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Principal ruled surfaces of a rectilinear congruence.

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Introduction.

Let p^{ij} be the Plücker coordinates of a line p in projective three dimensional space $\boldsymbol{R}_3$. If p $(p^{01}, p^{02}, p^{03}, p^{12}, p^{13}, p^{23})$ is a function of two parameters u^1 and u^2, the line p describes a rectilinear congruence K when u^1 and u^2 vary. Now put[1)]

$$p^i=\frac{\partial p}{\partial u^i}(i=1,2),\quad -((p_i\,p_j))=H_{ij}\quad (i,j=1,2).$$

If the determinant determined by the elements H_{ij} $(i,j=1,2)$ does not vanish identically, the congruence K has two focal surfaces S_0 and S_1. We restrict ourselves in this case.

Let us consider the image of a line p in the projective five dimensional space $\boldsymbol{R}_5$, the plane $\boldsymbol{S}_2$ determined by the three points p, p_1 and p_2 is the tangential plane of the image V of K at p, and the plane $\boldsymbol{S}_4$ determined by $\boldsymbol{S}_2$ and its conjugate $\boldsymbol{S}_2'$ with respect to the quadric of Plücker $\boldsymbol{Q}_4$ is the polar plane of $\boldsymbol{Q}_4$ at p, that is, the tangential plane of $\boldsymbol{Q}_4$ at p. Let p_5 be a point which does not lie in this tangential hyperplane $\boldsymbol{S}_4$, the plane $pp_1p_2p_5$ has no common point with the conjugate $\boldsymbol{S}_1'$ with respect to $\boldsymbol{Q}_4$, $\boldsymbol{S}_1'$ intersects with $\boldsymbol{Q}_4$ at two different points p_3 and p_4. Then p_3 and p_4 lie on the tangential hyperplane $pp_1p_2p_3p_4$, and the lines pp_k, p_kp_5 $(k=3,\ 4)$ are not conjugate to each other. Moreover, to determine uniquely the point p_5, we select p_5 as the intersection of $\boldsymbol{Q}_4$ and the line joining the point p and $\dfrac{1}{2}H^{\sigma\tau}\dfrac{\bar{\partial}^2 p}{\partial u^\sigma\partial u^\tau}$ ($\bar{d}$ denotes absolute differentiation). Then we have the fundamental equations for the given congruence K as follows[2)]:

$$
\text{(I)}\qquad
\begin{cases}
dp = du^\sigma p_\sigma\,, \\
\bar{d}p_i = E_{i\sigma}du^\sigma p + F_{i\sigma}du^\sigma p_3 + G_{i\sigma}du^\sigma p_4 + H_{i\sigma}du^\sigma p_5 \qquad (i=1,\ 2)\,, \\
dp_3 = M_\sigma du^\sigma p + G^\rho_\sigma du^\sigma p_\rho + L_\sigma du^\sigma p_3\,, \\
dp_4 = N_\sigma du^\sigma p + F^\rho_\sigma du^\sigma p_\rho - L_\sigma du^\sigma p_4\,, \\
dp_5 = E^\rho_\sigma du^\sigma p_\rho - N_\sigma du^\sigma p_3 - M_\sigma du^\sigma p_4\,.
\end{cases}
$$

The frame of tetrahedron thus constructed by p, p_1, p_2, p_3, p_4 and p_5 in $\boldsymbol{R}_5$ is called the fundamental coordinate tetrahedron R_0.

Now let the curves $u^1=$const. and $u^2=$const. on the image V in $\boldsymbol{R}_5$ represent the developable surfaces of K in $\boldsymbol{R}_3$, then the equations (I) become[3)]

$$
\text{(I}'\text{)}\qquad
\begin{cases}
dp = du\,p_u + dv\,p_v\,, \\
dp_u = (E_{11}du + E_{12}dv)\,p + \theta_u du\,p_u + F_{11}du\,p_3 + G_{11}du\,p_4 + H_{12}dv\,p_5\,, \\
dp_v = (E_{12}du + E_{22}dv)\,p + \theta_v dv\,p_v + F_{22}dv\,p_3 + G_{22}dv\,p_4 + H_{12}du\,p_5\,, \\
dp_3 = (M_1du + M_2dv)\,p + G^1_2dv\,p_u + G^2_1du\,p_v + (L_1du + L_2dv)\,p_3\,, \\
dp_4 = (N_1du + N_2dv)\,p + F^1_2dv\,p_u + F^2_1du\,p_v - (L_1du + L_2dv)\,p_4, \\
dp_5 = (E^1_1du + E^1_2dv)\,p_u + (E^2_1du + E^2_2dv)\,p_v - (N_1du + N_2dv)\,p_3 \\
\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad - (M_1du + M_2dv)\,p_4.
\end{cases}
$$

Such a specialization of the frame of coordinate tetrahedron R_0 to R_a enables us to obtain one of the suitable methods for the interpretation of the relation between $\boldsymbol{R}_5$ and $\boldsymbol{R}_3$, consequently the properties of a rectilinear congruence K in $\boldsymbol{R}_3$ is easily considered and calculated as the special variety in[2)] $\boldsymbol{R}_5$.

The conditions of integrability are given by

$$
\text{(II)}\qquad
\begin{cases}
E_{11v} - E_{12u} + E_{12}\theta_u + F^2_1G_{22}m_1 + F^1_2G_{11}n_1 = 0\,, \\
E_{22u} - E_{12v} + E_{12}\theta_v + F^1_2G_{11}m_2 + F^2_1G_{22}\,n_2 = 0\,,
\end{cases}
$$

$$
\text{(III)}\qquad E_{12} = \frac{1}{2}\,(F^2_1G_{22} + F^1_2G_{11} + \theta_{uv})\,,
$$

$$
\text{(IV)}\qquad n_1 = -(\log F_{22})_u - L_1, \quad n_2 = -(\log F_{11})_v - L_2,
$$

$$
\text{(V)}\qquad m_1 = -(\log G_{22})_u + L_1, \quad m_2 = -(\log G_{11})_v + L_2,
$$

$$
\text{(VI)}\qquad \frac{a_{11}}{F_{11}} = \frac{a_{22}}{F_{22}}\,,
$$

(VII) $$\frac{b_{11}}{G_{11}}=\frac{b_{22}}{G_{22}},$$

(VIII) $$L_{1v}-L_{2u}=w \qquad (w=H_{12}W=H_{12}(F_1^2G_2^1-F_2^1G_1^2)),$$

with respect to R_a, where

(IX) $$\begin{cases} a_{11}=E_{11}+n_{1u}-n_1\theta_u-(n_1)^2, & a_{22}=E_{22}+n_{2v}-n_2\theta_v-(n_2)^2, \\ b_{11}=E_{11}+m_{1u}-m_1\theta_u-(m_1)^2, & b_{22}=E_{22}+m_{2v}-m_2\theta_v-(m_2)^2. \end{cases}$$

The congruence K in question is also expressed by the form[2)]

(X) $$\begin{cases} z^3=\dfrac{1}{2}F_{\sigma\tau}z^\sigma z^\tau+\dfrac{1}{6}\left(\dfrac{\bar{\partial}F_{\sigma\tau}}{\partial u^\rho}+F_{\sigma\tau}L_\rho-H_{\sigma\tau}N_\rho\right)z^\sigma z^\tau z^\rho+\cdots\cdots, \\ z^4=\dfrac{1}{2}G_{\sigma\tau}z^\sigma z^\tau+\dfrac{1}{6}\left(\dfrac{\bar{\partial}G_{\sigma\tau}}{\partial u^\rho}-G_{\sigma\tau}L_\rho-H_{\sigma\tau}M_\rho\right)z^\sigma z^\tau z^\rho+\cdots\cdots, \end{cases}$$

with respect to R_0 which is also written in the form

(X′) $$\begin{cases} z^3=\dfrac{1}{2}[F_{11}f_1(z^1)^2+F_{22}(z^2)^2] \\ \qquad +\dfrac{1}{6}[2F_{11}f_1(z_1)^3-3F_{11}n_2(z^1)^2z^2-3F_{22}n_1z^1(z^2)^2+2F_{22}f_2(z^2)^3]+\cdots, \\ z^4=\dfrac{1}{2}[G_{11}(z^1)^2+G_{22}(z^2)^2] \\ \qquad +\dfrac{1}{6}[2G_{11}g_1(z^1)^3-3G_{11}m_2(z^1)^2z^2-3G_{22}m_1z^1(z^2)^2+2G_{22}g_2(z^2)^3]+\cdots, \end{cases}$$

with respect to R_a.

1. Principal ruled surfaces of a rectilinear congruence. The tangential plane of the image V (of a rectilinear congruence K) in $\boldsymbol{R}_5$ is determined by three points p, p_1 and p_2. The intersection of this tangential plane and Q_4 is the two generators pp_u, pp_v on Q_4 with respect to R_a, which represent the *focal pencils* of the congruence K in $\boldsymbol{R}_3$. Now consider the space $S(2)$ determined by p, p_i, p_{ij} $(i, j=1, 2)$, where $p_{ij}=\dfrac{\partial^2 p}{\partial u^i \partial u^j}$. If these six points are independent $[S(2)=S_5]$, that is, if

(1.1) $$\Delta=|\, p \;\; p_1 \;\; p_2 \;\; p_{11} \;\; p_{12} \;\; p_{22}\,|\neq 0$$

is satisfied, the image V (consequently the congruence K) is called *normal*. This equation can be also written, by means of (I), in the form

$$\Delta=\begin{vmatrix} F_{11} & G_{11} & H_{11} \\ F_{12} & G_{12} & H_{12} \\ F_{22} & G_{22} & H_{22} \end{vmatrix} \neq 0$$

with respect to R_0. It can be also rewritten

$$w \neq 0 \tag{1.1'}$$

with respect to R_a, where w is given by (VIII). It is easy to see[2] that $w=0$ is the necessary and sufficient condition that the congruence K is reduced to w congruence, hence we have the

THEOREM 1. *A rectilinear congruence K is a W congruence if and only if K is not normal*[4].

In this sense, the theory of W congruence is trivial in the general theory of a rectilinear congruence, consequently we exclude hereafter the W congruence, and consider the normal congruence only.

The quadric complex $\mathfrak{C}_2$ having the contact of fourth order with K along p is given, by means of (IX'), by the form

$$G_{11}G_{22}(z^3)^2-(F_{11}G_{22}+F_{22}G_{11})z^3z^4+F_{11}F_{22}(z^4)^2+\frac{1}{4}(F^2_1G_{22}-F^1_2G_{11})^2(z^5)^2=0$$

with respect to R_a. Hence we have the

THEOREM 2. *Consider the osculating quadric complex $\mathfrak{C}_2$ having the contact of fourth order with a rectilinear congruence K along a line p of K. Then K has the five ruled surfaces having the contact of fith order with $\mathfrak{C}_2$ along p, defined by the equation*

$$ak_1du^5-\frac{3}{2}as_2du^4dv-\pi_1du^3dv^2+\pi_2du^2dv^3+\frac{3}{2}bs_1dudv^4-bk_2dv^5=0\,, \tag{1.2}$$

where

$$a=F^2_1G_{11}\,,\ b=F^1_2G_{22}\,,$$

$$k_1=f_1-g_1=\frac{1}{2}(\log F_{11}:G_{11})_a+L_1\,,$$

$$k_2=f_2-g_2=\frac{1}{2}(\log F_{22}:G_{22})_v+L_2\,,$$

$$s_1=n_1-m_1=-(\log F_{22}:G_{22})_u-2L_1\,,$$

$$s_2=n_2-m_2=-(\log F_{11}:G_{11})_v-2L_2\,,$$

$$\pi_1=F_1^2G_{22}\Big(f_1-\frac{3}{2}m_1\Big)-F_2^1G_{11}\Big(g_1-\frac{3}{2}n_1\Big)\,,$$

$$\pi_2=F_2^1G_{11}\Big(f_2-\frac{3}{2}m_2\Big)-F_1^2G_{22}\Big(g_2-\frac{3}{2}n_2\Big)\,.$$

These five ruled surfaces are called the *principal ruled surfaces* of the congruene K. The principal ruled surfaces play the fundamental rôle on the general theory of a rectilinear congruence.

Now we introduce the relation between the principal line, which is the well known curve on the hypersurface in $\boldsymbol{R}_5$, and the principal ruled surface given above.

THEOREM 3. *The images of the principal ruled surfaces of a rectilinear congruence K defined by Theorem 2 coincide with the principal lines of the image V of the congruence.*

PROOF. The principal line of V at a point p is defined by

$$|\,p\;\;p_1\;\;p_2\;\;p_{1\sigma}du^\sigma\;\;\;p_{2\sigma}du^\sigma\;\;\;p_{\sigma\tau\rho}du^\sigma du^\tau du^\rho\,|=0\qquad\left(p_{ijk}=\frac{\partial^3p}{\partial u^i\partial u^j\partial u^k}\right),$$

which is also written, by means of (X$'$), in the form

$$\text{(1.3)}\qquad\begin{vmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\ *&*&*&F_{11}du&G_{11}du&H_{12}dv\\ *&*&*&F_{22}dv&G_{22}du&H_{12}du\\ *&*&*&\overline{\tau^3}&\overline{\tau^4}&\overline{\tau^5}\end{vmatrix}=0$$

with respect to R_a, where

$$\begin{cases}\overline{\tau^3}=F_{11}(2f_1+3\theta_u)du^3-3F_{11}n_2du^2dv-3F_{22}n_1dudv^2+F_{22}(2f_2+3\theta_v)dv^3,\\ \overline{\tau^4}=G_{11}(2g_1+3\theta_u)du^3-3G_{11}m_2du^2dv-3G_{22}m_1dudv^2+G_{22}(2g_2+3\theta_v)dv^3,\\ \overline{\tau^5}=3H_{12}(\theta_udu^2dv+\theta_vdudv^2)\,.\end{cases}$$

It is easy to see that the equation (1.3) is equivalent to (1.2), which proves the theorem.

Now we introduce two important special rectilinear congruences k and s obtained directly from the principal ruled surfaces and state several properties on them.

2. s congruence.

Definition. A rectilinear congruence K whose principal ruled surfaces along a line p have the directions:

1. Two of them are harmonic with respect to the directions of developable surfaces of K along p;
2. The remaining three are apolar to the two directions of developable surfaces of K,

is called s *congruence*, which is characterized by the conditions

$$s_1 = s_2 = 0 \,. \tag{2.1}$$

THEOREM 4. *The characteristic property of s congruence is that it has the sequence of Laplace of period four.*

To demonstrate this property, we shall give some preliminary notes.

Consider the sequence of Laplace of a given congruence K. The first sequence of Laplace is given by the congruence $\{p_4\}$ or $\{p_3\}$. Let the second sequences of K be

$\{p_6\}$: (in the direction of p_4),
$\{p_7\}$: (in the direction of p_3),

and let the focal surface of $\{p_4\}$ (different from S_0) be S_4 (cf. fig) and the focal point of p_4 on S_4 be P_1. The tangential plane of S_4 at P_1 is given by $A_0P_1P_2$, where P_2 is the intersection of this tangential plane and the line p_3. The point of Laplace P_1 has the coordinates $(-n_2,\ 0,\ 0,\ H_{12})$, while the asymptotic tangents of S_4 at P_1 are given by

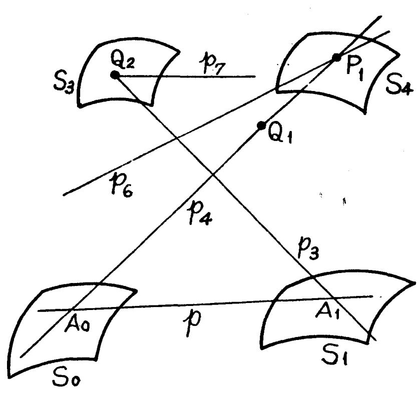

Fig.

$$F_{11}n_{21}du^2 - F_{22}n_{12}dv^2 = 0 \,, \tag{2.2}$$

where $n_{21} = E_{12} + n_{2u}$, $n_{12} = E_{12} + n_{1v}$. And the generator p_6 has the form

(2.3) $$F_2^1 n_1 n_2 p + F_2^1 n_2 p_u + F_2^1 n_1 p_v - a_{22} p_4 + F_{22} p_5 ,$$

Then we have the

LEMMA. *The generator p_6 coincides with the line $P_1 P_2$ if and only if*

(2.4) $$a_{22}=0 .$$

This condition is equivalent to $a_{11}=0$, owing to the existence of the condition of integrability (VI).

Similarly, let S_3 be the focal surface of $\{p_3\}$ (different from S_1), and the focal point of p_3 on S_3 be Q_2. Then the tangential plane of the focal surface S_3 at Q_2 is given by $A_1Q_1Q_2$, where Q_1 is the intersection of this tangential plane and the line p_4, and the asymptotic curves on S_3 are determined by

(2.5) $$G_{11} m_{21} du^2 - G_{22} m_{12} dv^2 = 0 ,$$

where $m_{21}=E_{12}+m_{2u}$, $m_{12}=E_{12}+m_{1v}$. The generator p_7 has the form

(2.6) $$G_1^2 m_1 m_2 p + G_1^2 m_2 p_u + G_1^2 m_1 p_v - b_{11} p_3 + G_{11} p_5 ,$$

and it coincides with the line Q_1Q_2 if and only if

(2.7) $$b_{11}=0$$
or
$$b_{22}=0 .$$

PROOF OF THEOREM 4. If p_6 and p_7 coincide with P_1P_2 and Q_1Q_2 simultaneously, we have

(2.8) $$a_{11}=a_{22}=b_{11}=b_{22}=0 ,$$

and then P_1 and Q_2 coincide with Q_1 and P_2 respectively. Then by the conditions (2.2), (2.5) and the conditions of integrability, we see that the congruence K has the Laplace sequence of period four. On the other hand, the conditions of s congruence (2.1) we obtain at once the equations (2.8), which demonstrates the theorem.

Note:—we state without demonstration the fact that the s congruence satisfying the condition $(U/a)_u=(V/b)_v$ permits the projective deformation of a rectilinear congruence, where U and V are functions of u and v respectively.

3. *k* congruence.

Definition. If two directions of the principal ruled surfaces of a rectilinear congruence coincide with those of the developable surfaces, the congruence is called *k congruence.* This condition is given by the equations

$$k_1=k_2=0 \tag{3.1}$$

From the definitions of (2.1) and (3.1), we have immediately the

THEOREM 5. *If a rectilinear congruence K has the property of k and s congruence, then K is not normal, that is, K is reduced to W congruence.*

Among the several properties concerning to k congruence, we introduce the most simple one in this paper.

THEOREM 6. *A rectilinear congruence is reduced to k congruence if and only if a pair of the osculating linear complexes of the developable surfaces along a line p is reduced to a pair of satellite complexes.*

PROOF. The image of the osculating linear complex b_1 of the developable surface $\mathfrak{L}_1$: $u=$const. ($u^1=$const.) is determined by the intersection of Q_4 and the hyperplane determined by the five points

$$p,\ p_u,\ F_{11}p_3+G_{11}p_4\,,\quad F_{11}k_1p_3+ap_v,\quad k_1p_3+G_1^2p_v\,.$$

Consequently the pole of b_1 is given by

$$k_1\Phi_{1u}p-k_1p_u+F_{11}p_3-G_{11}p_4\,, \tag{3.2}$$

where $\Phi_1=[\log(k_1/\sqrt{F_1^2G_1^2})]_u$. Similarly the pole of the osculating linear complex b_2 of the developable surface $\mathfrak{L}_2$: $v=$const. ($u^2=$const.) is determined by

$$k_2\Phi_{2v}p-k_2p_v+F_{22}p_3-G_{22}p_4\,, \tag{3.3}$$

where $\Phi_2=[\log(k_2/\sqrt{F_2^1G_2^1})]_v$. On the other hand, the poles of the satellite complexes K of are given by $F_{11}p_3-G_{11}p_4$, $F_{22}p_3-G_{22}p_4$, which demonstrates the theorem.

4. Quasi asymptotic ruled surfaces of a congrence. Now we use the concept of quasi asymptotic γ_{13} introduced by E. Bompiani[5], which is defined by the matrix

$$\| p \ p_1 \ p_2 \ p_{\sigma\tau\rho}du^\sigma du^\tau \ p_{\sigma\tau\rho}du^\sigma du^\tau du^\rho + 3p_{\sigma\tau}du^\sigma d^2u^\tau \| = 0\,, \tag{4.1}$$

where p_{ijl} is given in § 1. The equation (4.1) is equivalent to

$$\varphi^3 = 0\,, \quad \varphi^4 = 0\,, \tag{4.2}$$

where

$$\varphi^3 = \frac{3}{2}(dvd^2u - dud^2v)(F_{11}du^2 - F_{22}dv^2) + F_{11}\left(2f_1 + \frac{3}{2}\theta_u\right)du^3$$

$$-3F_{11}\left(n_2 + \frac{1}{2}\theta_v\right)du^2dv - 3F_{22}\left(n_1 + \frac{1}{2}\theta_u\right)dudv^2 + F_{22}\left(2f_2 + \frac{3}{2}\theta_v\right)dv^3,$$

$$\varphi^4 = \frac{3}{2}(dvd^2u - dud^2v)(G_{11}du^2 - G_{22}dv^2) + G_{11}\left(2g_1 + \frac{3}{2}\theta_v\right)du^3$$

$$-3G_{11}\left(m_2 + \frac{1}{2}\theta_v\right)du^2dv - 3G_{22}\left(m_1 + \frac{1}{2}\theta_u\right)dudv^2 + G_{22}\left(2g_2 + \frac{3}{2}\theta_v\right)dv^3,$$

and the condition of compatibility of these two equations is represented by (1.2). Hence if we define the ruled surface of a given congruence K, whose image is the quasi asymptotic curve in $\boldsymbol{R}_5$, the *quasi asymptotic ruled surface* of K, then the quasi asymptotic ruled surface has the direction defined by the principal ruled surfaces of K.

Now we know, by the equations (I′), that the osculating linear congruence $\mathfrak{K}$ of the asymptotic ruled surface is determined by the intersection of $\boldsymbol{Q}_4$ and the plane determined by p, p_u, p_v and

$$(F_{11}du^2 + F_{22}dv^2)p_3 + (G_{11}du^2 + G_{22}dv^2)p_4 + 2H_{12}dudv\,,$$

hence $\mathfrak{K}$ contains the focal pencils pp_u and pp_v of K along p. And this property holds only when (4.2) are satisfied. Hence we have the

THEOREM 7. *The quasi asymptotic ruled surface of a congruence K is characterized by the fact that its osculating linear congruence contains the focal pencils of K.*

In this paper we eliminate further considerations concerning to this item.

I dedicate my hearty thanks to Prof. Sasaki (Tôhoku University) for his kind advices for the publication of this paper.

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Notes.

1) $((p\,q)) = p^{01}q^{23} - p^{02}q^{13} + p^{03}q^{12} + p^{12}q^{03} - p^{13}q^{02} + p^{23}q^{01}$.

2) As for the details, reference is to be made to our paper: Takeda, K., On line congruence, I, Tôhoku, 44 (1938), 356-69.

3) As for the details and notations, references are to be made to our paper: Takeda, K., On line congruence, II, Tôhoku, 45 (1938), 103-110.

4) Here we exclude the trivial cases, where the focal surfaces are reduced to special forms, that is, the cases $F_{11}=G_{11}=0$, $F_{11}F_{22}=0$ and $G_{11}=G_{22}=0$.

5) Bompiani, E., Proprietà differenziale caratteristica delle superficie che rappresentano la totalità delle curve piane algebriche di dato ordine, Lincei, 30 (1921), 248-51.