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Betti numbers and exact differential forms.

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In this paper we consider an orientable compact positive definite Riemannian space R_n. The relations between exact differential forms and Betti numbers are known as de Rham's theorem. We shall give some applications of this theorem.

In § 1 and § 2, we shall find the conditions that the Betti number be greater than a certain number. In § 3, we consider the conditions that the equations of harmonic tensors become some total differential equations which enable us to evaluate the Betti numbers.

§ 1. LEMMA 1. *Let $A_{i(1)\cdots i(p)}$ and $C^{i(1)\cdots i(p)}$ be skew-symmetric and satisfy the conditions*

$$A_{i(1)\cdots i(p)}=B_{[i(1)\cdots i(p-1);\, i(p)]} \tag{1.1}$$

and

$$C^{i(1)\cdots i(p)}{}_{;\, i(p)}=0\,, \tag{1.2}$$

where $B^{i(1)\cdots i(p-1)}$ is a certain tensor. Then it follows that

$$\int A_{i(1)\cdots i(p)}C^{i(1)\cdots i(p)}dv=0\,, \tag{1.3}$$

where dv is the volume element and the integral extends over the whole space.

PROOF. By Green's theorem we have

$$\begin{aligned}0&=\int (B_{i(1)\cdots i(p-1)}C^{i(1)\cdots i(p)})_{;\, i(p)}dv=\int B_{i(1)\cdots i(p-1);\, i(p)}C^{i(1)\cdots i(p)}dv\\&+\int B_{i(1)\cdots i(p-1)}C^{i(1)\cdots i(p)}{}_{;\, i(p)}dv=\int B_{[i(1)\cdots i(p-1);\, i(p)]}C^{i(1)\cdots i(p)}\, dv\\&=\int A_{i(1)\cdots i(p)}C^{i(1)\cdots i(p)}dv\,.\end{aligned} \tag{1.4}$$

THEOREM 1. *Let $H_{(A)i(1)\cdots i(p)}$ $(A=1,\ 2,\ \cdots,\ s)$ and $H_{(B)i(p+1)\cdots i(n)}$ $(B=1,\ 2,\ \cdots,\ t)$ be exact and put*

(1.5) $$a_{AB}=\int H_{(A)i(1)\cdots i(p)}\varepsilon^{i(1)\cdots i(n)}H_{(B)i(p+1)\cdots i(n)}dv=\int H_{(A)}\vee H_{(B)},$$

then the p-th Betti number exceeds the rank of the matrix

(1.6) $$\| a_{AB} \| .$$

PROOF. By the assumption, $H_{(B)i(p+1)\cdots i(n)}$'s are exact. Hence their dual tensors

$$C_{(B)}^{i(1)\cdots i(p)}=\varepsilon^{i(1)\cdots i(n)}H_{(B)i(p+1)\cdots i(n)}$$

satisfy

(1.7) $$C_{(B)}^{i(1)\cdots i(p)}{}_{;\,i(p)}=0. \qquad \text{(Hodge [1])}$$

If any r tensors of $H_{(A)}$'s satisfy the relations of the form

(1.8) $$\sum_{A=A(1)}^{A(r)} P^{(A)}H_{(A)i(1)\cdots i(p)}=B_{[i(1)\cdots i(p-1);\,i(p)]},$$

where $P^{(A)}$'s are certain constants and B is a certain tensor, it follows from Lemma 1 that

(1.9) $$0=\sum_{A=A(1)}^{A(r)} P^{(A)}\int H_{(A)i(1)\cdots i(p)}C_{(B)}^{i(1)\cdots i(p)}dv$$

$$=\sum P^{(A)}\int H_{(A)i(1)\cdots i(p)}\varepsilon^{i(1)\cdots i(n)}H_{(B)i(p+1)\cdots i(n)}\,dv$$

$$=\sum P_{(A)}a_{AB}.$$

Hence any minor determinant of degree r of the matrix $\| a_{AB} \|$ is zero, i.e. the rank of the matrix $\| a_{AB} \|$ is less than r.

Therefore, if the rank of $\| a_{AB} \|$ is r, there exists at least one set of tensors

$$H_{(A)i(1)\cdots i(p)} \qquad (A=A(1), \cdots, A(r))$$

which does not satisfy any relation of the form (1.8).

Hence we have from de Rham's theorem that

$$B_p \geqq r$$

where B_p denotes the p-th Betti number.

§ **2.** By Bianchi's identity we can prove that the tensors

(2.1) $$K_{2m}=R^{a(1)}{}_{a(2)[i(1)i(2)}R^{a(2)}{}_{a(3)i(3)i(4)}\cdots R^{a(m)}{}_{a(1)i(2m-1)i(2m)]} \qquad (m\geqq 2)$$

are exact But we must remark that if the Riemannian space under consideration is of class 1, the tensor K_{2m}'s are identically zero, for, in this case the curvature tensor takes the form

$$R^{i}{}_{jkl}=H^{i}{}_{k}H_{jl}-H^{i}{}_{l}H_{jk}\,. \tag{2.2}$$

Next we can prove that K_{4m+2}'s ($m\geqq 1$) are identically zero, for example, as follows:

$$\begin{aligned}K_6&=R^{a(1)}{}_{a(2)[i(1)i(2)}R^{a(2)}{}_{a(3)i(3)i(4)}R^{a(3)}{}_{a(1)i(5)i(6)]}\\&=R^{a(3)}{}_{a(1)[i(5)i(6)}R^{a(2)}{}_{a(3)i(3)i(4)}R^{a(1)}{}_{a(2)i(1)i(2)]}\\&=-R^{a(1)}{}_{a(3)[i(5)i(6)}R^{a(3)}{}_{a(2)i(3)i(4)}R^{a(2)}{}_{a(1)i(1)i(2)]}\\&=-R^{a(1)}{}_{a(3)[i(1)i(2)}R^{a(3)}{}_{a(2)i(3)i(4)}R^{a(2)}{}_{a(1)i(5)i(6)]}\\&=-K_6\end{aligned} \tag{2.3}$$

Hence we can find only the following exact tensors.

$$\begin{aligned}&K_4 \quad \text{(degree 4)}\,,\\&K_8,\ K_4\times K_4 \quad \text{(degree 8)}\,,\\&K_{12},\ K_8\times K_4,\ K_4\times K_4\times K_4 \quad \text{(degree 12)}\,,\end{aligned} \tag{2.4}$$

etc., where the symbols $\times$ denote exterior products.

When $n=16$ we take as sets of matrices of the type (1.6) the followings:

$$\left(\int K_4\vee K_{12} \quad \int K_4\vee(K_8\times K_4) \quad \int K_4\vee(K_4\times K_4\times K_4)\right), \tag{2.5}$$

$$\begin{pmatrix}\int K_8\vee K_8 & \int (K_4\times K_4)\vee K_8\\ \int K_8\vee(K_4\times K_4) & \int (K_4\times K_4)\vee(K_4\times K_4)\end{pmatrix}. \tag{2.6}$$

Therefore, if the rank of the matrix (2.5) is 1, we have

$$B_4=B_{12}\geqq 1\,.$$

If the rank of the matrix (2.6) is r (1 or 2), we have

$$B_8\geqq r\,.$$

In special cases we can find exact tensors other than (2.4).

(I) *The case in which* $R^{a}{}_{ijk;a}=0$.

In this case it holds that

(2.7) $$R_{ij;k}-R_{ik;j}=0.$$

Hence we have the following exact tensors:

(2.8) $K_4,\ L_4\equiv R_{a[i}R^{a}{}_{\underline{b}jk}R^{\underline{b}}{}_{l]}$ (degree 4),

(2.9) $K_8,\ K_4\times K_4,\ K_4\times L_4,\ L_4\times L_4,$

$L_8\equiv R_{a(1)[i(1)}R^{a(1)}{}_{a(2)i(2)i(3)}\cdots R^{a(3)}{}_{a(4)i(6)i(7)}R^{a(4)}{}_{i(8)]}$ (degree 8),

(2.10) $K_{12},\ K_8\times K_4,\ K_4\times K_4\times K_4,\ L_8\times L_4,\ L_4\times L_4\times L_4,$

$K_8\times L_4,\ K_4\times L_4\times L_4,\ K_4\times L_8,\ L_4\times K_4\times K_4,$

$L_{12}\equiv R_{a(1)[i(1)}R^{a(1)}{}_{a(2)i(2)i(3)}\cdots R^{a(5)}{}_{a(6)i(10)i(11)}R^{a(6)}{}_{i(12)]}$ (degree 12),

etc., provided that n is greater than 4, 8 and 12 respectively.

(II.) *The case in which a skew-symmetric tensor H_{ij} satisfying*

(2.11) $$H_{[ij;k]}=0$$

exists.

In this case the following tensors are exact:

(2.12) H_{ij}, ... (degree 2),

(2.13) $K_4,\ H\times H$, ... (degree 4),

(2.14) $H\times H\times H,\ K_4\times H$, ... (degree 6),

(2.15) $K_8,\ K_4\times K_4,\ K_4\times H\times H,\ H\times H\times H\times H$ (degree 8),

etc., provided that n is greater than 2, 4, 6 and 8 respectively.

(III.) *The case in which a tensor $S^i{}_{jk}$ satisfying*

(2.16) $$S^i{}_{jk;l}-S^i{}_{jl;k}=0$$

exists.

In this case we get the following exact tensors:

(2.17) $S^a{}_{ai}$ (degree 1),

(2.18) $S^a{}_{b[i}S^{\underline{b}}{}_{\underline{c}j}S^{\underline{c}}{}_{\underline{a}k]},\ R^a{}_{b[ij}S^{\underline{b}}{}_{\underline{a}k]}$ (degree 3),

(2.19) $K_4,\ S^a{}_{b[i}R^{\underline{b}}{}_{\underline{c}jk}S^{\underline{c}}{}_{\underline{a}l]}$ (degree 4),

(2.20) $S^{a(1)}{}_{a(2)[i(1)}S^{\underline{a(2)}}{}_{\underline{a(3)}i(2)}\cdots S^{a(5)}{}_{\underline{a(1)}i(5)]}$,

$R^a{}_{b[ij}R^{\underline{b}}{}_{\underline{c}kl}S^{\underline{c}}{}_{ah]}$,

$R^{a(1)}{}_{a(2)[i(1)i(2)}S^{a(2)}{}_{a(3)i(3)}S^{a(3)}{}_{\underline{a(4)}i(4)}S^{\underline{a(4)}}{}_{\underline{a(1)}i(5)]}$ (degree 5),

$$(2.21)\qquad R^{a(1)}{}_{a(2)[i(1)i(2)}R^{\underline{a(2)}}{}_{\underline{a(3)}i(3)i(4)}S^{\underline{a(3)}}{}_{\underline{a(4)}i(5)}S^{\underline{a(4)}}{}_{\underline{a(1)}i(6)]},$$

$$R^{a(1)}{}_{a(2)[i(1)i(2)}S^{\underline{a(2)}}{}_{\underline{a(3)}i(3)}S^{\underline{a(3)}}{}_{a(4)i(4)}S^{\underline{a(4)}}{}_{\underline{a(5)}i(5)}S^{\underline{a(5)}}{}_{a(1)i(6)]},$$

$$R^{a(1)}{}_{a(2)[i(1)i(2)}S^{\underline{a(2)}}{}_{\underline{a(1)}i(3)}S^{b(1)}{}_{\underline{b(2)}i(4)}S^{\underline{b(2)}}{}_{\underline{b(3)}i(5)}S^{\underline{b(3)}}{}_{\underline{b(1)}i(6)]}\quad \text{(degree 6)},$$

etc., provided that n is greater than 3, 4, 5 and 6 respectively.

Generally, if there exists a tensor $Q^a{}_{bi(1)\cdots i(p)}$ which is skew-symmetric with respect to $i(1), \cdots, i(p)$ and satisfies

$$(2.22)\qquad Q^a{}_{b[i(1)\cdots i(p);\, i(p+1)]}=0,$$

then we have the following exact tensors:

$$(2.23)\quad Q^{a(1)}{}_{a(2)[i(1)\cdots i(p)}Q^{\underline{a(2)}}{}_{\underline{a(3)}i(p+1)\cdots i(2p)}\cdots Q^{\underline{a(m)}}{}_{\underline{a(1)}i(mp-p+1)\cdots i(mp)]}\ (m=1, 2, \cdots).$$

But in the following cases above tensors become identically zero.

(1) p is odd and m is even.

(2) Q is symmetric with respect to a and b and p is odd and

$$m=3,\ 7,\ \cdots,\ 4k-1.$$

(3) Q is skew-symmetric with repect to a and b and p is even and m is odd.

(4) Q is skew-symmetric with respect to a and b and p is odd and

$$m=\ 5,\ 9,\ \cdots,\ 4k+1.$$

From these tensors and K_{2m}'s we get many exact tensors as in the case (III).

Let

$$H_{(A)i(1)\cdots i(q)}\qquad (A=1,\ 2,\ \cdots,\ s)$$

and

$$H_{(B)i(q+1)\cdots i(n)}\qquad (B=1,\ 2,\ \cdots,\ t)$$

be these tensors. Then the rank of the matrix

$$\left\| \int H_{(A)i(1)\cdots i(q)} \vee H_{(B)i(q+1)\cdots i(n)} \right\|$$

gives us a lower bound of the q-th Betti number.

§ **3.** Let a skew-symmetric tensor $\xi_{i(1)\cdots i(p)}$ be harmonic, i. e. satisfy the conditions

$$(3.1)\qquad \begin{cases} \xi_{[i(1)\cdots i(p);\, r]}=0, \\ \xi_{i(1)\cdots i(p);\, r}\, g^{i(p)r}=0. \end{cases}$$

It is known that the conditions (3.1) are equivalent with

(3.2) $$\Delta\xi_{i(1)\cdots i(p)}=K_{i(1)\cdots i(p)}{}^{j(1)\cdots j(p)}\xi_{j(1)\cdots j(p)}\,,$$

where

$$K_{i(1)\cdots i(p)j(1)\cdots j(p)}=p(p-1)R_{[i(1)[j(1)i(2)j(2)}g_{i(3)j(3)}\cdots g_{i(n)]j(n)]}$$
$$+pR_{[i(1)[j(1)}g_{i(2)j(2)}\cdots g_{i(n)]j(n)]}\,.$$

In our previous paper ([2]) we have proved that

(3.4) $$0=\int\Delta(\xi_{i(1)\cdots i(p)}\xi^{i(1)\cdots i(p)})dv=2\int K_{i(1)\cdots i(p)j(1)\cdots i(p)}\xi^{i(1)\cdots i(p)}\xi^{i(1)\cdots i(p)}dv$$
$$+2\int\xi_{i(1)\cdots i(p);\,r}\xi^{i(1)\cdots i(p);\,r}dv$$
$$=2\int\{p(p-1)R_{a(1)b(1)a(2)b(2)}+pg_{a(1)b(1)}R_{a(2)b(2)}\}\xi^{a(1)a(2)a(3)\cdots a(p)}\xi^{b(1)b(2)}{}_{a(3)\cdots a(p)}dv$$
$$+2\int\xi_{i(1)\cdots i(p);\,r}\xi^{i(1)\cdots i(p);\,r}dv\,.$$

Hence, if the quadratic form

(3.5) $$\{(p-1)R_{acbd}+g_{ac}R_{bd}\}f^{ab}f^{cd}\quad(f^{ab}=-f^{ba})$$

is everywhere positive semi-definite, it follows that

(3.6) $$\xi^{i(1)\cdots i(p);\,r}=0\,.$$

The solution of the equation (3.6) involves at most $\binom{n}{p}$ arbitrary constants. On the other hand, we know from Hodge's theorem that any harmonic tensor is a linear combination (with constant coefficients) of B_p fundamental harmonic tensors. Hence we have the

THEOREM 2 (Bochner-Lichnerowicz). *If the quadratic form* (3.5) *is everywhere positive semi-definite, then it follows that*

$$B_p\leqq\binom{n}{p}.$$

Next we consider an harmonic vector ξ_i satisfying

(3.7) $$\xi_{i;\,j}=\xi_{j;\,i}\,,\qquad\xi_{i;\,j}g^{ij}=0$$

and a certain tensor A_{ijk} satisfying

(3.8) $$A_{ijk}=A_{jik}.$$

By Green's theorem we have

(3.9) $$0=\int\Delta\{(\xi_{i;\,j}+A_{ijk}\xi^k)(\xi^{i;\,j}+A^{ij}{}_m\xi^m)\}dv$$
$$=2\int\Delta(\xi_{i;\,j}+A_{ijk}\xi^k)(\xi^{i;\,j}+A^{ij}{}_m\xi^m)dv$$
$$+2\int(\xi_{i;\,j}+A_{ijk}\xi^k)_{;\,r}(\xi^{i;\,j}+A^{ij}{}_m\xi^m)^{;\,k}dv\,.$$

If in this case the following relation is satisfied

(3.10) $$\Delta(\xi_{i;\,j}+A_{ijk}\xi^k)=c(\xi_{i;\,j}+A_{ijk}\xi^k) \qquad (c>0),$$

where c is a certain positive constant, then (3.9) becomes

(3.11) $$0=c\int(\xi_{i;\,j}+A_{ijk}\xi)(\xi^{i;\,j}+A^{ij}{}_m\xi^m)dv$$
$$+\int(\xi_{i;\,j}+A_{ijk}\xi^k)_{;\,r}(\xi^{i;\,j}+A^{ij}{}_m\xi^m)^{;r}dv\,.$$

Hence we have in this case

(3.12) $$\xi_{i;\,j}+A_{ijk}\xi^k=0\,.$$

Since the solution of the equation (3.12) involves at most n arbitrary constants, we have in this case

(3.13) $$B_1\leqq n\,.$$

In order that the relation (3.10) be satisfied by any harmonic vector, it is sufficient that

(3.14) (a) $$\Delta A_{ijk}+A_{ijm}R^m{}_k+R_{ki;\,j}+R_{kj;\,i}-R_{ij;\,k}=cA_{ijk} \quad (c>0),$$

(b) $$\sum_{(k,l)}\left(A_{ijk;\,l}-R_{kijl}+\frac{R_{ki}g_{lj}+R_{lj}g_{ik}-cg_{ik}g_{jl}}{2}\right)=0\,,$$

where $\sum_{(k,l)}$ denotes $(k,l)+(l,k)$

Thus we have the

THEOREM 3. *If there exist a tensor A_{ijk} and a positive constant c satisfying the equations* (3.14), *it follows that*

$$B_1 \leqq n.$$

Next, if the relations (3.14) are satisfied by $c=0$ and a certain A we have from (3.11) that

$$(\xi_{i;\,j}+A_{ijm}\xi^m)_{;r}=0\,. \tag{3.15}$$

The last equation and

$$\xi_{i;\,j}=\frac{\partial\xi_i}{\partial x^j}-\left\{\begin{matrix}m\\ ij\end{matrix}\right\}\xi_m$$

constitute a system of differential equations for ξ_i and $\xi_{i;\,j}$. Its solution involves at most

$$\frac{n(n+1)}{2}+n-1$$

arbitrary constants. Thus we have the

THEOREM 4. *If the equations* (3.14) *are satisfied by $c=0$ and a certain A_{ijk}, it follows that*

$$B_1 \leqq \frac{n(n+1)}{2}+n-1\,. \tag{3.16}$$

Let $\xi_{i(1)\cdots i(p)}$ be harmonic and $A_{i(1)\cdots i(p)r}{}^{j(1)\cdots j(p)}$ be *a* certain tensor. By Green's theorem we have

$$\begin{aligned}0=&\int\Delta\{\xi_{i(1)\cdots i(p);\,r}+A_{i(1)\cdots i(p)r}{}^{j(1)\cdots j(p)}\xi_{j(1)\cdots j(p)})\\ &\qquad\times(\xi^{i(1)\cdots i(p);\,r}+A^{i(1)\cdots i)p)r}{}_{a(1)\cdots a(p)}\xi^{a(1)\cdots a(p)})\}dv\\ =&2\int\Delta(\xi_{i(1)\cdots i(p);\,r}+A_{i(1)\cdots i(p)r}{}^{j(1)\cdots j(p)}\xi_{j(1)\cdots j(p)})\\ &\qquad\times(\xi^{i(1)\cdots i(p);\,r}+A^{i(1)\cdots i(p)r}{}_{a(1)\cdots a(p)}\xi^{a(1)\cdots a(p)})dv\\ &+2\int(\xi_{i(1)\cdots i(p);\,r}+A_{i(1)\cdots i(p)r}{}^{j(1)\cdots j(p)}\xi_{j(1)\cdots j(p)})_{;\,s}\\ &\qquad\times(\xi^{i(1)\cdots i(p);\,r}+A^{i(1)\cdots i(p)r}{}_{a(1)\cdots a(p)}\xi^{a(1)\cdots a(p)})^{;\,s}dv\,.\end{aligned} \tag{3.17}$$

We assume that

(3.18) $$A_{[i(1)\cdots i(p)r]}{}^{j(1)\cdots j(p)}=0\,.$$

Moreover, if the following relation is satisfied

(3.19) $$\Delta(\xi_{i(1)\cdots i(p);\,r}+A_{i(1)\cdots i(p)r}{}^{j(1)\cdots j(p)}\xi_{j(1)\cdots j(p)}) \\ =c(\xi_{i(1)\cdots i(p);\,r}+A_{i(1)\cdots i(p)r}{}^{j(1)\cdots j(p)}\xi_{j(1)\cdots j(p)})$$

provided that c is a certain positive constant, then we have from (3.17)

(3.20) $$\xi_{i(1)\cdots i(p);\,r}+A_{i(1)\cdots i(p)r}{}^{j(1)\cdots j(p)}\xi_{j(1)\cdots j(p)}=0\,.$$

The solution of the last equation involvs at most $\binom{n}{p}$ arbitrary constants. Hence we have in this case

$$B_p\leqq\binom{n}{p}.$$

The sufficient conditions that (3.16) become (3.19) are as follows:

(3.21) $$\begin{cases} \text{(a)} \quad \Delta A_{i(1)\cdots i(p)r\,j(1)\cdots j(p)}+A_{i(1)\cdots i(p)r}{}^{a(1)\cdots a(p)}K_{a(1)\cdots a(p)j(1)\cdots j(p)} \\ \qquad\qquad +B_{i(1)\cdots i(p)r\,j(1)\cdots j(p)}cA_{i(1)\cdots i(p)r\,j(1)\cdots j(p)}\,, \\ \text{(b)} \quad \sum\limits_{(j(1)\cdots j(p)s)}\Big(A_{i(1)\cdots i(p)r\,j(1)\cdots j(p)s}+C_{i(1)\cdots i(p)r\,j(1)\cdots j(p)s} \\ \qquad\qquad -\dfrac{c}{2(p!)^2}g_{[i(1)[j(1)}\cdots g_{i(p)]j(p)]}g_{rs}\Big)=0\,, \end{cases}$$

where

$$B_{i(1)\cdots i(p)r\,j(1)\cdots j(p)}=K_{i(1)\cdots i(p)j(1)\cdots j(p);\,r}+pR_{[i(1)[j(1)rs}{}^{;s}g_{i(2)j(2)}\cdots g_{i(p)]j(p)]}\,,$$

$$C_{i(1)\cdots i(p)r\,j(1)\cdots j(p)s}=\frac{1}{2}K_{i(1)\cdots i(p)j(1)\cdots j(p)}g_{rs}+pR_{[i(1)[j(1)rs}g_{i(2)j(2)}\cdots g_{i(p)]j(p)]} \\ +\frac{1}{2}g_{[i(1)[j(1)}\cdots g_{i(p)]j(p)]}R_{rs}$$

and $\sum_{(j(1)\cdots j(p)s)}$ denotes

$$j(1)\cdots j(p)s+sj(2)\cdots j(p)j(1)+j(1)sj(3)\cdots j(p)j(2)+\cdots\cdots+j(1)j(2)\cdots sj(p).$$

Thus we have the

THEOREM 5. *If the equations* (3.18) *and* (3.21) *are satisfied by a certain tensor A and a positive constant c, then we have* $B_p \leqq \binom{n}{p}$. *If the equations are satisfied by* $c=0$, *then we have*

$$B_p \leqq \binom{n}{p} + n\binom{n}{p} - \binom{n}{p-1} - \binom{n}{p+1}.$$

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