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On the uniform distribution of numbers mod. 1.

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1. Let $\{x_n\}$ $(n=1, 2, \cdots)$ be a sequence of real numbers and put

$$\bar{x}_n = x_n - [x_n], \quad 0 \leqq \bar{x}_n < 1. \tag{1}$$

Let I be an interval in $[0, 1]$ and $|I|$ be its length and $n(I)$ be the number of $\bar{x}_\nu$ $(\nu=1, 2, \cdots, n)$ contained in I. If for any I

$$\lim_{n\to\infty} \frac{n(I)}{n} = |I|, \tag{2}$$

then $\{x_n\}$ is called to be uniformly distributed mod. 1.

The following theorems are known.

THEOREM 1 (Weyl)[1]. *The necessary and sufficient condition that $\{x_n\}$ is uniformly distributed mod.* 1 *is that for any R-integrable function $f(x)$ in* $[0, 1]$,

$$\lim_{n\to\infty} \frac{f(\bar{x}_1) + \cdots + f(\bar{x}_n)}{n} = \int_0^1 f(x)dx.$$

THEOREM 2 (Weyl)[2]. *The necessary and sufficient condition that $\{x_n\}$ is uniformly distributed mod.* 1 *is that for* $m=0, \pm 1, \pm 2, \cdots$

$$\sum_{\nu=1}^{n} e^{2\pi m x_\nu i} = o(n).$$

THEOREM 3 (van der Corput)[3]. *Let $g_h(t) = g(t+h) - g(t)$ $(h=1, 2, \cdots)$. If $\{g_h(n)\}$ is uniformly distributed mod.* 1 *for any h, then $\{g(n)\}$ is uniformly distributed mod.* 1.

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1), 2). H. Weyl: Über die Gleichverteilung von Zahlen mod. 1, Math. Ann. 77 (1916).

3) J. G. van der Corput: Diophantische Ungleichungen, I, Zur Gleichverteilung modulo Eins, Acta Math. 56 (1931).

THEOREM 4 (Fejér)[4]. *Let $g(t)>0$ be a continuous increasing function with a continuous derivative $g'(t)$ for $1\leqq t<\infty$ and satisfy the following conditions:*

(i) $g(t)\rightarrow\infty$, *as* $t\rightarrow\infty$,

(ii) $g'(t)\rightarrow 0$ *monotonically, as* $t\rightarrow\infty$,

(iii) $tg'(t)\rightarrow\infty$, *as* $t\rightarrow\infty$.

Then $\{g(n)\}$ *is uniformly distributed mod.* 1.

We shall give a simple proof of Theorem 4.

PROOF. By Euler's summation formula, if we put $P_1(t)=[t]-t+\frac{1}{2}$,

$$\sum_{\nu=1}^{n} e^{2\pi mg(\nu)i}=\int_1^n e^{2\pi mg(t)i}dt+\frac{1}{2}(e^{2\pi mg(n)i}+e^{2\pi mg(1)i})-\int_1^n P_1(t)\frac{d}{dt}(e^{2\pi mg(t)i})dt$$

$$=\int_1^n e^{2\pi mg(t)i}dt+O(1)+O\left(\int_1^n g'(t)dt\right). \qquad (1)$$

By condition (ii),

$$\int_1^n g'(t)dt=o(n). \qquad (2)$$

Since by condition (ii) $1/g'(t)$ is monotone,

$$2\pi mi\int_1^n e^{2\pi mg(t)i}dt=2\pi mi\int_1^n e^{2\pi mg(t)i}\frac{g'(t)}{g'(t)}dt$$

$$=\left[\frac{e^{2\pi mg(t)i}}{g'(t)}\right]_1^n-\int_1^n e^{2\pi mg(t)i}\frac{d}{dt}\left(\frac{1}{g'(t)}\right)dt$$

$$=O\left(\frac{1}{g'(n)}\right)+O\left(\int_1^n\frac{d}{dt}\left(\frac{1}{g'(t)}\right)dt\right)$$

$$=O\left(\frac{1}{g'(n)}\right)+O\left(\frac{1}{g'(n)}\right)=O\left(\frac{1}{g'(n)}\right)=o(n) \qquad (3)$$

by condition (iii).
Hence by (1), (2) and (3)

$$\sum_{\nu=1}^{n} e^{2\pi mg(\nu)i}=o(n), \qquad \text{q. e. d.}$$

REMARK. Hence $\{an^\sigma\}$ $(a>0,\ 0<\sigma<1)$ and $\{a(\log n)^\sigma\}$ $(a>0,\ \sigma>1)$ are uniformly distributed mod. 1.

4) Pólya-Szegö: Aufgaben u. Lehrsätze, I, S. 72, Berlin (1926).

Let $g(t)=at^\sigma$ ($a>0$, $\sigma>0$) and $\Delta_h g=g_h(t)=g(t+h)-g(t)$. If σ is not an integer, then if we apply the operation Δ_h $[\sigma]$-times, then σ can be reduced to $0<\sigma<1$. Hence by Theorem 3, $\{an^\sigma\}$ ($a>0$, $\sigma>0$) is uniformly distributed mod. 1, if σ is not an integer. If σ is an integer and a is irrational, then as is well known, $\{an^\sigma\}$ is uniformly distributed mod. 1.

$\{\log n\}$ is not uniformly distributed mod. 1 as is seen as follows. We put $g(t)=\log t$, then as before,

$$\sum_{\nu=1}^{n} e^{2\pi m g(\nu) i}=\int_1^n e^{2\pi m g(t) i}dt+o(n)=\frac{n^{2\pi mi+1}}{2\pi mi+1}+o(n)\neq o(n).$$

Concerning the distribution of $\{\log n\}$, cf. Theorem 9.

2. We generalize the notion of uniform distribution mod. 1 as follows. Let $\lambda_n>0$ be a sequence, which satisfies the following

$$\text{condition (A): (i)} \quad \lambda_1\geqq\lambda_2\geqq\cdots\geqq\lambda_n>0, \quad \text{(ii)} \quad \sum_{n=1}^{\infty}\lambda_n=\infty. \tag{1}$$

Let I be an interval in $[0, 1]$ and $\varphi(x)$ be its characteristic function, such that $\varphi(x)=1$ for $x\in I$ and $\varphi(x)=0$ elsewhere. If for any I

$$\lim_{n\to\infty}\frac{\lambda_1\varphi(\bar{x}_1)+\cdots+\lambda_n\varphi(\bar{x}_n)}{\lambda_1+\cdots+\lambda_n}=|I|, \tag{2}$$

then we say that $\{x_n\}$ is $\{\lambda_n\}$-uniformly distributed mod. 1. The uniform distribution mod. 1 is a special case, where $\lambda_n=1$ ($n=1, 2, \cdots$).

Similarly as Weyl, we can prove the following theorems.

THEOREM 5. *The necessary and sufficient condition that* $\{x_n\}$ *is* $\{\lambda_n\}$*-uniformly distributed mod.* 1 *is that, for any R-integrable function* $f(x)$ *in* $[0, 1]$,

$$\lim_{n\to\infty}\frac{\lambda_1 f(\bar{x}_1)+\cdots+\lambda_n f(\bar{x}_n)}{\lambda_1+\cdots+\lambda_n}=\int_0^1 f(x)dx.$$

THEOREM 6. *The necessary and sufficient condition that* $\{x_n\}$ *is* $\{\lambda_n\}$*-uniformly distributed mod.* 1 *is that for* $m=0, \pm 1, \pm 2, \cdots$

$$\sum_{\nu=1}^{n}\lambda_\nu e^{2\pi m x_\nu i}=o\Big(\sum_{\nu=1}^{n}\lambda_\nu\Big).$$

We shall prove

THEOREM 7. *Let $\{x_n\}$ be uniformly distributed mod. 1. If $\{\lambda_n\}$ satisfies the condition (A), then $\{x_n\}$ is $\{\lambda_n\}$-uniformly distributed mod. 1. More generally, let $\{\lambda_n\}$ and $\{\mu_n\}$ satisfy the condition (A) and $\lambda_n=a_n\mu_n$, where $a_1\geqq a_2\geqq\cdots\geqq a_n>0$. If $\{x_n\}$ is $\{\mu_n\}$-uniformly distributed mod. 1, then $\{x_n\}$ is $\{\lambda_n\}$-uniformly distributed mod. 1.*

PROOF. Let $\sigma_n=\sum_{\mu=1}^{n}\mu_\nu e^{2\pi m x_\nu i}$, then $\sigma_n=o\left(\sum_{\nu=1}^{n}\mu_\nu\right)$.

$$\left|\sum_{\nu=1}^{n}\lambda_\nu e^{2\pi m x_\nu i}\right|=\left|\sum_{\nu=1}^{n}a_\nu\mu_\nu e^{2\pi m x_\nu i}\right|=|a_1\sigma_1+a_2(\sigma_2-\sigma_1)+\cdots+a_n(\sigma_n-\sigma_{n-1})|$$

$$=|\sigma_1(a_1-a_2)+\cdots+\sigma_{n-1}(a_{n-1}-a_n)+\sigma_n a_n|$$

$$\leqq|\sigma_1|(a_1-a_2)+\cdots+|\sigma_{n-1}|(a_{n-1}-a_n)+|\sigma_n|a_n$$

$$=o[(\mu_1(a_1-a_2)+(\mu_1+\mu_2)(a_2-a_3)+\cdots+(\mu_1+\cdots+\mu_{n-1})(a_{n-1}-a_n)$$

$$+(\mu_1+\cdots+\mu_n)a_n)]=o\left(\sum_{\nu=1}^{n}a_\nu\mu_\nu\right)=o\left(\sum_{\nu=1}^{n}\lambda_\nu\right),\ \text{q. e. d.}$$

3. Let $\lambda_n=\lambda(n)$, where $\lambda(t)>0$ is a continuous decreasing function with a continuous derivative $\lambda'(t)$ for $1\leqq t<\infty$, such that

$$\sum_{\nu=1}^{n}\lambda_\nu\sim\int_1^n\lambda(t)dt\to\infty\qquad(n\to\infty).\tag{1}$$

We shall prove an analogue of Theorem 4.

THEOREM 8. *Let $g(t)>0$ be a continuous increasing function with a continuous derivative $g'(t)$ for $1\leqq t<\infty$ and satisfy the following conditions:*

(i) $g(t)\to\infty$, *as* $t\to\infty$.

(ii) $g'(t)\to 0$ *monotonically, as* $t\to\infty$.

(iii) $g'(t)/\lambda(t)$ *is monotone for* $t\geqq t_0$,

(iv) $\dfrac{g'(t)}{\lambda(t)}\displaystyle\int_1^t\lambda(t)dt\to\infty$, *as* $t\to\infty$.

Then $\{g(n)\}$ is $\{\lambda_n\}$-uniformly distributed mod. 1.

PROOF. As in the proof of Theorem 4,

$$\sum_{\nu=1}^{n}\lambda(\nu)e^{2\pi m g(\nu)i}=\int_1^n\lambda(t)e^{2\pi m g(t)i}dt+\frac{1}{2}[(\lambda(n)e^{2\pi m g(n)i}+\lambda(1)e^{2\pi m g(1)i})]$$

$$-\int_1^n P_1(t)\frac{d}{dt}(\lambda(t)e^{2\pi mg(t)i})dt$$

$$=\int_1^n \lambda(t)e^{2\pi mg(t)i}dt+O(1)+O\Big(\int_1^n(|\lambda'(t)|+\lambda(t)g'(t))dt\Big)$$

$$=\int_1^n \lambda(t)e^{2\pi mg(t)i}dt+O(1)+o\Big(\int_1^n \lambda(t)dt\Big)$$

$$=\int_1^n \lambda(t)e^{2\pi mg(t)i}dt+o\Big(\int_1^n \lambda(t)dt\Big).$$

$$2\pi mi\int_1^n \lambda(t)e^{2\pi mg(t)i}dt=2\pi mi\int_1^n \frac{\lambda(t)}{g'(t)}e^{2\pi mg(t)i}g'(t)dt$$

$$=\Big[\frac{\lambda(t)}{g'(t)}e^{2\pi mg(t)i}\Big]_1^n-\int_1^n e^{2\pi mg(t)i}\frac{d}{dt}\Big(\frac{\lambda(t)}{g'(t)}\Big)dt$$

$$=O\Big(\frac{\lambda(n)}{g'(t)}\Big)+O\Big(\int_1^n\frac{d}{dt}\Big(\frac{\lambda(t)}{g'(t)}\Big)dt\Big)$$

$$=O\Big(\frac{\lambda(n)}{g'(n)}\Big)+O\Big(\frac{\lambda(n)}{g'(n)}\Big)+O(1)=O\Big(\frac{\lambda(n)}{g'(n)}\Big)+O(1)=o\Big(\int_1^n\lambda(t)dt\Big).$$

Hence

$$\sum_{\nu=1}^n\lambda(\nu)e^{2\pi mg(\nu)i}=o\Big(\int_1^n\lambda(t)dt\Big)=o\Big(\sum_{\nu=1}^n\lambda_\nu\Big), \qquad \text{q. e. d.}$$

4. As a special case, we take

$$\lambda(t)=\frac{1}{t},\quad \int_1^t\lambda(t)dt=\log t, \tag{1}$$

then conditions (iii), (iv) become:

(iii) $tg'(t)$ is monotone for $t\geqq t_0$,

(iv) $t\log t\ g'(t)\to\infty$, as $t\to\infty$.

Next we take

$$\lambda(t)=\frac{1}{t\log t\log_2 t\cdots\log_{k-1}t},\quad \int^t\lambda(t)dt=\log_k t, \tag{2}$$

where $\log_k t=\log(\log_{k-1}t)$. Then conditions (iii), (iv) become:

(iii) $t\log t\cdots\log_{k-1}t\ g'(t)$ is monotone for $t\geqq t_0$,

(iv) $t\log t\cdots\log_k t\ g'(t)\to\infty$, as $t\to\infty$.

Hence we have

THEOREM 9. *Let $g(t)>0$ be a continuous increasing function with a continuous derivative $g'(t)$ for $1\leqq t<\infty$ and satisfy the following conditions:*

(i) $g(t)\to\infty$, *as* $t\to\infty$,

(ii) $g'(t)\to 0$ *monotonically, as* $t\to\infty$,

(iii) $tg'(t)$ *is monotone for* $t\geqq t_0$,

(iv) $t\log t\ g'(t)\to\infty$, *as* $t\to\infty$.

Then $\{g(n)\}$ *is* $\left\{\frac{1}{n}\right\}$*-uniformly distributed mod.* 1.

If conditions (iii), (iv) *be replaced by*

(iii) $t\log t\cdots\log_{k-1}t\ g'(t)$ *is monotone for* $t\geqq t_0$,

(iv) $t\log t\cdots\log_k t\ g'(t)\to\infty$, *as* $t\to\infty$,

then $\{g(n)\}$ *is* $\left\{\frac{1}{n\log n\cdots\log_{k-1}n}\right\}$*-uniformly distributed mod.* 1.

Hence if $g(t)=at^\sigma(\log t)^{\sigma_1}(a>0,\ 0\leqq\sigma<1)\to\infty$ as $t\to\infty$, i.e. the first one of σ, σ_1, which is not zero, is positive, then $\{g(n)\}$ is $\left\{\frac{1}{n}\right\}$-uniformly distributed mod. 1.

If $g(t)=at^\sigma(\log t)^{\sigma_1}\cdots(\log_k t)^{\sigma_k}(a>0,\ 0\leqq\sigma<1)\to\infty$ as $t\to\infty$, i. e. the first one of σ, σ_1, $\cdots$, σ_k, which is not zero, is positive and other σ_i may be $\gtreqqless 0$, then $\{g(n)\}$ is $\left\{\frac{1}{n\log n\cdots\log_{k-1}n}\right\}$-uniformly distributed mod. 1. In Theorem 11, we shall prove that σ may be $\geqq 1$.

5. In order to prove an analogue of Theorem 3, we shall first prove an extension of van der Corput's lemma by modifying his proof.

LEMMA. *Let $u(x)$ be defined for $x=1, 2, \cdots, n$ and $\bar{u}(x)$ be its conjugate complex and $\lambda_\nu>0$ ($\nu=1, 2, \cdots$). Then for $q=1, 2, \cdots$*

$$q^2\left|\sum_{\nu=1}^{n}\lambda_\nu u(\nu)\right|^2\Big/\sum_{\nu=1}^{n+q-1}\lambda_\nu\leqq\sum_{\nu=1}^{n}\left(\frac{1}{\lambda_\nu}+\cdots+\frac{1}{\lambda_{\nu+q-1}}\right)\lambda_\nu^2|u(\nu)|^2$$

$$+2\Re\left[\sum_{h=1}^{q-1}\sum_{\nu=1}^{n-h}\lambda_\nu\lambda_{\nu+h}\left(\frac{1}{\lambda_{\nu+h}}+\cdots+\frac{1}{\lambda_{\nu+q-1}}\right)u(\nu)\bar{u}(\nu+h)\right].$$

PROOF. We extend the domain of definition of $u(x)$ outside $[1, n]$

by putting $u(x)=0$ for $x<1$ and $x>n$. Then

$$q\sum_{\nu=1}^{n}\lambda_\nu u(\nu)=\sum_{\sigma=1}^{n+q-1}\sum_{\mu=0}^{q-1}\lambda_{\sigma-\mu}u(\sigma-\mu), \tag{1}$$

so that

$$q^2\left|\sum_{\nu=1}^{n}\lambda_\nu u(\nu)\right|^2\leqq\left(\sum_{\sigma=1}^{n+q-1}\left|\sum_{\mu=0}^{q-1}\lambda_{\sigma-\mu}u(\sigma-\mu)\right|\right)^2\leqq\sum_{\sigma=1}^{n+q-1}\lambda_\sigma\left(\sum_{\sigma=1}^{n+q-1}\frac{1}{\lambda_\sigma}\left|\sum_{\mu=0}^{q-1}\lambda_{\sigma-\mu}u(\sigma-\mu)\right|^2\right). \tag{2}$$

Hence

$$q^2\left|\sum_{\nu=1}^{n}\lambda_\nu u(\nu)\right|^2\Big/\sum_{\nu=1}^{n+q-1}\lambda_\nu\leqq\sum_{\sigma=1}^{n+q-1}\frac{1}{\lambda_\sigma}\left(\sum_{\mu=0}^{q-1}\lambda_{\sigma-\mu}u(\sigma-\mu)\sum_{\nu=0}^{q-1}\lambda_{\sigma-\nu}\bar{u}(\sigma-\nu)\right)$$

$$=\sum_{\sigma=1}^{n+q-1}\frac{1}{\lambda_\sigma}\left(\sum_{\mu=0}^{q-1}\sum_{\nu=0}^{q-1}\lambda_{\sigma-\mu}\lambda_{\sigma-\nu}u(\sigma-\mu)\bar{u}(\sigma-\nu)\right)=\sum_{\sigma=1}^{n+q-1}\frac{1}{\lambda_\sigma}\left(\sum_{\substack{\mu=0\\(\mu=\nu)}}^{q-1}\sum_{\nu=0}^{q-1}\right)$$

$$+\sum_{\sigma=1}^{n+q-1}\frac{1}{\lambda_\sigma}\left(\sum_{\substack{\mu=0\\(\nu<\mu)}}^{q-1}\sum_{\nu=0}^{q-1}\right)+\sum_{\sigma=1}^{n+q-1}\frac{1}{\lambda_\sigma}\left(\sum_{\substack{\mu=0\\(\nu>\mu)}}^{q-1}\sum_{\nu=0}^{q-1}\right)=\sum\nolimits_1+\sum\nolimits_2+\sum\nolimits_3$$

$$=\sum\nolimits_1+2\Re(\sum\nolimits_2). \tag{3}$$

$$\sum\nolimits_1=\sum_{\sigma=1}^{n+q-1}\frac{1}{\lambda_\sigma}\left(\sum_{\mu=0}^{q-1}\lambda^2_{\sigma-\mu}|u(\sigma-\mu)|^2\right)=\sum_{\sigma=1}^{n+q-1}\frac{1}{\lambda_\sigma}\left(\sum_{x=\sigma-q+1}^{\sigma}\lambda_x^2\,|\,u(x)\,|^2\right)$$

$$=\sum_{x=1}^{n}\left(\frac{1}{\lambda_x}+\cdots+\frac{1}{\lambda_{x+q-1}}\right)\lambda_x^2\,|\,u(x)\,|^2. \tag{4}$$

If we put $\sigma-\mu=x$, $\sigma-\nu=x+h$, $h=\mu-\nu$ $(h=1, 2, \cdots, q-1)$ in

$$\sum\nolimits_2=\sum_{\sigma=1}^{n+q-1}\frac{1}{\lambda_\sigma}\left(\sum_{\substack{\mu=0\\(\nu<\mu)}}^{q-1}\sum_{\nu=0}^{q-1}\lambda_{\sigma-\mu}\lambda_{\sigma-\nu}u(\sigma-\mu)\bar{u}(\sigma-\nu)\right),$$

then

$$\sum\nolimits_2=\sum_{h=1}^{q-1}\left(\sum_{\sigma=1}^{n+q-1}\frac{1}{\lambda_\sigma}\left(\sum_{x=\sigma-q+1}^{\sigma-h}\lambda_x\lambda_{x+h}u(x)\bar{u}(x+h)\right)\right)$$

$$=\sum_{h=1}^{q-1}\sum_{x=1}^{n-h}\lambda_x\lambda_{x+h}\left(\frac{1}{\lambda_{x+h}}+\cdots+\frac{1}{\lambda_{x+q-1}}\right)u(x)\bar{u}(x+h). \tag{5}$$

By (3), (4) and (5), the lemma is proved.

6. Now we shall prove an analogue of Theorem 3.

THEOREM 10. *Let* $\{\lambda_n\}$ *satisfy the condition* (A) *and further the condition that* $\frac{\lambda_n}{\lambda_{n+k}}$ *is a decreasing function of* n *for* $k=1, 2, \cdots$. *Let* $g_h(t)=g(t+h)-g(t)$ $(h=1, 2, \cdots)$. *If* $\{g_h(n)\}$ *is* $\{\lambda_n\}$*-uniformly distributed mod.* 1 *for any* h, *then* $\{g(n)\}$ *is* $\{\lambda_n\}$*-uniformly distributed mod.* 1.

PROOF. We put $u(x)=e^{-2\pi m g(x)i}$, then $u(x)\bar{u}(x+h)=e^{2\pi m g_h(x)i}$. Hence by the lemma, since $|u(x)|=1$,

$$q^2\left|\sum_{\nu=1}^{n}\lambda_\nu u(\nu)\right|^2\Big/\sum_{\nu=1}^{n+q-1}\lambda_\nu \leqq \sum_{\nu=1}^{n}\left(\frac{1}{\lambda_\nu}+\cdots+\frac{1}{\lambda_{\nu+q-1}}\right)\lambda_\nu^2$$

$$+2\Re\left[\sum_{h=1}^{q-1}\sum_{\nu=1}^{n-h}\lambda_\nu\lambda_{\nu+h}\left(\frac{1}{\lambda_{\nu+h}}+\cdots+\frac{1}{\lambda_{\nu+q-1}}\right)e^{2\pi m g_h(\nu)i}\right]=\sum\nolimits_1+2\Re\left(\sum\nolimits_2\right). \qquad (1)$$

Since $\frac{\lambda_\nu}{\lambda_{\nu+k}}$ is a decreasing function of ν,

$$\sum\nolimits_1 \leqq qO\left(\sum_{\nu=1}^{n}\lambda_\nu\right). \qquad (2)$$

By the hypothesis,

$$s_n=\sum_{\nu=1}^{n}\lambda_\nu e^{2\pi m g_h(\nu)i}=o\left(\sum_{\nu=1}^{n}\lambda_\nu\right). \qquad (3)$$

If we put

$$a_\nu=\lambda_{\nu+h}\left(\frac{1}{\lambda_{\nu+h}}+\cdots+\frac{1}{\lambda_{\nu+q-1}}\right), \qquad (4)$$

then since $\frac{\lambda_\nu}{\lambda_{\nu+k}}$ is a decreasing function of ν, a_ν is a decreasing function of ν, so that

$$\left|\sum_{\nu=1}^{n-h}\lambda_\nu\lambda_{\nu+h}\left(\frac{1}{\lambda_{\nu+h}}+\cdots+\frac{1}{\lambda_{\nu+q-1}}\right)e^{2\pi m g_h(\nu)i}\right|=\left|\sum_{\nu=1}^{n-h}a_\nu\lambda_\nu e^{2\pi m g_h(\nu)i}\right|$$

$$=|a_1s_1+a_2(s_2-s_1)+\cdots+a_{n-h}(s_{n-h}-s_{n-h-1})|$$

$$=|s_1(a_1-a_2)+\cdots+s_{n-h-1}(a_{n-h-1}-a_{n-h})+s_{n-h}a_{n-h}|$$

$$\leqq|s_1|(a_1-a_2)+\cdots+|a_{n-h-1}|(a_{n-h-1}-a_{n-h})+|s_{n-h}|a_{n-h}$$

$$=o[(\lambda_1(a_1-a_2)+(\lambda_1+\lambda_2)(a_2-a_3)+\cdots$$
$$+(\lambda_1+\cdots+\lambda_{n-h-1})(a_{n-h-1}-a_{n-h})+(\lambda_1+\cdots+\lambda_{n-h})a_{n-h})]$$
$$=o\Big(\sum_{\nu=1}^{n-h}a_\nu\lambda_\nu\Big)=o\Big(\sum_{\nu=1}^{n}\lambda_\nu\Big). \qquad (5)$$

Hence by (1), (2) and (5)

$$q^2\Big(\Big|\sum_{\nu=1}^{n}\lambda_\nu u(\nu)\Big|\Big/\sum_{\nu=1}^{n}\lambda_\nu\Big)^2\leq O(q)+o(1) \text{ as } n\to\infty,$$

so that

$$q^2\varlimsup_{n\to\infty}\Big(\Big|\sum_{\nu=1}^{n}\lambda_\nu u(\nu)\Big|\Big/\sum_{\nu=1}^{n}\lambda_\nu\Big)^2\leqq O(q).$$

Hence, for $q\to\infty$, we have

$$\sum_{\nu=1}^{n}\lambda_\nu u(\nu)=o\Big(\sum_{\nu=1}^{n}\lambda_\nu\Big), \qquad \text{q. e. d.}$$

7. As an application of Theorem 10, we shall prove

THEOREM 11. *Let* $g(t)=at^\sigma(\log t)^{\sigma_1}(a>0,\ |\sigma_1|\neq 0)\to\infty$ *as* $t\to\infty$, *i.e., the first one of* σ, σ_1, *which is not zero, is positive. Then* $\{g(n)\}$ *is* $\left\{\frac{1}{n}\right\}$*-uniformly distributed mod.* 1.

Let $g(t)=at^\sigma(\log t)^{\sigma_1}\cdots(\log_k t)^{\sigma_k}$ $(a>0,\ |\sigma_1|+\cdots+|\sigma_k|\neq 0)\to\infty$ *as* $t\to\infty$, *i.e., the first one of* σ, σ_1, $\cdots$, σ_k, *which is not zero, is positive and other* σ_i *may be* $\gtreqless 0$. *Then* $\{g(n)\}$ *is* $\left\{\frac{1}{n\log n\cdots\log_{k-1}n}\right\}$*-uniformly distributed mod.* 1.

PROOF. We have already proved the case $0\leqq\sigma<1$, hence we assume that $\sigma\geqq 1$. Let

$$\lambda(t)=\frac{1}{t\log t\cdots\log_{k-1}t}, \qquad (1)$$

then, for $a=1, 2, \cdots$,

$$\frac{\lambda(t)}{\lambda(t+a)}=\frac{t+a}{t}\,\frac{\log(t+a)}{\log t}\cdots\frac{\log_{k-1}(t+a)}{\log_{k-1}t}. \qquad (2)$$

Since each factor of (2) is a decreasing function of t, $\dfrac{\lambda(t)}{\lambda(t+a)}$ is a decreasing function of t, so that $\lambda_n = \lambda(n)$ satisfies the condition of Theorem 10.

$$\Delta_h g(t) = g_h(t) = g(t+h) - g(t) = \sum a' t^{\sigma'} (\log t)^{\sigma'_1} \cdots (\log_k t)^{\sigma'_k} + o(1), \quad (3)$$

where $0 \leqq \sigma' \leqq \sigma - 1$ and $o(1) \to 0$ as $t \to \infty$.

We put $\Delta^k_{h_1 \cdots h_k}(g) = \Delta_{h_k}(\Delta^{k-1}_{h_1 \cdots h_{k-1}}(g))$. We consider three cases:

(i) σ is not an integer. Then for $k = [\sigma]$, $\Delta^k_{h_1 \cdots h_k}$ becomes of the form (3), where each σ' is $0 \leqq \sigma' < 1$ and for one of σ''s, $0 < \sigma' < 1$. Then, by Theorem 9, we see easily that the sequence $\sum a' n^{\sigma'} (\log n)^{\sigma'_1} \cdots (\log_k n)^{\sigma'_k}$ is $\{\lambda_n\}$-uniformly distributed mod. 1. Since the term $o(1)$ has no influence on the uniform distribution, $\{\Delta^k_{h_1 \cdots h_k} g(n)\}$ is $\{\lambda_n\}$-uniformly distributed mod. 1. Hence, by Theorem 10, $\{g(n)\}$ is $\{\lambda_n\}$-uniformly distributed mod. 1.

(ii) σ is an integer and the first one of $\sigma_1, \cdots, \sigma_k$, which is not zero, is positive. In this case we see similarly $\{\Delta^k_{h_1 \cdots h_k} g(n)\}$ $(k = \sigma)$ is $\{\lambda_n\}$-uniformly distributed mod. 1. Hence $\{g(n)\}$ is $\{\lambda_n\}$-uniformly distributed mod. 1.

(iii) σ is an integer and the first one of $\sigma_1, \cdots, \sigma_k$, which is not zero, is negative. Then, by Theorem 4, we see easily that $\{\Delta^{k-1}_{h_1 \cdots h_{k-1}} g(n)\}$ $(k = \sigma)$ is uniformly distributed mod. 1, so that, by Theorem 3, $\{g(n)\}$ is uniformly distributed mod. 1, a fortiori, $\{g(n)\}$ is $\{\lambda_n\}$-uniformly distributed mod. 1. Hence our theorem is proved.

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