a(Y+A) = p_a(Y'+A) = p_a(W')$ by [Z-4], part III. Moreover, we may assume that there is an everywhere biregular birational correspondence T between V and a non-singular $\bar{V}$ such that $|A|$ is transformed to the complete linear system on $\bar{V}$ formed by a hyperplane sections of it by the remark at the end of no. 1. Since $\bar{V}$ is non-singular and T is everywhere biregular, $p_a(W) = p_a(T(W))$, $p_a(W') = p_a(T(W'))$ by [M-Z]-th. 2 and [W-1]-th. 17, ch. IV. Moreover, it holds $T(W) \sim T(Y)+T(A)$, $T(W') \sim T(Y')+T(A)$ by [W-1]-th. 7, VIII. This implies that $p_a(T(Y)+T(A)) = p_a(T(W))$, $p_a(T(W')) = p_a(T(Y')+T(A))$ by [Z-3], part III. Now replacing $T(A)$ by a generic hyperplane section C of $\bar{V}$ over a common field of definition for V, T, $T(Y)$ and $T(Y')$ containing K, we have

$$p_a(T(Y)+T(A)) = p_a(T(Y)+C)$$

$$= p_a(T(Y)) + p_a(C) + p_a(T(Y) \cdot C),$$

$$p_a(T(Y')+T(A)) = p_a(T(Y')+C)$$

$$= p_a(T(Y')) + p_a(C) + p_a(T(Y') \cdot C).$$

Since Y and Y' are algebraically equivalent to each other, $T(Y)$, $T(Y')$ are also algebraically equivalent to each other. Hence $T(Y) \cdot C$ and $T(Y') \cdot C$ are also algebraically equivalent to each other. We have $p_a(T(Y) \cdot C) = p_a(T(Y') \cdot C)$ by our induction assumption and hence $p_a(T(Y)) = p_a(T(Y'))$. Since Y and Y' are non-singular, and T is everywhere biregular, we have $p_a(Y) = p_a(Y')$ by [W-1]-cor. 4, th. 17, ch. IV and [M-Z]-th. 2. This shows that $p_a(C_s+X) = p_a(C_s+X')$ and from the modular property of the arithmetic genus,

$$p_a(C_s) + p_a(X) + p_a(X \cdot C_s) = p_a(C_s) + p_a(X') + p_a(C_s \cdot X')$$

whenever C_s is generic over K. Since C_s is non-singular by [N], and $X \cdot C_s$, $X' \cdot C_s$ are algebraically equivalent to each other, $p_a(X \cdot C_s)$ $= p_a(X' \cdot C_s)$ by our induction assumption. This proves our theorem.

<h2 style="text-align:center">§ 2.</h2>

LEMMA 5. *Let* U *be a non-singular abstract Variety and* W *be a non-singular projective Variety ; let* Z *be a positive* $U \times W$*-cycle. There is a frontier* $\widetilde{\mathfrak{F}}$ *on* U *such that, the fact that a Point* x *on* U *is not contained in any component of* $\widetilde{\mathfrak{F}}$*, is the necessary and sufficient condition for* $Z \cdot (x + W)$ *to be defined.*

PROOF. We may assume that Z is a Variety and reduce this lemma to the case when Z is the graph of the function. Let K be a common field of definition for U, W and Z and x^* the Chow-Point of $Z(x)$ defined by $Z(x \times W) = x \times Z(x)$. $Z(x)$ is rational over $K(x)$ and hence x^* is rational over it. Therefore, x^* has the Locus U^* over K and $g(x) = x^*$ is the function defined on U with values on U^*. Let x' be a Point on U at which g is defined. Then x^* has the uniquely determined specialization $x^{*\prime}$ over $x \to x'$ with reference to K (cf. [W-2]-no. 1, ch. I). This shows that $Z \cdot (x' \times W)$ is defined. Assume conversely that $Z \cdot (x' \times W) = x' \times Z(x')$ is defined, then $Z(x')$ is the uniquely determined specialization of $Z(x)$ over $x \to x'$ with reference to K by [M-2] prop. 2., i. e., x^* has the uniquely determined specialization $x^{*\prime}$ over $x \to x'$ with reference to K. When that is so, if $\mathit{\Omega}$ is the graph of g

$$(x' \times U^*) \frown \mathit{\Omega}$$

reduces to a component $x' \times x^{*\prime}$. As U is non-singular, the projection of $\mathit{\Omega}$ on U is regular at x', by the main theorem of birational transformations (cf. [Z-4]-main th.) and g is defined at x'. Thus, the fact that $Z \cdot (x' \times W)$ is defined and the fact that g is defined at x' are equivalent. Hence we may assume that Z is the graph of the function and this case is reduced to the case when Z is the graph of the numerical function, returning to the definitions of function (cf. [W-2]-no. 1, ch. I). Then our lemma follows from [W-1]-cor. 2, th. 1, ch. VIII.

LEMMA 6. *Let $\{X\}$ be a regular maximal algebraic family of positive V-divisors. There is an integer m_0 such that for every integer $m \geq m_0$ we have*

$$l(X + C_m) = l(X' + C_m)$$

for every X, X' in $\{X\}$.

PROOF. Let K be a common field of definition for V^r, the Picard Variety P of V and for $\{X\}$ and X a generic divisor of it over K. Let x be the Chow-Point of X and C_n a generic divisor of L_n over $K(x)$.

Let W be the associated-Variety of $\{X\}$. We shall show that, there is a frontier $\mathfrak{F}_n$ on W algebraic over K for the given integer n such that

$$l(C_n + X) = l(C_n + X')$$

when the Chow-Point of X' is on $W - \mathfrak{F}_n$, and

$$l(C_n + X) < l(C_n + X')$$

otherwise. There is a finite number of maximal algebraic families of positive V-divisors of the same degree as $\{X\}$, whose divisors are mutually algebraically equivalent. Denote the associated-Variety of them by W', W'', $\cdots$. They are clearly defined over $\overline{K}$, and $W \frown W'$, $W \frown W''$, $\cdots$ are algebraic over K. Put $\mathfrak{F}' = (W \frown W') \smile (W \frown W'') \cdots$. Let X_0 be a fixed rational divisor of $\{X\}$ over $\overline{K}$ and ξ the class of $X - X_0$ on P. We may assume that ξ is rational over $\overline{K}(x)$ by [M-4]-th. 3 and when that is so, there is a function g defined on W with values on P such that $g(x) = \xi$. Let Z be the graph of g, i.e., the Locus of $\xi \times x$ over $\overline{K}$. By [M-4]-prop. 10, and its corollary, $Z(\xi) = pr_W[Z \cdot (\xi + L)]$ is the associated-Variety of $|X|$, where L is the ambient projective space of W. There is a bunch $\overline{\varPsi}$ algebraic over K on P such that

$$(\xi' \times L) \cdot Z = \xi' \times Z(\xi')$$

is defined when ξ' is on $P - \overline{\varPsi}$ and is not defined when ξ' is on some component of $\overline{\varPsi}$ by lemma 5. Put

$$\mathfrak{F}'' = \mathrm{proj}_L[(\overline{\varPsi} \times L) \frown Z],$$

$$\mathfrak{F} = \mathfrak{F}' \smile \mathfrak{F}''$$

We shall prove that when x' is on $W-\mathfrak{F}$, the corresponding divisor X' is such that $l(X')=l(X)$ and when x' is on $\mathfrak{F}$, $l(X)<l(X')$. Let $T(X)$ and $T(X')$ be associated-Varieties of $|X|$ and $|X'|$. We have $Z(\xi)=T(X)$ by [M-4]-cor., prop. 10. Let $\left(X', T(X)'\right)$ be a specialization of $\left(X, T(X)\right)$ over $\overline{K}$ on ambient projective spaces. Since specializations of divisors of functions are also divisors of functions (cf. [M-4]-lemma 3 and [W-2]-lemma 10) every component of $T(X)'$ is contained in $T(X')$. As x' is not on $\mathfrak{F}''$, and as $(P\times x')\frown Z$ reduces to a component $\xi'\times x'$ by [M-4]-prop. 10, ξ' is not on the frontier $\overline{\Psi}$ and ξ' is the class of $X'-X_0$ on P by [M-4] prop. 10 and th. 1. By the definition of $\overline{\Psi}$,

$$(\xi'\times L)\cdot Z=\xi'\times Z(\xi')$$

is defined and $Z(\xi')$ is the uniquely determined specialization of $Z(\xi)$ over $\xi\to\xi'$, that is, over $x\to x'$ with reference to $\overline{K}$ by [M-2]-th. 2 and [S]-th. 2, ch. VI. Therefore $Z(\xi')=T(X)'$ and from the property of Z, $W\frown T(X')$ and $T(X)'$ coincide from the point set theoretical point of view. (cf. [M-4]-prop. 10 and its corollary). There is a certain $W^{(j)}$ such that $T(X')$ is the Subvariety of it. Then x' is in $W\frown W^{(j)}$ and so $W^{(j)}=W$ as $x'\notin\mathfrak{F}'$. This proves that $T(X')$ is a component of $T(X)'$ and so $\dim T(X')=\dim T(X)'=\dim T(X)$. Therefore $l(X)=l(X')$.

Assume that x' is on $\mathfrak{F}'$ and assume further, that it is a generic Point of its component $\overline{W}$ over $\overline{K}$. There is a certain $W^{(i)}$ such that $\overline{W}$ is a component of $W\frown W^{(i)}$. As we have seen above, $T(X)'$ is contained in $T(X')$. Assume that $\dim T(X)'=\dim T(X')$ then, there is a positive integer α such that $T(X)'=\alpha\cdot T(X')$. Since $T(X)'$ is on W, $T(X')$ is on W. Let x_i be a generic Point of $W^{(i)}$ over $\overline{K}$, corresponding to X_i. Since $\{X\}$ is regular, there is a divisor X_i' in $\{X\}$ such that $X_i-X_0\sim X_i'-X_0$, i. e., $X_i\sim X_i'$ by [M-4]-th. 1. Hence $l(X_i)\geq l(X)$. As $\dim T(X)=\dim T(X)=\dim T(X')$, we have $l(X)=l(X')$, and as X' is a specialization of X_i over $\overline{K}$, $l(X_i)\leq l(X')=l(X)$. This proves that $l(X)=l(X_i)=l(X_i')$. But when that is so, we can easily see that $|X_i'|=|X_i|$ is totally contained in $\{X\}$, and this shows that $W\succ W^{(j)}$. Therefore we must have $l(X')>l(X)$.

Assume that x' is on $\mathfrak{F}''$. Then ξ' is on $\overline{\Psi}$ and hence

$$(\xi' \times L) \frown Z$$

has a component $\xi' \times \overline{W}$ having the greater dimension than $\xi \times Z(\xi)$ $= \xi \times T(X)$. From the property of Z (cf. [M-4]-prop. 10 and its corollary), every divisor whose Chow-Point is on $\overline{W}$ is mutually linearly equivalent. Therefore, $\overline{W}$ is the Subvariety of $T(X')$ i. e., $l(X') > l(X)$.

Now we shall prove the following lemma, known as a lemma of Castelnuovo in the classical case of algebraic surfaces. Let D be a positive divisor on V, rational over a field k and $C = C_1$ be a generic divisor of L_1 over k. Then for sufficiently large h, $|D + h C|$ induces on C a complete linear system. For

$$\dim(Tr_C |D + h C|) = (-1)^r \{p_a(V) + p_a(-D - h C) - p_a(V)$$

$$-p_a(-D-(h-1)C)\} - 1 = (-1)^r \{p_a(-D-h C)$$

$$-p_a(-D-(h-1)C)\} - 1$$

when h is sufficiently large by Zariski's theorem (cf. [Z-3]-th. 5). Moreover, when h is sufficiently large, we have

$$\dim|D \cdot C + h C^2| = (-1)^{r-1}\{p_a(C) + p_a(-D \cdot C - h C^2)\} - 1$$

by the same theorem. But from the modular property of the arithmetic genus, we have

$$p_a(-D \cdot C - h C^2) = p_a(-D-(h-1)C) - p_a(C) - p_a(-D-h C)$$

and hence

$$\dim|D \cdot C + h C^2| = (-1)^{r-1}\{p_a(D-(h-1)C) - p_a(-D-h C)\} - 1.$$

This proves our assertion.

We complete our proof with induction on r. Assume that our lemma is proved already for non-singular projective Varieties of dimensions less than r. Let C be a generic divisor of L_1 over $K(x)$ and h_0 a positive integer such that $|X + h C|$ induces on C a complete linear system when $h > h_0$ and that every divisor in a maximal algebraic family of C-divisors containing $X \cdot C$ has also the property enunciated in our lemma, when $h > h_0$. Let $\widetilde{\mathfrak{F}}_h$ be the bunch on W already

defined and x' be a Point on $W - \widetilde{\mathfrak{F}}_h$ corresponding to X' such that $X' \cdot C$ is defined. We have

$$\dim \left(Tr_C | X + (h+1) \, C | \right) = \dim \left(| X + (h+1) \, C | \right) - \dim \left(| X + h \, C | \right)$$

$$\dim \left(Tr_C | X' + (h+1) \, C | \right) = \dim \left(| X' + (h+1) \, C | \right) - \dim \left(| X' + h \, C | \right)$$

By our choice of h, we have $\dim \left(Tr_C | X + (h+1) \, C | \right) \geqq \dim \left(Tr_C | X' + (h+1) \, C | \right)$ and as $x' \notin W - \widetilde{\mathfrak{F}}_h$, it holds

$$\dim | X + h \, C | = \dim | X' + h \, C |$$

Therefore, $\dim | X + (h+1) \, C | > \dim | X' + (h+1) \, C |$. This proves that it must hold the following equality

$$\dim | X + (h+1) \, C | = \dim | X' + (h+1) \, C |$$

Repeating this, we conclude that

$$\dim | X + h \, C | = \dim | X' + h \, C |$$

for any $h > h_0$ whenever $x' \in W - \widetilde{\mathfrak{F}}_h$.

Let $\widetilde{\mathfrak{F}}_h = \cup \, U_i$ and x_i be a generic Point of U_i over $\bar{K}$, corresponding to X_i. One can find a positive integer $h_1 > h_0$ such that for integers $t \geqq h_1$ we have

$$l (C_t + X) = (-1)^r \{ p_a (V) + p_a (-C_t - X) \}$$

$$l (C_t + X_i) = (-1)^r \{ p_a (V) + p_a (-C_t - X_i) \}$$

by Zariski's theorem. By th. 1, we get $l (C_t + X) = l (C_t + X_i)$. When the Chow-Point of X' is on $W - \widetilde{\mathfrak{F}}_{h_1}$

$$l (C_t + X) = l (C_t + X') = l (C_t + X_i) \, .$$

This shows that the frontier $\widetilde{\mathfrak{F}}_{h_1}$ corresponding to h_1 is such that every component of it is properly contained in a component of $\widetilde{\mathfrak{F}}_{h_0}$. Therefore when h is sufficiently large, $\widetilde{\mathfrak{F}}_h$ will be empty, which completes the proof of our lemma, since our lemma holds on non-singular Curves by virtue of the theorem of Riemann-Roch.

3. Now we shall state and prove the main theorem of this §.

THEOREM 2. *Let* $\{X\}$ *be a regular maximal algebraic family of*

positive divisors on V, $\{Y\}$ and $\{X\}$ a maximal algebraic family of positive V-divisors such that X and Y are algebraically equivalent to each other. When $\{X\}$ and $\{Y\}$ are distinct, $\{Y\}$ cannot be a regular family.

When $\{X\}$ is any maximal algebraic family of positive divisors of V, there is a positive integer m'_0 independent of X such that for integers $m > m'_0$, there is a maximal algebraic family $\{X+C_m\}$ containing $X+C_m$ for arbitrary X in $\{X\}$, such that it is regular.

Moreover, there is a positive integer m_0, such that for any integer $m > m_0$, a maximal algebraic family $\{X+C_m\}$ is complete and regular. But in this case m_0 depends on $\{X\}$.

PROOF. We prove our first assertion. Let K be a common field of definition for V, the Picard Variety P of V, $\{X\}$ and for $\{Y\}$. Let W, U be associated-Varieties of $\{X\}$, $\{Y\}$ and X_0, X'_0 be fixed divisors of $\{X\}$, $\{Y\}$ such that $X_0 \sim X'_0$. We assume that K contains the coordinates of the Chow-Point of X_0 and that $\{Y\}$ is also a regular family. Let y be a generic Point of U over K, corresponding to the divisor Y. Since $\{Y\}$ is regular, there is a function g defined on U with values on P defined over K and $g(y) = \eta$ is a generic Point of P over K, which is the class of $Y - X_0$ on P with respect to linear equivalence (cf. [M-4]-prop. 10 and th. 2). In the same way, there is a function h defined on W with values on P, having the same property as g. As $\{X\}$ is regular, there is a divisor X in $\{X\}$ such that

$$Y - X_0 \sim X - X_0, \quad \text{i. e.,} \quad Y \sim X.$$

This proves that, if Z is the graph of h and x is the Chow-Point of X, the intersection $(x \times P) \frown Z$ reduces to $x \times \eta$ (cf. [M-4]-prop. 10). Therefore x is a Point of $Z'(\eta)$ defined by $(W \times \eta) \cdot Z = Z'(\eta) \times \eta$ which is clearly defined on $W \times P$ since η is a generic Point of P over K (cf. [W-1]-th. 12, ch. VIII). Moreover, from the property of h, $Z'(\eta)$ is the associated-Variety of $|X|$ by [M-4]-cor. of prop. 10 and prop. 7, and $Z'(\eta)$ contains y. But this is a contradiction to our hypothesis that $\{Y\}$ and $\{X\}$ are distinct. This proves our first assertion. (cf. [Z-5]-no. 3, ch. V).

Now let $\{A\}$ be any regular maximal algebraic family of positive V-divisors and $\{X\}$ an empty or arbitrary maximal algebraic family of positive V-divisors. Let K be a common field of definition for V, $\mathfrak{F}$,

$\{A\}$ and $\{X\}$ and A a generic divisor of $\{A\}$ over K. One can find a positive integer m_0' such that for any integer $m > m_0'$, L_m is complete and contains partially A (hence, a fortiori, special divisors of $\{A\}$). Let A_0 be a divisor in $\{A\}$ rational over $\bar{K}$ and X_0 a rational divisor of $\{X\}$ over $\bar{K}$. The divisor

$$X_0 + C_m + A - A_0$$

is linearly equivalent to a positive divisor Y since L_m contains partially A_0, i. e. $X_0 + C_m + A \sim Y + A_0$. It can be easily seen that a maximal algebraic family $\{X_0 + C_m + A\}$ containing $X_0 + C_m + A$ is regular, and $X_0 + C_m + A$ is linearly equivalent to a generic divisor of it over $\bar{K}$ by the same reasoning as above. Consider a maximal algebraic family $\{Y\}$ containing Y. We have

$$X_0 + C_m - Y \sim A_0 - A$$

and $X_0 + C_m$ may be assumed to be rational over $\bar{K}$. Since $X_0 + C_m$ is algebraically equivalent to Y, we conclude that $\{Y\}$ is a regular family.

By lemma 6, there is a positive integer $m_0 \geq m_0'$ such that for any integer $m > m_0$ we have $l(Y + C_m) = l(Y' + C_m)$ for any Y' in $\{Y\}$. We shall prove that when D is any positive V-divisor algebraically equivalent to $Y + C_m$, D is a divisor of a maximal algebraic family containing $Y + C_m$. Let W be the associated-Variety of $\{Y + C_m\}$. We can easily see that it is defined over $\bar{K}$ and $Y + C_m$ is linearly equivalent to a generic divisor $\overline{Y}$ of $\{Y + C_m\}$ over $\bar{K}$. Let Y_0 be a rational divisor of $\{Y\}$ over $\bar{K}$. There is, as before, a function g defined on W with values on P defined over $\bar{K}$, and if $\bar{y}$ is the Chow-Point of $\overline{Y}$, it is a generic Point of W over $\bar{K}$ and $g(\bar{y}) = \eta$ is the class of $\overline{Y} - (Y_0 + C_m)$ on P with respect to linear equivalence: let Z be the graph of g, then $Z(\eta)$ defined by $(\eta \times W) \cdot Z = \eta \times Z(\eta)$ is the associated-Variety of $|\overline{Y}|$ by [M-4]-cor., prop. 10. th. 2 and prop. 7. We have $\dim Z(\eta) = l(\overline{Y}) - 1$ and as $\overline{Y} \sim Y + C_m$, $\dim Z(\eta) = l(Y + C_m)$ $-1 = l(Y' + C_m) - 1$ for any Y' in $\{Y\}$. Denote by η' the class of $D - (Y_0 + C_m)$ on P with respect to linear equivalence, and consider the intersection

$$(\eta' \times W) \frown Z.$$

When $\eta' \times \bar{y}'$, $\eta' \times \bar{y}''$ are in that intersection, the divisors $\overline{Y}'$, $\overline{Y}''$ corresponding to $\bar{y}'$, $\bar{y}''$ are mutually linearly equivalent by [M-4]-th. 1, and cor. prop. 10. Hence every component of that intersection is contained in $\eta' \times T(D)$ where $T(D)$ is the associated-Variety of $|D| = |\overline{Y}|$. On the other hand, every component of a specialization $\eta' \times Z(\eta)'$ of $\eta \times Z(\eta)$ over $\eta \to \eta'$ with reference to $\overline{K}$ is contained in a component of $(\eta' \times W) \frown Z$: moreover, $Z(\eta)'$ is contained in $T(D)$ by [M-4]-lemma 3 and cor. prop. 10. But we have $\dim Z(\eta) = \dim Z(\eta') = l(Y + G_m) - 1 = l(D) - 1 = \dim T(D)$; since $D \sim Y' + C_m$ for a certain Y' in $\{Y\}$. Hence $Z(\eta') = \alpha \cdot T(D)$ for a certain positive integer α and this shows that $T(D) < W$. This completes our proof.

REMARK. Let X be a positive V-divisor and $\{A\}$ a regular maximal algebraic family of positive V-divisors. Let A be a generic divisor of $\{A\}$ over an algebraically closed common field of definition for V, $\{A\}$ over which X is rational, and A_0 a rational divisor of $\{A\}$ over K. When $l(X + A - A_0) \geq 1$, i. e., $|X + A - A_0|$ exists and non-empty, we can prove the existence of a regular maximal algebraic family containing X as its divisor, in the same way as in the proof of the above theorem. Conversely, when X belongs to a maximal regular family, then we must have $l(X + A - A_0) \geq 1$. In this case, $l(X + A' - A_0) \geq 1$ for any A' in $\{A\}$ since A' is a specialization of A over K. Hence, in order that X belongs to maximal regular algebraic family, it is necessary and sufficient that $l(X + A' - A_0) \geq 1$ for any A' in $\{A\}$. Moreover, when that is so, $l(X + Y) \geq 1$ for any divisor Y, which is algebraically equivalent to zero. A part of the above theorem is a special case of this. Severi and Italian geometers called a complete linear system $|X|$ on a non-singular surface in a projective space as " arithmetically effective ", when

$$I(X \cdot X) - \pi_X + p_a + 1 - l(K - X) \geq 0$$

where K is the canonical divisor, π_X the virtual arithmetic genus of X and p_a the arithmetic genus of the surface. Such X satisfies the above condition and hence belongs to a regular family. (cf., [Severi-1], and [Z-5]-no. 3, ch. V).

§ 3.

4. Let $|X|$ be a complete linear system on a normal projective Variety V and $L(X)$ the module of the function on V such that $(f) \succ -X$. $L(X)$ has a finite basis $\varphi_0, \varphi_1, \cdots, \varphi_m$. Let K be a common field of definition for V and for every φ_i. Denote by $(u_0, \cdots, u_m)$ a generic point of a representative of L^m over K and put $Z = (\sum_{i=0}^{m} u_i \varphi_i)_0$, considering $\sum u_i \varphi_i$ as a function on $L^m \times V$. Intersection-product $(v \times V) \cdot Z$ is defined for any Point v on L since $\varphi_0, \cdots, \varphi_m$ are linearly independent. One can see easily that there is a fixed divisor D on V such that every divisor of $|X|$ is of the form $Z(v) + D$, or $|X|$ consists of all the divisors of the form $\overline{Z}(v)$ where $\overline{Z} = Z + L \times D$. We shall say that $|X|$ is defined over a field K' or K' is a field of definition for $|X|$ when $\overline{Z}$ is rational over K'. D is the fixed component for $|X|$. It is easy to see that when a certain divisor of it is rational over a field K', K' is a field of definition for $|X|$ (cf. [W-1]-th ch. VIII). Given a simple Point Q on V, not contained in any component of X, the divisor $\overline{Z}(v) = (\sum v_i \varphi_i)_0 + D$ of $|X|$ contains Q if and only if $(v_0, \cdots, v_m)$ satisfies the linear condition $\sum v_i \varphi_i(Q) = 0$, i. e., if and only if v is on the hyperplane defined by $\sum \varphi_i(Q) X_i = 0$ defined over $K'(Q)$.

LEMMA 7. *Let $Q_1, \cdots, Q_r$ be a set of independent generic Points of V over a field K. Let $|X|$ be a complete linear system on V defined over K. When $l(X) = r + 1$, there is the uniquely determined divisor in $|X|$ passing through Q_i, for every i, which is rational over $K(Q_1, \cdots, Q_r)$.*

PROOF. Let Z be a divisor on $L^r \times V$ such that every divisor of $|X|$ is of the form $Z(v)$, where v is a Point on L. Since $|X|$ is defined over K, we may assume that Z is rational over K, and hence $|X|$ has a rational divisor X_0 over K. Let $\varphi_0, \cdots, \varphi_r$ be a base of $L(X_0)$. Q_i is not contained in any component of X_0 and we may assume further, that $Z = (\sum u_i \varphi_i)_0$ where $(u_0, \cdots, u_r)$ is a representative of a generic Point of L over K. A divisor $Z(v)$ passing through Q_i is such that v is on the hyperplane $H(Q_i)$ defined by $\sum \varphi_j(Q_i) X_j = 0$. We may choose φ_i in such a way that it is defined over K by [W-1]-

th. 12. ch. VIII and hence the matrix $\|\varphi_j(Q_i)\|$ has the maximum rank r since $Q_1, \cdots, Q_r$ are independent generic Points of V over K; this shows that the intersection-product $H(Q_1) \cdots H(Q_r)$ is defined and reduces to a Point v which is rational over $K(Q_1, \cdots, Q_r)$. Then $Z(v)$ is rational over it.   q. e d.

5. Let W be the associated-Variety of a regular maximal algebraic family of positive V-divisors $\{X\}$ and X a generic divisor of it over an algebraically closed field of definition k for V and X. Let M be the Chow-Point of the associated-Variety $T(X)$ of $|X|$ and U the Locus of M over k. Let X_0 be a rational divisor of $\{X\}$ and $Y_0, \cdots, Y_n$ a set of independent generic divisors of $\{X\}$ over $k'(x)$ where x is the Chow-Point of X and k' a field containing k. Denote by $N_0, \cdots, N_n$ the Chow-Point of $T(Y_0), \cdots, T(Y_n)$. Then, for sufficiently large n, U is birationally equivalent to the Picard Variety P of V over k' ($N_0, \cdots, N_n$) (cf. [M-4]-prop. 10, th. 2 and [M-5], § 2). More precisely, there is a frontier $\mathfrak{F}_\alpha$, normally algebraic over k on a representative U of U and a coherent birational transformation $T_{\beta\alpha}$ between U and U biregular at every point on $U-\mathfrak{F}_\alpha$ and $U-\mathfrak{F}_\beta$ such that the abstract Variety

$$[U_\alpha \cdot (=U) : \mathfrak{F}_\alpha : T_{\beta\alpha}]$$

is the Picard Variety P when n is large. The mode of construction of an Abelian Variety in this way from a Variety having the normal law of composition is due to A. Weil (cf. [W-2]-th. 15 and its proof.). As is shown in that proof of Weil, a certain representative of any Point on P is a generic Point of U over k'. The birational transformation $T_{\beta\alpha}$ is defined in the following way. Let X be a generic divisor of $\{X\}$ over $k'(N_0, \cdots, N_n)$. Put

$$X + Y_\alpha - X_0 \sim X_\alpha \qquad (0 \leq \alpha \leq n)$$

and M_α the Chow-Point of $T(X_\alpha)$. Then M_α and M_β are regularly corresponding generic Points of U_α and U_β by $T_{\beta\alpha}$.

THEOREM 3. *The associated-Variety* W *of a regular maximal algebraic family* $\{X\}$ *of positive V-divisors is birationally equivalent to* $P \times L$ *where* L *is a projective space of a certain dimension.*

PROOF. Let, as above, k be an algebraically closed common field of definition for V, W and X_0 a rational divisor of $\{X\}$ over k. Let $Q_1, \cdots, Q_r$ be r independent generic Points of V over k: put $k' = k(Q_1, \cdots, Q_r) = k(Q)$ and X, $Y_0, \cdots, Y_n$ $n+2$ independent generic divisors of $\{X\}$ over k'. Denote by N_i the Chow-Points of $T(Y_i)$ and M the Chow-Point of $T(X)$. M has the Locus U over $k'(N_0, \cdots, N_n) = k'(N)$ and over $k(N)$. Put $r = l(X) - 1$. $Q_1, \cdots, Q_r$ are r independent generic Points of V over $k(N, M)$ and $T(X)$ is defined over it. Let x' be the Chow-Point of the divisor X' in $\{X\}$, then $|X|$ is defined over $k(N, M, Q, x')$ and by lemma 7, the divisors $\overline{X}$ of $|X|$ going through $Q_1, \cdots, Q_r$ is determined uniquely and is rational over $k(N, M, Q, x')$. Hence the Chow-Point of it is rational over $k(N, M, Q, x')$ by [C-2]. Now let x'' be a generic Point of $T(X)$ over $k(N, M, Q, x')$. Then it is also rational over $k(N, M, Q, x'')$ and hence the divisor $\overline{X}$ is rational over $k(N, M, Q)$ i.e., $|X|$ is defined over it and the basis $\varphi_0, \cdots, \varphi_r$ of $L(\overline{X})$ can be taken from the functions defined over it. There is, then, a divisor Z on $L^r \times V$ rational over $k(N, M, Q) = k'(N, M)$ which defines $|X|$. We may assume that x is a generic Point of $T(X)$ over $k'(N, M)$. There is a generic Point u of L^r over $k'(N, M)$ such that $Z(u) = X$. Then as X is rational over $k'(N, M, u)$ by [W-1]-th. 12, ch. VII, we have $k'(N, M, x) \subset k'(N, M, u)$ and u is purely inseparable over the former. For, let u' be a specialization of u over $k'(N, M, x)$: then it is easily seen that $Z(u') = X = Z(u)$ and if $u \neq u'$, it implies $\sum (u_i - c\,u_i')\,\varphi_i = 0$ for a certain c which contradicts to the linear independency of $\varphi_0, \cdots, \varphi_r$. Now we shall prove that

$$k'(N, M, x) = k'(N, M, u) \, .$$

As X is rational over $k'(N, M, x)$, there is a function ψ on V defined over it such that $(\psi) = X - \overline{X}$ by [W-1]-th. 10, VIII and hence $\psi = \sum c_i\,\varphi_i$. But as $\varphi_0, \cdots, \varphi_r$ are linearly independent we conclude that $\sum u_i\,\varphi_i$ is defined over $k'(N, M, x)$. Let Q be a generic Point of V over it and put

$$\varphi_i(Q) = z_i \, , \qquad \sum u_i\,\varphi_i(Q) = w \, .$$

z_i and w are in $k'(N, M, x)(Q)$ and $(z_0, \cdots, z_r, w)$ is linearly dependent over $k'(N, M, u)$. But as $k'(N, M, u)$ and $k'(N, M, x, Q)$ are linearly

disjoint over $k'(N, M, x)$, it must be linearly dependent over $k'(N, M, x)$ by [W-1]-prop. 3, ch. I. Therefore u is rational over $k'(N, M, x)$ and this proves our assertion.

x is a generic Point of W over $k'(N)$ and $k'(N, x) \supset k'(N, M)$. The Locus U of M over $k'(N)$ is birationally equivalent to P over it. The Locus $T(X)$ of x over $k'(N, M)$ is birationally equivalent to L over it by what we have proved above. Therefore W is birationally equivalent to $P \times L$ over $k'(N)$. This proves our theorem.

6. We shall state some remarks and add some appendices about this paper. It will be desirable to extend our th. 2, replacing hyperplane sections by larger classes of positive V-divisors. It might be possible to do so, for non-singular algebraic surfaces, by examining carefully the generalized theorem of Riemann-Roch (cf. [K] and [Z-3]).

Now we shall add a certain result without proofs. Divide the algebraic families of positive V-divisors $\{X\}$ (not necessarily maximal) into three classes:

    i) the first class: a generic divisor of $\{X\}$ over a field of definition for $\{X\}$ is isolated with respect to linear equivalence,

    ii) the second class: $\{X\}$ contains all the divisors of the complete linear system determined by a generic divisor of $\{X\}$ over a field of definition for it (where we assume that that complete linear system has the dimension > 1),

'   iii) the third class: all the other algebraic family.

An maximal algebraic family $\{X\}$ belongs to the second class if and only if a generic divisor of it is not isolated with respect to linear equivalence, otherwise it belongs to the first class.

Let U be any Variety. We shall say that U is a *minimum model*, when, given any Variety W and a function g defined on W with values on U, g is defined at every simple Point of W (cf. [W-2]-no. 15, ch. II). Now the following holds:

Associated-Varieties of the algebraic families of the first class on a normal projective Variety have minimum models, but on the contrary, associated-Varieties of algebraic families of the second class have no minimum models.

The last half of the above assertion follows immediately when we observe that the projective straight line has no minimum model.

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