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Conformal rigidity of Riemann surfaces.

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1. The following theorem of T. Radó [3] is well-known:

Let G be a planar region bounded by $n\ (\geqq 2)$ Jordan curves $C_1, \cdots, C_n$, and G' be a proper subregion of G bounded by n Jordan curves $C'_1, \cdots, C'_n$, such that C'_k is homotopic to C_k in $\overline{G}$ $(k=1, \cdots, n)$. Then, G admits no one-to-one conformal mapping onto G'.

In the present paper we shall consider, instead of a planar region of finite connectivity, a Riemann surface G (of finite or infinite connectivity and genus) bounded partly by a finite number of Jordan curves. Under the assumption that G admits a one-to-one conformal mapping onto a proper subregion of itself satisfying some topological conditions, we shall prove that G must be of some particularly simple structure (Theorem 2), a result which constitutes a generalization of Radó's theorem.

First, in § 2, we prove a general selection theorem on single-valued (not necessarily one-to-one) analytic mappings of a Riemann surface into another Riemann surface. In § 3, the above mentioned Theorem 2 is stated and proved, to which a remark is added in § 4. Finally, in § 5, we prove a rigidity theorem without any topological restrictions on the subregion.

2. THEOREM 1. *Let F, F^* be two Riemann surfaces whose universal covering surfaces are of hyperbolic type*[1], *and $\{f_\nu\}_{\nu=1}^{\infty}$ be a sequence of single-valued analytic mappings of F into F^*. Then, either*

i) *there exists a subsequence $\{f_{\nu_k}\}_{k=1}^{\infty}$ which converges, uniformly in the wider sense in F (with respect to the uniform topology of F^* defined by means of Poincaré's hyperbolic metric), to a limit analytic mapping f of F into F^*; or else*

ii) *for any point p on F the point sequence $\{f_\nu(p)\}$ on F^* tends to the ideal boundary of F^* uniformly in the wider sense in F.*

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1) In point of fact, it suffices merely to assume that the universal covering surface of F^* is of hyperbolic type.

The statement ii) means: if K, K^* are compact point sets on F, F^* respectively, then $f_\nu(K) \cap K^* = \phi$ for sufficiently large ν. Further, in the case ii), it is easily proved that a suitable subsequence $\{f_{\nu_k}(p)\}$ tends, uniformly in the wider sense in F, to a single ideal boundary component[2] of F^*. It suffices for this purpose to take a subsequence $\{f_{\nu_k}\}$, such that, for a point p_0 on F, the point sequence $\{f_{\nu_k}(p_0)\}$ tends to a single ideal boundary component of F^*.

PROOF. Suppose that ii) does not hold. Then, there exist a subsequence of $\{f_\nu\}$ (which we denote also by $\{f_\nu\}$) and a point sequence $\{p_\nu\}$ on F, such that $\{p_\nu\}$ tends to a point q on F and $\{f_\nu(p_\nu)\}$ tends to a point q^* on F^*.

We map the universal covering surfaces of F and F^* one-to-one conformally onto the discs $|z| < 1$ and $|z^*| < 1$ respectively, so that $z = 0$ corresponds to q on F and $z^* = 0$ to q^* on F^*, and let $\mathfrak{G}$, $\mathfrak{G}^*$ be the corresponding Fuchsian or Fuchsoid groups. We denote by $\{z_\nu\}$ the image of $\{p_\nu\}$ in a neighbourhood of $z = 0$, so that $z_\nu \to 0$ for $\nu \to \infty$.

In a neighbourhood of $z = 0$ the composed mapping $z \to p \to f_\nu(p) = p^* \to z^*$ defines an analytic function element, which can be analytically continued along any path in $|z| < 1$ and defines a single-valued function $z^* = \varphi_\nu(z)$ analytic in $|z| < 1$. Obviously $|\varphi_\nu(z)| < 1$ in $|z| < 1$. While choosing a suitable branch of the mapping $p^* \to z^*$ for each ν, we can assume that $\varphi_\nu(z_\nu) \to 0$ for $\nu \to \infty$.

The functions $\varphi_\nu(z)$ have the following property: if z, z' ($|z|$, $|z'| < 1$) are equivalent to each other with respect to $\mathfrak{G}$, then $\varphi_\nu(z)$, $\varphi_\nu(z')$ are equivalent with respect to $\mathfrak{G}^*$. f_ν is interpreted as the composed mapping $p \to z \to \varphi_\nu(z) = z^* \to p^*$.

Since $\{\varphi_\nu(z)\}$ is uniformly bounded, a suitable subsequence $\{\varphi_{\nu_k}(z)\}$ converges, uniformly in the wider sense in $|z| < 1$, to a limit function $\varphi(z)$. Since $z_{\nu_k} \to 0$ and $\varphi_{\nu_k}(z_{\nu_k}) \to 0$, $\varphi(z)$ can not be a constant of modulus one, so that $\varphi(z)$ is analytic and of modulus less than one in $|z| < 1$ (φ may be $\equiv 0$).

Let z, z' be a pair of points in $|z| < 1$ equivalent with respect to $\mathfrak{G}$. Suppose that $\varphi(z)$, $\varphi(z')$ were not equivalent with respect to $\mathfrak{G}^*$. Then there would exist neighbourhoods U, U' of $\varphi(z)$, $\varphi(z')$ respectively, such that any point of U would have no equivalent in U' with respect

2) As for the precise definition of ideal boundary components (éléments-frontières), cf. S. Stoïlow [4], M. Ohtsuka [2].

to $\mathfrak{G}^*$. This contradicts the fact that $\varphi_{\nu_k}(z)$, $\varphi_{\nu_k}(z')$ are equivalent with respect to $\mathfrak{G}^*$ and tend to $\varphi(z)$, $\varphi(z')$ respectively for $k\to\infty$. Hence, if z, z' are equivalent with respect to $\mathfrak{G}$, then $\varphi(z)$, $\varphi(z')$ are equivalent with respect to $\mathfrak{G}^*$.

Now consider the composed mapping $f: p\to z\to\varphi(z)=z^*\to p^*$. f is a single-valued analytic mapping of F into F^*. Since $\{\varphi_{\nu_k}(z)\}$ converges to $\varphi(z)$ uniformly in the wider sense in $|z|<1$, we see that $\{f_{\nu_k}\}$ converges to f uniformly in the wider sense in F with respect to the mentioned uniform topology of F^*, q. e. d.

3. Let G be a Riemann surface bounded partly by a finite number of Jordan curves C. More precisely stated: let G be a subregion of a Riemann surface, such that the relative boundary of G with respect to the surface consists of a finite number of Jordan curves C.

For simplicity, we use the following terminology in the sequel: for a subregion $\varDelta$ of G, we shall understand by the "*exterior complement*" of $\varDelta$ (with respect to the region G and the boundary part C) the connected components of the complement $G\cup C-\varDelta$ containing some points of C, and by the "*exterior boundary*" of $\varDelta$ the boundary components of $\varDelta$ belonging to the exterior complement of $\varDelta$.

We remark that, if $\varDelta$ is contained in a simply connected subregion of G, the exterior boundary of $\varDelta$ consists of a single connected component.

Our main theorem is now formulated in the following form:

THEOREM 2. *Let G be a Riemann surface bounded partly by a finite number of Jordan curves C. Let G' be a proper subregion of G with the exterior boundary C', such that the exterior complement of G' is compact. Suppose that G admits a one-to-one conformal mapping $p\to\psi(p)$ onto G' in such a manner that C corresponds to C'. Then, either*

a) *G is simply connected; or else*

b) *G is conformally equivalent to a simply connected region pricked at a single point; or else*

c) *G is of infinite genus and has precisely one ideal boundary component*[3] *of harmonic measure zero, and there exists an exhaustion*

3) I. e., among the connected components of the complement of any compact point set in G, there is precisely one component with non-compact closure.

$\{G_k\}_{k=0}^{\infty}$ *of* G *such that* $C\subset\overline{G}_0$ *and* G_k-G_{k-1} *admits a one-to-one conformal mapping onto* $G_{k+1}-G_k$ $(k=1, 2, \cdots)$.

Let $\varDelta$ be a subregion of G. For later use, we shall make here some obvious remarks on the region $\psi(\varDelta)$ and the mapping $\varDelta\to\psi(\varDelta)$:

i) if the exterior complement of $\varDelta$ is compact, the exterior complement of $\psi(\varDelta)$ is also compact;

ii) the exterior boundary of $\varDelta$ corresponds by ψ to the exterior boundary of $\psi(\varDelta)$.

PROOF OF THEOREM 2. Let $\{\psi_\nu\}_{\nu=1}^{\infty}$ be the sequence of iterates of the mapping ψ:

$$\psi_1=\psi,\quad \psi_{\nu+1}=\psi(\psi_\nu) \qquad (\nu=1, 2, \cdots).$$

Applying Theorem 1 to $\{\psi_\nu\}$ with G as F and with a Riemann surface containing $G\cup C$ as F^*, we see that either (A): a suitable subsequence $\{\psi_{\nu_k}\}$ converges uniformly in the wider sense in G to a mapping Ψ which either maps G one-to-one conformally onto a subregion of G or transforms G into a single point p_0 on $G\cup C$; or else (B): $\{\psi_\nu(p)\}$ tends to the ideal boundary of G uniformly in the wider sense in G.

Suppose that (A) *is the case.* We shall first show that $\Psi(G)$ reduces really to a single point.

The image $\psi_\nu(G)$ of G decreases monotonously for $\nu\to\infty$. We put $E=\bigcap_1^{\infty}\psi_\nu(G)$. Then $\psi(E)=\bigcap_1^{\infty}\psi_{\nu+1}(G)=E$, i.e. E is invariant under the mapping ψ[4].

Suppose now that $\Psi(G)$ were a subregion of G. Let p denote a generic point in G and U be a neighbourhood of p contained wholly in G. $\Psi(U)$ would then be a neighbourhood of the point $\Psi(p)$. Since $\psi_{\nu_k}\to\Psi$ uniformly in U, we should have $\Psi(p)\in\psi_{\nu_k}(U)\subset\psi_{\nu_k}(G)$ for sufficiently large k. Then, since $\psi_\nu(G)$ decreases for $\nu\to\infty$, $\Psi(p)\in\psi_\nu(G)$ must remain valid for $\nu=1, 2, \cdots$, so that $\Psi(p)\in E$. Hence $\Psi(G)\subset E$. Since $\psi_{\nu_k}(p)\to\Psi(p)$, we should have $\psi_{\nu_k}(p)\in\Psi(U)\subset\Psi(G)\subset E$ for some sufficiently large k. Since the mapping ψ_{ν_k} is one-to-one and $\psi_{\nu_k}(E)=E$, it would follow that $p\in E$. Hence $G\subset E$, so that $G=E$. This contradicts the hypothesis that $G'=\psi(G)$ is a proper subregion of G. Hence $\Psi(G)$ must reduce to a single point p_0 on $G\cup C$.

4) E is the maximal invariant set in G. In some cases, E contains almost all points of G, e.g.: if G is a region on the z-plane star-shaped with respect to the origin and $\psi(z)\equiv kz$ $(0<k<1)$, any half straight-line $\arg z=$const. contained in G belongs to E. (Also cf. J. Wolff [5].)

Next, let Δ be a subregion of G contained wholly in G, and γ be the exterior boundary of Δ. Since $\psi_{\nu_k}(p) \to p_0$ for $k \to \infty$ uniformly in Δ, $\psi_{\nu_k}(\Delta)$ is contained in a simply connected subregion of G for sufficiently large k, regardless of whether p_0 lies in G or on C. Hence $\psi_{\nu_k}(\Delta)$ must be of genus zero and the exterior boundary $\psi_{\nu_k}(\gamma)$ of $\psi_{\nu_k}(\Delta)$ must consist of a single connected component, so that Δ is of genus zero and γ consists of a single connected component. Since this is true for any such Δ, we see that G is of genus zero and C consists of a single connected component.

Further, if we take a Δ with a compact exterior complement, the exterior complement of $\psi_\nu(\Delta)$ is also compact for any ν by the remark i). Since, for sufficiently large k, the exterior complement of $\psi_{\nu_k}(\Delta)$ contains all points of G except those of a neighbourhood of the point p_0, we see that $G \cup C$ itself is compact.

Thus, in the case (A), G is a simply connected region.

Next, suppose that (B) *is the case.*

Let γ_0 be a Jordan curve in G separating C from the ideal boundary of G, G_0 be the part of G bounded by C and γ_0, and Δ_0 be the part of G separated from C by γ_0. γ_0 is the exterior boundary of Δ_0.

Since $\psi_\nu(\gamma_0)$ tends to the ideal boundary of G for $\nu \to \infty$, we have $\psi_N(\gamma_0) = \gamma_1 \subset \Delta_0$ for a sufficiently large integer N. We put $\psi_{kN}(\gamma_0) = \gamma_k$, $\psi_{kN}(\Delta_0) = \Delta_k$, for $k = 1, 2, \cdots$. Since γ_1 is the exterior boundary of Δ_1 and $\gamma_1 \subset \Delta_0$, we have $\Delta_1 \cup \gamma_1 \subset \Delta_0$, so that $\Delta_k \cup \gamma_k \subset \Delta_{k-1}$ for $k = 1, 2, \cdots$.

Let σ_1 be the exterior complement of Δ_1 with respect to the region Δ_0 and the boundary part γ_0. σ_1 is compact and is bounded by γ_0 and γ_1 only, and $C \cup G_0 \cup \sigma_1$ is the exterior complement of Δ_1 with respect to G and C.

Since, by the mapping ψ_N, Δ_0 is mapped onto Δ_1 and Δ_1 onto Δ_2, σ_1 must be mapped onto the exterior complement σ_2 of Δ_2 with respect to the region Δ_1 and the boundary part γ_1. σ_2 is compact and is bounded by γ_1 and γ_2 only, and $C \cup G_0 \cup \sigma_1 \cup \sigma_2$ is the exterior complement of Δ_2 with respect to G and C.

Similarly, denoting by σ_k the exterior complement of Δ_k with respect to the region Δ_{k-1} and the boundary part γ_{k-1}, we see successively that $\sigma_k = \psi_N(\sigma_{k-1})$ and that $C \cup G_0 \cup \sigma_1 \cup \cdots \cup \sigma_k$ is compact and constitutes the exterior complement of Δ_k with respect to G and C.

We put $G_k = G_0 \cup \sigma_1 \cup \cdots \cup \sigma_k - \gamma_k$, so that $G_{k-1} \subset G_k$ and $G_k - G_{k-1} = \sigma_k - \gamma_k$. Hence we have $\psi_N(G_k - G_{k-1}) = G_{k+1} - G_k$. On the other hand, since $\overline{G}_k$ is the exterior complement of Δ_k with respect to G and C, and since the exterior boundary γ_k of Δ_k tends to the ideal boundary of G, we see that $\cup_0^\infty G_k = G$.

Thus the existence of an exhaustion $\{G_k\}_0^\infty$ of G as mentioned in c) is proved. Since $G_k - G_{k-1}$ is connected and has one and the same modulus for $k=1, 2, \cdots$, it follows easily that G has precisely one ideal boundary component of harmonic measure zero.

It remains to be proved that, if especially G is of finite genus, G is conformally equivalent to a simply connected region pricked at a single point.

If G is of finite genus, there exists a Riemann surface $\tilde{G}$ bounded by a finite number of Jordan curves $\tilde{C}$ such that $\tilde{G} \cup \tilde{C}$ is compact and G is conformally equivalent to $\tilde{G}$ less a single point $\tilde{p}_0$. Correspondingly, ψ is transformed into a one-to-one conformal mapping $\tilde{\psi}$ of $\tilde{G} - \{\tilde{p}_0\}$ onto its proper subregion. $\tilde{\psi}$ admits analytic continuation also at $\tilde{p}_0$ and maps $\tilde{G}$ onto a proper subregion $\tilde{G}'$ of $\tilde{G}$. For $\tilde{G}$, $\tilde{G}'$ and $\tilde{\psi}$, the conditions of Theorem 2 are satisfied, and, since $\tilde{G} \cup \tilde{C}$ is compact, the case (B) can not occur. Hence, as is already proved, $\tilde{G}$ must be simply connected.

Thus Theorem 2 is proved.

4. The exhaustion $\{G_k\}_0^\infty$ of G constructed above in the case (B) satisfies $\psi_N(G_k - G_{k-1}) = G_{k+1} - G_k$. We can also construct *an exhaustion* $\{G_k^*\}_0^\infty$ *of* G *with the property* $\psi(G_k^* - G_{k-1}^*) = G_{k+1}^* - G_k^*$.

Let γ be a system of a finite number of Jordan curves in G separating C from the ideal boundary of G, and Δ be the part of G separated from C by γ. As is seen from the above construction, it suffices for our purpose to find a system γ such that $\psi(\gamma) \subset \Delta$.

We remark that the relative boundary of G' with respect to $G \cup C$ consists only of the exterior boundary C'. In fact, if G' had a relative boundary component C'' other than C', the component of $G \cup C - G'$ containing C'' would not be compact, since G' has only one (ideal) boundary component other than C'. Hence C'' would contain a continuum, so that C'' would have a positive harmonic measure with respect to G'. This contradicts that the ideal boundary of G is of harmonic measure zero. Since G' is a proper subregion of G, it follows in particular that C' is not coincident with C.

For the exhaustion $\{G_k\}_0^\infty$ constructed above, each $G_k-\overline{G}_{k-1}$ is a connected region and is conformally equivalent to $G_1-\overline{G}_0$. Hence it follows, by a result in [1] (Theorem 13.1), that *the ideal boundary of G is of harmonic dimension one* in the sense of M. Heins [1]. In other words, there exists one and only one positive harmonic function $u(p)$ (normalized minimal) in G vanishing continuously on C, such that its conjugate harmonic function has the modulus of periodicity 2π around the ideal boundary, i. e. $\int_\alpha (\partial u/\partial n)ds=2\pi$ for any simple curve α in G separating C from the ideal boundary of G where n denotes the inner normal with respect to the subregion of G separated from C by α.

It is known that $\lim u(p)=+\infty$ for p tending to the ideal boundary, and that, if $v(p)$ is a positive harmonic function in G continuous on $G\cup C$ whose conjugate harmonic function has the modulus of periodicity 2π around the ideal boundary, $v(p)-u(p)$ is bounded, and hence remains non-negative on $G\cup C$ (cf. [1]).

We put $u'(p)=u(\psi^{-1}(p))$ in $G'=\psi(G)$. Since $u'(p)$ is the normalized minimal positive harmonic function of G' vanishing continuously on C', we have $u(p)-u'(p)\geqq 0$ on $G'\cup C'$. Since, at the points of C' lying in G, $u(p)>0$ and $u'(p)=0$, we have $u(p)>u'(p)$ in G'.

Let δ be a positive number, $\varDelta$ be the part of G where $u(p)>\delta$, and γ be the niveau curve $u(p)=\delta$. Since $\lim u(p)=+\infty$ at the ideal boundary, γ is compact and constitutes the exterior boundary of $\varDelta$. On $\psi(\gamma)$ we have $u(p)>u'(p)=\delta$. Hence $\psi(\gamma)\subset\varDelta$, as was desired.

5. Finally we remark that, without any topological restrictions on the subregion, the following theorem holds good.

THEOREM 3. *A Riemann surface G of finite positive genus admits no one-to-one conformal mapping ψ onto any proper subregion of itself.*

PROOF. If G is closed, the theorem is trivial. If G is open, let it be represented as a proper subregion of a closed Riemann surface $\tilde{G}$. Suppose that such a mapping ψ did exist, and denote by $\{\psi_\nu\}$ the sequence of iterates of ψ.

Let q be a point of $\tilde{G}-G$. Since $\tilde{G}$ is of positive genus, the universal covering surface of $\tilde{G}-\{q\}$ is of hyperbolic type. Applying Theorem 1 to $\{\psi_\nu\}$ with G as F and with $\tilde{G}-\{q\}$ as F^*, we see that either a suitable subsequence $\{\psi_{\nu_k}\}$ would converge to an analytic mapping Ψ of G into $\tilde{G}-\{q\}$, or else $\{\psi_\nu(p)\}$ would tend to the point q, both uniformly in the wider sense in G. In the former case, Ψ must transform G into a single point on $\tilde{G}-\{q\}$, as was proved in the case (A) of the proof of Theorem 2.

Hence, in any case, a suitable subsequence $\{\psi_{\nu_k}(p)\}$ would converge, uniformly in the wider sense in G, to a single point p_0 on $\tilde{G}$. Let Δ be a subregion of G of positive genus contained wholly in G. Then, for sufficiently large k, $\psi_{\nu_k}(\Delta)$ would be contained in a neighbourhood of p_0, so that it would be of genus zero, which is a contradiction.

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