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Myrberg's approximation theorem on Fuchsian groups.

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Let G be a Fuchsian group of linear transformations, which make $|z|<1$ invariant and D_0 be its fundamental domain. We assume that D_0 has a finite number of sides, such that D_0 lies entirely in $|z|<1$, or has a finite number of parabolic vertices on $|z|=1$. It can be proved that this is equivalent to that the non-euclidean area of D_0 is finite[1]. Then Myrberg[2] proved the following approximation theorem.

THEOREM. *There exists a set E of measure 2π on $|z|=1$, which satisfies the following condition. Let $L=L(\theta)$ be a diameter of $|z|=1$ through $e^{i\theta}$ and L_ν $(\nu=0, 1, 2, \cdots)$ be its equivalents by G. Let C be any orthogonal circle to $|z|=1$. If $e^{i\theta}\in E$, then we can find ν_k, such that $L_{\nu_k}\to C\ (k\to\infty)$.*

We shall prove this theorem simply by means of Hopf's ergodic theorem.

PROOF. We denote an orthogonal circle to $|z|=1$, whose end points on $|z|=1$ are $e^{i\theta}$, $e^{i\varphi}$ by $C(\theta, \varphi)$. Now (θ, φ) can be considered as a point on a torus Ω: $0\leqq\theta\leqq 2\pi$, $0\leqq\varphi\leqq 2\pi$ and the measure mE of a measurable set E on Ω is defined by

$$mE=\iint_E d\theta\, d\varphi, \tag{1}$$

so that $m\Omega=4\pi^2$.

1) C. L. Siegel: Some remarks on discontinuous groups. Ann. of Math. 46 (1945). M. Tsuji: Theory of Fuchsian groups. Jap. Journ. Math. 20 (1952).

2) P. J. Myrberg: Ein Approximationssatz für die Fuchsschen Gruppen. Acta Math. 57 (1931).

Let S be a transformation of G, then (θ, φ) is transformed into (θ', φ') by

$$e^{i\theta'}=S(e^{i\theta}),\ e^{i\varphi'}=S(e^{i\varphi}). \tag{2}$$

We denote this transformation also by S and let $(\theta_\nu, \varphi_\nu)$ $(\nu=0, 1, 2, \cdots)$ be equivalents of (θ, φ).

Let E be a measurable set on Ω, which is invariant by G. If $mE>0$, then Hopf[3] proved that $mE=4\pi^2$.

Let M be the set of (θ, φ), such that $(\theta_\nu, \varphi_\nu)$ $(\nu=0, 1, 2, \cdots)$ are not everywhere dense on Ω, then we shall prove that $mM=0$.

First we shall prove that M is measurable. Let $r_\nu (\nu=1, 2, \cdots)$ be all rational numbers in $[0, 2\pi]$ and

$$\Delta_{ij,kl}:\ r_i\leqq\theta\leqq r_j,\ r_k\leqq\varphi\leqq r_l \tag{3}$$

be an interval on Ω and $M^s_{ij,kl}$ be the set of (θ, φ), such that $(\theta_\nu, \varphi_\nu)$ $(\nu=0, 1, 2, \cdots, s)$ lie outside of $\Delta_{ij,kl}$. Then $M^s_{ij,kl}$ is an open set, so that $M_{ij,kl}=\prod_{s=0}^{\infty} M^s_{ij,kl}$ is a G_δ-set, hence $N=\sum_{i,j,k,l} M_{ij,kl}$ is measurable. We see easily that $N=M$, hence M is measurable.

Now we shall prove that $mM=0$. Suppose that $mM>0$, then $mM_{ij,kl}>0$ for some i, j, k, l. Let $\tilde{M}_{ij,kl}$ be the sum of all equivalents of $M_{ij,kl}$, then $\tilde{M}_{ij,kl}$ is invariant by G and $m\tilde{M}_{ij,kl}>0$, so that by Hopf's theorem, $m\tilde{M}_{ij,kl}=4\pi^2$. But by definition, $\tilde{M}_{ij,kl}$ has no points in $\Delta_{ij,kl}$, so that $m\tilde{M}_{ij,kl}<4\pi^2$, which is absurd. Hence $mM=0$.

Let $C(\theta_0, \varphi_0)$ be any orthogonal circle to $|z|=1$. We shall prove that there exists a set E of measure 2π on $|z|=1$, which depends on (θ_0, φ_0), such that if $e^{i\theta}\in E$, then a suitable sequence from the equivalents of a diameter $L(\theta)$ tends to $C(\theta_0, \varphi_0)$.

Since $mM=0$, there exists a set e_1 on the θ-axis, which is of measure 2π, such that if $\theta\in e_1$, then the line through $(\theta, 0)$ and parallel

3) E. Hopf: Ergodentheorie. Berlin (1937).

M. Tsuji: Hopf's ergodic theorem. Jap. Journ Math. 19 (1945). In this paper, I have proved that Hopf's result holds, if $\overline{\lim}_{r\to 1} n(r)(1-r)>0$, where $n(r)$ is the number of equivalents of $z=0$, which lie in $|z|<r$. Hence Myrberg's theorem holds for such a Fuchsian group.

to the φ-axis meets M in a null set $e_2(\theta)$.

Let $(\theta, \varphi) \bar{\in} M$ $(\theta \neq \varphi)$, then $(\theta_\nu, \varphi_\nu)$ $(\nu=0, 1, 2, \cdots)$ are everywhere dense on Ω, so that we can find ν_k, such that $(\theta_{\nu_k}, \varphi_{\nu_k}) \to (\theta_0, \varphi_0)$ $(k \to \infty)$, or the orthogonal circles

$$C(\theta_{\nu_k}, \varphi_{\nu_k}) \to C(\theta_0, \varphi_0) \ (k \to \infty). \tag{4}$$

Let κ be a small orthogonal circle to $|z|=1$, which contains $e^{i\theta}$ in its inside and z be the point of intersection of κ with $C(\theta, \varphi)$. Similarly we define κ', z' for $e^{i\varphi}$. We take κ, κ' so small that they lie outside each other. Let z_ν, z_ν' be equivalent of z, z' respectively, then

$$z_{\nu_k}, z'_{\nu_k} \to e^{i\theta_0} \text{ or } e^{i\varphi_0}.$$

If
$$z_{\nu_k} \to e^{i\varphi_0}, \tag{5}$$

then the equivalents of κ and hence the equivalents of the outside of κ tend to $e^{i\varphi_0}$, so that the equivalents of a diameter $L(\theta)$ tend to $C(\theta_0, \varphi_0)$.

If
$$z_{\nu_k} \to e^{i\theta_0}, \text{ then } z'_{\nu_k} \to e^{i\theta_0}. \tag{6}$$

For, if otherwise, $z'_{\nu_k} \to e^{i\varphi_0}$, then the non-euclidean distance of z_{ν_k}, z'_{ν_k} tends to ∞, while it is equal to that of z, z', which is finite. Hence $z'_{\nu_k} \to e^{i\theta_0}$, so that the equivalents of a diameter $L(\varphi)$ tend to $C(\theta_0, \varphi_0)$.

Suppose that for every $\theta \in e_1$, (5) holds for at least one $\varphi \in [0, 2\pi] - e_2(\theta)$, then we put $E=e_1$.

If this is not the case, there exists $\theta_1 \in e_1$, such that for $\theta=\theta_1$, (6) holds for all $\varphi \in [0, 2\pi] - e_2(\theta_1)$, then we put $E=[0, 2\pi]-e_2(\theta_1)$.

In each case, E is of measure 2π and if $e^{i\theta} \in E$, then a suitable sequence from the equivalents of $L(\theta)$ tends to $C(\theta_0, \varphi_0)$, q. e. d.

Now we take a countable set of points (θ_n, φ_n) $(n=1, 2, \cdots)$ on Ω, which is everywhere dense on Ω and let E_n be the corresponding set of measure 2π on $|z|=1$ and put $E=\prod_{n=1}^{\infty} E_n$, then E is of measure 2π. If $e^{i\theta} \in E$, then for any n, a suitable sequence from the equivalents of a diameter $L(\theta)$ tends to $C(\theta_n, \varphi_n)$. Since (θ_n, φ_n) are everywhere dense on Ω, a suitable sequence from the equivalents of $L(\theta)$ tends to any orthogonal circle $C(\theta_0, \varphi_0)$. Hence our theorem is proved.

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