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Supplementary remarks on the Schur relations for a Frobenius algebra.

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The present paper is a continuation of an earlier investigation [4]. We shall derive some results from the Schur relations obtained in [4].

Let A be a Frobenius algebra over an algebraically closed field K, and let

$$A=\bar{A}+N$$

be a splitting of A into a direct sum of a semisimple subalgebra $\bar{A}$ and the radical N of A. We shall denote by

$$\bar{A}=\bar{A}_1+\bar{A}_2+\cdots+\bar{A}_n$$

the unique splitting of $\bar{A}$ into a direct sum of simple invariant subalgebras $\bar{A}_\lambda$. Let $e_{\lambda,\alpha\beta}\,(\alpha,\beta=1,2,\cdots,f(\lambda))$ denote a set of matrix units for the simple algebra $\bar{A}_\lambda$, we set $e_\lambda=e_{\lambda,11}$ and $E_\lambda=\sum_\alpha e_{\lambda,\alpha\alpha}$. Let $F_1, F_2,\cdots,F_n$ be the distinct irreducible representations of A. Let $U_1, U_2,\cdots,U_n$ be the indecomposable constituents of the left regular representation of A. Then $U_\lambda\cong V_{\pi(\lambda)}$, where the V_λ are the indecomposable constituents of the right regular representation and $(\pi(1),\pi(2),\cdots,\pi(n))$ is a permutation of $(1,2,\cdots,n)$. As is well known, A has a Nakayama's automorphism φ: $a\rightarrow a^\varphi$ which is completely determined by A, apart from an inner automorphism. In the following we shall consider a special Nakayama's automorphism φ which satisfies

$$e_{\pi(\lambda),\alpha\beta}\equiv e^{\varphi}_{\lambda,\alpha\beta}\quad(\mathrm{mod}\ N)\,. \tag{1}$$

We obtain the irreducible representation $a\rightarrow F_\lambda(a^{\varphi^{-1}})$ which will be denoted by $F_{\lambda*}(a)$. Then, by (1) we have $F_{\lambda*}(a)=F_{\pi(\lambda)}(a)$. Let

$$(e_{\rho u,\alpha 1}b_u e_{\sigma u,1\beta}) \tag{2}$$

be the Cartan basis of A, and let U_λ be the indecomposable constituents of the left regular representation defined by the basis (2). We may set

$$U_\lambda=\begin{pmatrix} H^{(1,1)} & & & \\ H^{(2,1)} & H^{(2,2)} & & \\ \cdots\cdots & & & \\ H^{(t,1)} & H^{(t,2)} & \cdots\cdots & H^{(t,t)} \end{pmatrix}, \tag{3}$$

where the $H^{(i,i)}$ are the irreducible representations and in particular, $H^{(1,1)}=F_\lambda$, $H^{(t,t)}=F_{\lambda^*}$. We write

$$H^{(i,j)}(a)=(h_{\mu\nu}^{(i,j)}(a))\,. \tag{4}$$

If (p_s) and (q_s) are corresponding bases belonging to φ, then we have

$$\sum_s h_{\mu\nu}^{(m,l)}(q_s)h_{\alpha\beta}^{(i,j)}(p_s)=0\,, \qquad \text{if } i<l\,. \tag{5}$$

$$\sum_s h_{\mu\nu}^{(m,i)}(q_s)h_{\alpha\beta}^{(i,j)}(p_s)=\begin{cases}0\,, & \text{if } 1<j, \text{ or if } j=1, m<t\\ c_\lambda\delta_{\alpha\nu}\delta_{\beta\mu}\,, & \text{if } j=1,\ m=t\,,\end{cases} \tag{6}$$

where the element $c_\lambda\neq0$ of K is independent of α, β, μ, ν and i. The Schur relations (5), (6) are the main results obtained in [4].

If u and v are any two linear functions of A, then, by [2] or by Lemma 6 [4], we have

$$u(q_s)\,v(p_s)=u(p_s^\varphi)\,v(q_s)\,. \tag{7}$$

Hence it follows from (5), (6) that

$$\sum_s h_{\mu\nu}^{(m,l)}(p_s^\varphi)h_{\alpha\beta}^{(i,j)}(q_s)=0\,, \qquad \text{if } i<l\,. \tag{8}$$

$$\sum_s h_{\mu\nu}^{(m,i)}(p_s^\varphi)h_{\alpha\beta}^{(i,j)}(q_s)=\begin{cases}0\,, & \text{if } 1<j, \text{ or if } j=1,\ m<t\\ c_\lambda\delta_{\alpha\nu}\delta_{\beta\mu}\,, & \text{if } j=1, m=t\,.\end{cases} \tag{9}$$

We may write (8), (9) as follows:

$$\sum_s h_{\mu\nu}^{(m,l)}(q_s)h_{\alpha\beta}^{(i,j)}(p_s^\varphi)=0\,, \qquad \text{if } m<j\,. \tag{8'}$$

$$\sum_s h_{\mu\nu}^{(j,l)}(q_s)h_{\alpha\beta}^{(i,j)}(p_s^\varphi)=\begin{cases}0\,, & \text{if } 1<l, \text{ or if } l=1,\ i<t\\ c_\lambda\delta_{\alpha\nu}\delta_{\beta\mu}\,, & \text{if } l=1,\ i=t\,.\end{cases} \tag{9'}$$

THEOREM 1. *If U_λ is written in the form* (3), *then for any element a in A*

$$\sum_s h_{\mu\nu}^{(m,l)}(q_s a)\, h_{\alpha\beta}^{(i,j)}(p_s)=0\,,\ \text{if } i<l. \tag{10}$$

$$\sum_s h_{\mu\nu}^{(m,i)}(q_s a)\, h_{\alpha\beta}^{(i,j)}(p_s)=\begin{cases}0\,, & \text{if } 1<j, \text{ or if } j=1,\ m<t\\ c_\lambda h_{\alpha\nu}^{(i,i)}(a)\delta_{\beta\mu}\,, & \text{if } j=1,\ m=t\,.\end{cases} \tag{11}$$

PROOF. Since $U_\lambda(q_s a)=U_\lambda(q_s)U_\lambda(a)$, we have from (3)

$$H^{(m,l)}(q_s a)=\sum_{k=l}^{m} H^{(m,k)}(q_s)H^{(k,l)}(a)\,,$$

so that

$$h_{\mu\nu}^{(m,l)}(q_s a)=\sum_{k=l}^{m}\Big(\sum_\rho h_{\mu\rho}^{(m,k)}(q_s)h_{\rho\nu}^{(k,l)}(a)\Big)\,.$$

Hence we see readily by (5), (6) that (10), (11) are valid.

Corresponding to (8), (9), we have

$$\sum_s h_{\mu\nu}^{(m,l)}(p_s^\varphi a)\, h_{\alpha\beta}^{(i,j)}(q_s)=0\,, \qquad \text{if } i<l. \tag{12}$$

$$\sum_s h_{\mu\nu}^{(m,i)}(p_s^\varphi a)\, h_{\alpha\beta}^{(i,j)}(q_s)=\begin{cases}0\,, & \text{if } 1<j, \text{ or if } j=1,\ m<t\\ c_\lambda\, h_{\alpha\nu}^{(i,i)}(a)\delta_{\beta\mu}, & \text{if } j=1,\ m=t\,.\end{cases} \tag{13}$$

We denote by $l(N)$ [$r(N)$] the set of all left [right] annihilators of N in A. Since A is a Frobenius algebra, we have $l(N)=r(N)$ and

$$l(N)=Ad=dA\,, \tag{14}$$

where $a^\varphi d=da$ for every a in A (see [3] Theorem 12). We may assume without restriction that

$$U_\lambda(d)=\begin{pmatrix}0 & \\ I_\lambda & 0\end{pmatrix} \qquad (\lambda=1,2,\cdots,n)\,, \tag{15}$$

where I_λ is the unit matrix of degree $f(\lambda)$. It follows from Theorem 1 and (12), (13) that

$$\sum_s q_s a p_s=\sum_s p_s^\varphi a q_s=\sum_\lambda c_\lambda \operatorname{tr}(U_\lambda(a))dE_\lambda\,. \tag{16}$$

THEOREM 2. *Let* (a_s) *and* (b_s) *be corresponding bases of* A *belonging to an arbitrary Nakayama's automorphism. If the underlying field* K *has characteristic* 0, *then* $\sum b_s a_s \neq 0$. *If* K *has characteristic* p, *then* $\sum b_s a_s = 0$, *if and only if the degree* $u(\lambda)$ *of* U_λ *is divisible by* p *for every* λ.

PROOF. We have $\sum b_s a_s = t \sum q_s p_s$, where t is a regular element of A (see [4] p. 4). It follows from (16) that $\sum q_s p_s = \sum_\lambda c_\lambda u(\lambda)\, dE_\lambda$ and hence our theorem is proved immediately.

THEOREM 3. *If* U_λ *is written in the form* (3), *then for any element* a *in* A

$$\sum_s h_{\mu\nu}^{(m,l)}(aq_s) h_{\alpha\beta}^{(i,j)}(p_s) = 0\,, \qquad \text{if } i < l. \tag{17}$$

$$\sum_s h_{\mu\nu}^{(m,i)}(aq_s)\, h_{\alpha\beta}^{(i,j)}(p_s) = \begin{cases} 0\,, & \text{if } 1 < j, \text{ or if } j=1,\ m < t \\ c_\lambda\, h_{\mu\beta}^{(t,l)}(a)\, \delta_{\alpha\nu}\,, & \text{if } j=1,\ m=t. \end{cases} \tag{18}$$

PROOF. From $U_\lambda(aq_s) = U_\lambda(a) U_\lambda(q_s)$, we have

$$H^{(m,l)}(aq_s) = \sum_{k=l}^{m} H^{(m,k)}(a) H^{(k,l)}(q_s)\,,$$

so that

$$h_{\mu\nu}^{(m,l)}(aq_s) = \sum_{k=l}^{m} \left(\sum_\rho h_{\mu\rho}^{(m,k)}(a) h_{\rho\nu}^{(k,l)}(q_s)\right).$$

By (5), (6) we have easily (17), (18).

Since $F_{\pi(\lambda)}(a) = (f_{\alpha\beta}^{\pi(\lambda)}(a)) = (f_{\alpha\beta}^{\lambda}(a^{\varphi^{-1}}))$, we have

$$a\left(\sum_s q_s p_s\right) = \sum_\lambda \left(\sum_{\alpha,\beta} c_\lambda\, u(\lambda) f_{\alpha\beta}^{\pi(\lambda)}(a) de_{\lambda,\alpha\beta}\right). \tag{19}$$

We may generalize the Schur relations (5), (6) to quasi-Frobenius algebras as in [1], but we shall not enter into this problem.

From now on we assume that A is a symmetric algebra. Then φ becomes the identical automorphism. The corresponding bases (p_s) and (q_s) belonging to the identical automorphism are called quasi-complementary bases. We have in U_λ, (3)

$$H^{(1,1)} = H^{(t,t)} = F_\lambda\,.$$

The Schur relations for a symmetric algebra A are given by

(20) $$\sum_s h_{\mu\nu}^{(m,l)}(q_s)\, h_{\alpha\beta}^{(i,j)}(p_s)=0\,, \qquad \text{if } i<l.$$

(21) $$\sum_s h_{\mu\nu}^{(m,i)}(q_s)\, h_{\alpha\beta}^{(i,j)}(p_s)=\begin{cases}0\,, & \text{if } 1<j, \text{ or if } j=1,\ m<t\\ c_\lambda\,\delta_{\alpha\nu}\,\delta_{\beta\mu}\,, & \text{if } j=1,\ m=t.\end{cases}$$

(22) $$\sum_s h_{\mu\nu}^{(m,i)}(q_s)\, h_{\alpha\beta}^{(i,j)}(p_s)=0\,, \qquad \text{if } m<j.$$

(23) $$\sum_s h_{\mu\nu}^{(j,l)}(q_s)\, h_{\alpha\beta}^{(i,j)}(p_s)=\begin{cases}0\,, & \text{if } 1<l, \text{ or if } l=1,\ i<t\\ c_\lambda\,\delta_{\alpha\nu}\,\delta_{\beta\mu}\,, & \text{if } l=1,\ i=t.\end{cases}$$

We obtain (22), (23) by putting $\varphi=1$ in (8′) and (9′).

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Added in proof

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