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On uniform topologies in general spaces.

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The important notion of the uniformity, introduced by A. Weil[1] and others, shows its full effectiveness, when the space is completely regular. However, we can define the "generalized uniformity" of any neighbourhood space, as an arbitrary collection of correspondences assigning to every point of the space a neighbourhood. We shall show in § 2 of this paper, that the most part of the theory of uniformity holds also in the spaces with the generalized uniformity. We can consider also the completion of such spaces in several manners (§§ 3, 4). The usual way of completion by means of Cauchy filters (we have named it C-extension, § 5) does not give a complete space in general cases. In § 6 we shall consider some additional conditions on such spaces, and investigate the behaviour of the C-extensions of spaces satisfying these conditions.

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§ 1 Generalized uniformity.

1.1 Definition. We say that X is a *space*, if X is an aggregate of points, where a closure operator is defined which assigns to each subset M of X a closure $\overline{M}$ with the following properties:

(1) $\overline{M}\supset M$, (2) $M_1\subset M_2$ implies $\overline{M}_1\subset \overline{M}_2$, (3) $\overline{\phi}=\phi$.

Topological concepts, such as *neighbourhood* (abbr. nbd) of a point, continuity of mappings, etc., may be defined in our space in the well-known way.[2]

1) A. Weil: Sur les espaces à structure uniforme et sur la topologie générale. Actual. Sci. Ind. 551 (1938).

2) cf. e.g. J. W. Tukey: Convergence and uniformity in topology. Princeton Univ. (1940).

Let φ be a correspondence, which assigns to each point x of X a nbd $\varphi(x)$ of x. We shall consider in general an aggregate Ψ of such correspondences, and call it a *generalized uniformity* (abbr. GU). For example, a GU may consist of only one correspondence assigning to every point the whole space. This GU is called the *trivial* GU. The *maximal* GU, which we denote by the special letter Φ, is the maximal collection of such correspondences.

A GU Ψ of a space X is called *basic* at x, if the family of sets $\{\varphi(x), \varphi \in \Psi\}$ is a nbd basis of the point x of X. A basic GU is a GU, which is basic at every point of X. We say that a GU Ψ is *additive*, if there exists an element φ of Ψ for any pair of elements φ_1, φ_2 of Ψ, such that $\varphi(x) \subset \varphi_1(x) \frown \varphi_2(x)$ for each point $x \in X$. A GU Ψ is called *open*, if $\varphi(x)$ is an open set of X for every $\varphi \in \Psi$ and $x \in X$. Clearly, the trivial GU is an open additive GU, and the maximal GU is basic.

The following proposition is evident.

PROPOSITION 1. *The closure operator of the space is additive if and only if there exists an additive basic* GU, *and a closure of any subset is closed if and only if there exists an open basic* GU.

Let Y be a subspace of a space X with a GU Ψ_X. We denote by φ_Y the contraction of an element φ_X of Ψ_X to Y, i. e., φ_Y is defined on Y and $\varphi_Y(x) = \varphi_Y(x) \frown Y$. The collection $\Psi_Y = \{\varphi_Y;\ \varphi_X \in \Psi_X\}$ is called a *contraction* of Ψ_X to Y.

Clearly, the contraction Ψ_Y of a GU Ψ_X to its subspace Y is a GU of Y, and conversely, any GU of Y is a contraction of a certain GU of X. If Ψ_X is basic (additive or open), then the contraction Ψ_Y of Ψ_X to Y is also basic (additive or open respectively)

1.2 **Equivalence of GU.** Now, let Ψ be any collection of correspondences, whose element φ assigns to each point x of X a subset $\varphi(x) \ni x$ of X (not necessarily a nbd of x). Then we can define a closure operator with the properties (1), (2) and (3) above, taking $\{\varphi(x);\ \varphi \in \Psi\}$ as a nbd basis of a point x, and speak of the topology of X induced by Ψ. The words such as "Ψ-closure" or "Ψ-converge" will indicate that they refer to this topology. It is evident that Ψ is a basic GU of X with respect to Ψ-topology.

For two collections Ψ_1 and Ψ_2, we denote:

$\Psi_1 < \Psi_2$ if there exists an element φ_2 of Ψ_2 for any element φ_1 of Ψ_1

such that $\varphi_2(x) \subset \varphi_1(x)$ to every point x of X.

$\Psi_1 \sim \Psi_2$ if $\Psi_1 < \Psi_2$ and $\Psi_1 > \Psi_2$.

Clearly, the relation "$<$" is reflexive and transitive, and the relation "$\sim$" is an equivalence relation.

Let $[\Psi] = \cup\{\Psi' ; \Psi' \sim \Psi\}$. Then the following proposition is proved easily.

PROPOSITION 2. (1) *$\Psi_1 \supset \Psi_2$ implies $\Psi_1 > \Psi_2$, and $\Psi_1 > \Psi_2$ if and only if $[\Psi_1] \supset [\Psi_2]$.*

(2) *If Ψ_1 is a* GU, *Ψ_2 a basic* GU, *and $\Psi_1 > \Psi_2$, then Ψ_1 is also basic.*

(3) *If $\Psi_1 > \Psi_2$, then the Ψ_1-topology is stronger than the Ψ_2-topology.*

(4) *If Ψ is a* GU *of a space X, then the Ψ-topology is weaker than the original topology of X.*

(5) *If Ψ is a basic* GU *of a space X, then the Ψ-topology is equivalent with the original topology of X.*

(6) *Let $\mathfrak{M}$ be a filter*[3] *and Ψ a basic* GU *of X, then*

$\mathfrak{M} \rightarrow x$ *if and only if $\varphi \in \Psi$ implies $\varphi(x) \in \mathfrak{M}$.*

$\mathfrak{M} \rightharpoonup x$ *if and only if $\varphi \in \Psi$ and $M \in \mathfrak{M}$ implies $\varphi(x) \cap M \neq \phi$.*

Here $\mathfrak{M} \rightarrow x$ means that x is a limit point of $\mathfrak{M}$, i.e., $\mathfrak{M}$ contains every nbd of x, and $\mathfrak{M} \rightharpoonup x$ means that x is contained in the intersection $\cap\{\bar{M} ; M \in \mathfrak{M}\}$.

§ 2 Some properties defined by GU.

2.1 Uniformly continuous mapping. Let $f : X \rightarrow Y$ be a mapping of a space X into another space Y, and Ψ_X, Ψ_Y be GU's of X and Y respectively. A mapping f is called *uniformly continuous* with respect to Ψ_X and Ψ_Y (abbr. $\Psi_X \cdot \Psi_Y$-u-continuous), if it satisfies the following condition:

(U) For any φ_Y of Ψ_Y, there exists an element φ_X of Ψ_X such that $f\varphi_X(x) \subset \varphi_Y f(x)$ for every point x of X.

The following propositions and theorem are proved easily.

PROPOSITION 3. *If $f : X \rightarrow Y$ and $g : Y \rightarrow Z$ are uniformly continuous mappings with respect to Ψ_X and Ψ_Y, Ψ_Y and Ψ_Z respectively, then the composite $gf : X \rightarrow Z$ is uniformly continuous with respect to Ψ_X and Ψ_Z.*

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3) Cf. e.g. N. Bourbaki: Topologie générale. Actual. Sci. Ind. 858 (1940).

PROPOSITION 4. *Let* $\Psi_X > \Psi'_X$ *and* $\Psi_Y > \Psi'_Y$. *If a mapping* f: $X \to Y$ *is* $\Psi'_X \cdot \Psi_Y$*-u-continuous, then* f *is* $\Psi_X \cdot \Psi_Y$*-u-continuous and* $\Psi'_X \cdot \Psi'_Y$*-u-continuous.*

THEOREM 1. *If* Ψ_Y *is a basic* GU *of a space* Y, *then we have:*

(1) *A* $\Psi_X \cdot \Psi_Y$*-u-continuous mapping* f *of a space* X *into* X *is a continuous mapping.*

(2) *A mapping* f *of a space* X *into* Y *is continuous if and only if* f *is* $\Phi_X \cdot \Psi_Y$*-u-continuous.* (Φ_X *means the maximal* GU *of* X).

A one-to-one onto mapping $f: X \to Y$ is called a *unimorphism*, when f and f^{-1} is uniformly continuous. Let f be any mapping of a space X into Y. We denote by $f^{-1}(\Psi_Y)$ the collection of correspondences φ_X of $x \in X$ to the subset $f^{-1}\varphi_Y f(x)$ of X, where $\varphi_Y \in \Psi_Y$, i.e. $f^{-1}(\Psi_Y) = \{\varphi_X; \varphi_X(x) = f^{-1}\varphi_Y f(x), \varphi_Y \in \Psi_Y\}$.

PROPOSITION 5. (1) *When* Ψ_Y *is basic, a mapping* $f: X \to Y$ *is continuous if and only if* $f^{-1}(\Psi_Y)$ *is a* GU *of the space* X.

(2) *A mapping* $f: X \to Y$ *is* $\Psi_X \cdot \Psi_Y$*-u-continuous if and only if* $\Psi_X > f^{-1}(\Psi_Y)$.

(3) *A one-to-one onto mapping* $f: X \to Y$ *is a unimorphism if and only if* $\Psi_X \sim f^{-1}(\Psi_Y)$.

2.2 **Totally bounded set.** A subset X_1 of a space X with a GU Ψ is called [*self*] *totally bounded* with respect to Ψ (abbr. [self] Ψ-*t*-bounded), if for any $\varphi \in \Psi$ there exist finite number of points $x_1, \cdots, x_n$ [$\in X_1$] such that $X_1 \subset \bigcup_i \varphi(x_i)$. We can prove easily the following properties.

PROPOSITION 6. (1) *If* $\Psi > \Psi'$, *then a* [*self*] Ψ*-t-bounded set is also* [*self*] Ψ'*-t-bounded.*

(2) *If a mapping* $f: X \to Y$ *is* $\Psi_X \cdot \Psi_Y$*-u-continuous and a subset* X_1 *of* X *is* [*self*] Ψ_X*-t-bounded, then the image* $f(X_1)$ *is* [*self*] Ψ_Y*-t-bounded.*

A subset X_1 of a space X is called *conditionally compact* [*compact*], if for any family $\mathfrak{M} \ni X_1$ of subsets of X with the finite intersection property, there exists a point x [$\in X_1$] such that $x \in \cap\{\overline{M}; M \in \mathfrak{M}\}$.

THEOREM 2. *A subset* X_1 *of a space* X *is conditionally compact* [*compact*] *if and only if* X_1 *is* [*self*] Φ*-t-bounded. Therefore every conditionally compact subset of a space* X *is totally bounded with respect to any* GU Ψ *of* X.

PROOF. As $\cap\{\overline{X_1 - \varphi(x)}, x \in X\} = \phi$ [$\cap\{\overline{X_1 - \varphi(x)}, x \in X_1\} \subset X - X_1$]

for any $\varphi \in \Phi$, the condition is necessary. Let $\{M_\alpha ; \alpha \in A\}$ be a family of subsets M_α of X_1, and $M=\cap\{\bar{M}_\alpha ; \alpha \in A\}=\phi$ $[M \cap X_1=\phi]$. Then there exists $\varphi \in \Phi$ such that $\varphi(x)=X-M_{\alpha(x)}$, $\alpha(x) \in A$ to each point $x\,[\in X_1]$. Thus the condition is also sufficient. The rest follows easily from the proposition 6.

This theorem shows that compactness can be defined as the totally boundedness with respect to the special GU Φ. It is a special case of (2) of the proposition 6, that a continuous image of a compact set is compact.

2.3 **Ψ-filter.** A filter $\mathfrak{M}$ in a space X with a GU Ψ is called a *Ψ-filter,* if for any element φ of Ψ there exists a point x such that $\varphi(x) \in \mathfrak{M}$. A maximal filter, which is also a Ψ-filter, is called a *maximal Ψ-filter.* Clearly, if $\Psi > \Psi'$, then a Ψ-filter is also a Ψ'-filter.

Now we can prove easily the following theorems.

THEOREM 3. *A filter $\mathfrak{M}$ has a limit point if and only if $\mathfrak{M}$ is a Φ-filter.*

THEOREM 4. *If a mapping $f: X \rightarrow Y$ is $\Psi_X \cdot \Psi_Y$-u-continuous and $\mathfrak{M}$ is a Ψ_X-filter of the space X, then, $f(\mathfrak{M})$ (i.e. the family of subsets N of Y such that $f^{-1}(N) \in \mathfrak{M}$) is a Ψ_Y-filter.*

THEOREM 5. *A subset X_1 is Ψ-t-bounded if and only if any maximal filter $\mathfrak{M} \ni X_1$ is a Ψ-filter. Therefore X_1 is conditionally compact if and only if any maximal filter $\mathfrak{M} \ni X_1$ has a limit point.*

PROOF. We shall prove only the theorem 5. If $X_1 \subset \cup\{\varphi(x_i); i=1, \cdots, n\}$, then a maximal filter $\mathfrak{M} \in X_1$ contains actually a certain $\varphi(x_i)$. Thus the condition is necessary. Let the condition be fulfilled, and $\varphi \in \Psi$. If $M_\alpha=X_1-\cup\{\varphi(x); x \in \alpha\} \neq \phi$ for any finite subset α of X, then M_α's belong to a certain maximal filter $\mathfrak{M} \ni X_1$, and there exists a point x_0 such that $\varphi(x_0) \in \mathfrak{M}$. But $M_{x_0}=X-\varphi(x_0) \notin \mathfrak{M}$ is in contradiction with $M_{x_0} \in \mathfrak{M}$. Thus the condition is also sufficient. The rest follows from the theorem 2 and the theorem 3.

2.4 **Ψ-completeness.** We call *Ψ-set* every non vacuous finite intersection of sets expressed as $\varphi(x)$ or $X-\varphi(x)$. A subset X_1 of a space X with a GU Ψ is called *conditionally Ψ-complete* [*Ψ-complete*] if any of the following three equivalent conditions is fulfilled.

(1) Any maximal Ψ-filter $\mathfrak{M} \ni X_1$ has a Ψ-limit point $x\,[\in X_1]$.

(2) For any Ψ-filter $\mathfrak{M} \ni X_1$, there exists a point $x\,[\in X_1]$ such that $\varphi \in \Psi$, $M \in \mathfrak{M}$ imply $\varphi(x) \cap M \neq \phi$.

(3) For any Ψ-filter $\mathfrak{M} \ni X_1$, there exists a point $x\,[\in X_1]$ such that $\varphi \in \Psi$, Ψ-set $M \in \mathfrak{M}$ imply $\varphi(x) \cap M \neq \phi$.

Clearly any subset of X is conditionally Φ-complete, and any closed subset is Φ-complete. If a space X is Ψ-complete with respect to a basic GU Ψ of X, then any maximal Ψ-filter has a limit point. A subset X_1 of X is Ψ-complete if and only if X_1 is Ψ_1-complete with respect to the contraction Ψ_1 of Ψ to X_1.

From theorem 2 and 5, we have directly.

THEOREM 6. *If a subset X_1 is conditionally compact [compact], then X_1 is [self] Ψ-t-bounded and conditionally Ψ-complete [Ψ-complete]. When Ψ is a basic* GU, *the converse is also true.*

Remark. A simple example shows that, in case Ψ is not basic, Ψ-t-bounded and Ψ-complete space is not necessarily compact.

PROPOSITION 7. *If $\Psi > \Psi'$ and Ψ' is a basic* GU, *then a conditionally Ψ'-complete set is conditionally Ψ-complete.*

§ 3 Completion of a space.

3.1 **Space Z.** Let Ψ_X be a GU of a space X. We introduce the following notations: For two Ψ-filters $\mathfrak{M}_1$, $\mathfrak{M}_2$,

$\mathfrak{M}_1 < \mathfrak{M}_2$ means that $\varphi(x) \in \mathfrak{M}_1$ implies $\varphi(x) \in \mathfrak{M}_2$ for any element φ of Ψ_X and any point x of X.

$\mathfrak{M}_1 \approx \mathfrak{M}_2$ means that $\mathfrak{M}_1 < \mathfrak{M}_2$ and $\mathfrak{M}_1 > \mathfrak{M}_2$. (We say that $\mathfrak{M}_1$ is equivalent to $\mathfrak{M}_2$).

For a maximal Ψ_X-filter $\mathfrak{M}$,

$[\mathfrak{M}]$ is the class of maximal Ψ_X- filters equivalent to $\mathfrak{M}$.

$[\mathfrak{M}] \ni X_1$ means that there exists a Ψ_X-set M of $\mathfrak{M}$ such that $M \subset X_1$.

$[\mathfrak{M}] \to x$ means that $N \ni [\mathfrak{M}]$ if N is a nbd of x.

Clearly two maximal Ψ_X-filters $\mathfrak{M}_1$, $\mathfrak{M}_2$ are equivalent to each other if and only if any Ψ_X-set M of $\mathfrak{M}_1$ belongs also to $\mathfrak{M}_2$. Therefore the definition of $[\mathfrak{M}] \ni X_1$ does not depend on the choice of $\mathfrak{M}$ from the class $[\mathfrak{M}]$. Obviously, $[\mathfrak{M}] \to x$ implies $\mathfrak{M} \to x$, and if Ψ_X is basic, then conversely $\mathfrak{M} \to x$ implies $[\mathfrak{M}] \to x$.

The class $[\mathfrak{M}]$ of maximal Ψ-filters is called a *non-convergent class,* if $[\mathfrak{M}] \to x$ for no point x of X. We consider a non-convergent class of maximal Ψ_X-filters as a point, and denote by Z_0 the set of all these classes, by Z the set-union of Z_0 and X.

PROPOSITION 8. *Let* $g(X_1)=X_1+\{z\,;\, z\in Z_0, z=[\mathfrak{M}]\ni X_1\}$, *then we have*

(1) $g(X_1)\frown X=X_1, \quad g(X)=Z, \quad g(\phi)=\phi$.

(2) $X_1\supset X_2$ *implies* $g(X_1)\supset g(X_2)$.

(3) $g(\frown_i X_i)=\frown_i g(X_i)$ *for any finite number of subsets* X_i *of* X.

(4) *If* $\{X_i\}$ *is a finite family of* Ψ_X*-sets, then* $Z-g(X_1)=g(X-X_1)$ *and* $g(\smile_i X_i)=\smile_i g(X_i)$.

PROOF. (1) and (2) are evident. (3) follows from (1) and (2), for a finite intersection of Ψ-sets is also a Ψ-set. (4) follows from (1) and (3), for the complement of Ψ-set X_1 is a finite union of Ψ-sets.

Now, we define a nbd system of Z as follows: a) For a point x of X, a nbd of x is a such subset U of Z that $g(N)\subset U$ for a certain nbd N of x in the space X. b) For a point z of Z_0, a nbd of $z=[\mathfrak{M}]$ is a such subset U of Z that $g(\frown_i \varphi_i(x_i))\subset U$ for some $\varphi_i\in\Psi_X$, $\varphi_i(x)\in[\mathfrak{M}]$, $i=1,\cdots,n$.

Then Z is a space, and the nbd system of a point z of Z_0 forms a filter. Clearly, for a subset M of X, the closure of M in the space X is the intersection of X and the closure $\overline{M}$ of M in the space Z. A subset $U=g(\frown\{\varphi_i(x_i)\,;\, i=1,\cdots,n\})$ is a nbd of any point z in $U\frown Z_0$. If z_1, z_2 are distinct points of Z_0, then there exist a point x in X and an element φ of Ψ_X such that $\varphi(x)\in z_i, \notin z_j\,;\,(i,j)=(1,2)$. This shows $\bar{z}_j\not\ni z_i$. Thus we have proved.

PROPOSITION 9. *The extended space* Z *has the following properties:*

(1) X *is a subspace of* Z.

(2) $\overline{X_1}\supset g(X_1)$ *for any subset* X_1 *of* X.

(3) $(\overline{K}\smile\overline{H})\frown Z_0=(\overline{K\smile H})\frown Z_0$ *for any two subsets* K, H *of* Z.

(4) *Any nbd* $g(\frown_i \varphi_i(x_i))$ *of a point* z *of* Z_0 *is relatively open in the subspace* Z_0.

(5) *The subspace* Z_0 *is a* T_0*-space.*

PROPOSITION 10. If $z=[\mathfrak{M}]\in Z_0$, *then* $\bar{z}=(\frown\{\overline{M}\,;\, M\in\mathfrak{M}\})\frown Z_0$, *especially* $\bar{z}=\frown\{\overline{M}\,;\, M\in\mathfrak{M}\}\subset Z_0$ *when* Ψ_X *is basic.*

PROOF. $\bar{z}\subset Z_0$, since to any point $x\in X$ there exists a nbd N of x in the space X such that $N\notin[\mathfrak{M}]$. Let $z'\in\bar{z}$, then any nbd $g(\frown_i \varphi_i(x_i))$ of z', contains z, that is $\frown_i \varphi_i(x_i)\in\mathfrak{M}$. Thus $M\in\mathfrak{M}$ implies $M\frown g(\frown_i \varphi_i(x_i))\neq\phi$. This shows that $\bar{z}$ is a subset of $\frown\{\overline{M}\,;\, M\in\mathfrak{M}\}$. Conversely, if $z'\in Z_0-\bar{z}$, then there exists a nbd $g(\frown_i \varphi_i(x_i))$ of z', which does not contain z. As $M=\frown_i \varphi_i(x_i)$ is a Ψ_X-set, $g(M)\not\ni z$ implies

$M \notin \mathfrak{M}$. Thus $X-M \in \mathfrak{M}$ and $X-M$ does not intersect with the nbd $g(M)$ of z'. Therefore we have $\bar{z} \supset (\cap \{\bar{M}; M \in \mathfrak{M}\} \cap Z_0$. When Ψ_X is basic, there exists for any $x \in X$ an element φ of Ψ_X such that $\varphi(x) \notin \mathfrak{M}$. Thus we have $x \notin \cap \{\bar{M}; M \in \mathfrak{M}\}$, since $X-\varphi(x) \in \mathfrak{M}$. Therefore $\cap \{\bar{M}; M \in \mathfrak{M}\} \subset Z_0$.

PROPOSITION 11. *For any* $\varphi \in \Psi$ *and* $x \in X$, *we have* $Z_0 \cap g\varphi (X-\varphi(x)) = Z_0 \cap \overline{X-\varphi(x)}$. *Especially* $g\varphi(x) = Z - \overline{X-\varphi(x)}$, *and this set is open in* Z, *when* $\varphi(x)$ *is an open subset of the space* X.

PROOF. Let $z=[\mathfrak{M}] \in Z_0$. If $z \in g(X-\varphi(x)) = Z - g\varphi(x)$, then $z \in \overline{X-\varphi(x)}$ by proposition 9. If $z \notin g(X-\varphi(x))$, then we have $z \notin \overline{X-\varphi(x)}$, since $g\varphi(x)$ is a nbd of z. Thus the first half is proved. The rest is an easy consequence of this.

3.2 **GU of the space** Z. Now we define a GU of Z as follows: For any finite system $\{\varphi_1, \cdots, \varphi_n\}$ of elements of Ψ_X, let ψ be a correspondence assigning to each point $x \in X$ a nbd $\psi(x) = g\varphi_i(x)$ of x, and to each point $z \in Z_0$ a nbd $\psi(z) = g(\underset{i}{\cap}\ \varphi_i(x_i))$ of z. Then the element φ_1 can be regarded as the contraction of ψ to X, so it is also denoted by ψ_X. The collection of all correspondences such as ψ, taking $\{\varphi_1, \cdots, \varphi_n\}$ in every possible way, constitutes clearly a GU of Z, and Ψ_X is a contraction of this GU to X. We denote this GU of Z by Ψ_Z. The space Z with this GU Ψ_Z is called a space *obtained by completion* of the space X with respect to Ψ_X.

PROPOSITION 12. *The space Z with the* GU Ψ_Z *has the following properties:*

(1) Ψ_Z *is basic at each point* z *of* Z_0. *Therefore* Ψ_Z *is basic if* Ψ_X *is basic in* X.
(2) *If* X_1 *is a* [*self*] Ψ_X-*t*-*bounded subset of* X, *then* $g(X_1)$ *is* [*self*] Ψ_Z-*t*-*bounded.*
(3) *Any maximal* Ψ_Z-*filter converges to a point* z *in* Z. *Therefore* $g(X_1)$ *is conditionally* Ψ_Z-*complete for any subset* X_1 *of* X, *and* $g(X_1)$ *is conditionally compact if* X_1 *is a* Ψ_X-*t*-*bounded subset of* X.
(4) *The completion of* Z *with respect to* Ψ_Z *adds no new point.*

PROOF. (1) and (2) are proved easily. (3): Let $\mathfrak{F}$ be a maximal Ψ_Z-filter containing $g(X_1)$ as an element. Then $\mathfrak{M} = \{M; M \subset X, g(M) \in \mathfrak{F}\}$ is a Ψ_X-filter. Therefore any maximal filter $\mathfrak{M}_1 \supset \mathfrak{M}$ is a maximal Ψ_X-filter containing X_1 as an element. If $\varphi(x) \notin \mathfrak{M}$, then $g\varphi(x) \notin \mathfrak{F}$ and

$X-\varphi(x)\in\mathfrak{M}$ follows from $g(X-\varphi(x))=Z-g\varphi(x)\in\mathfrak{F}$. Similarly $X-\varphi(x)\notin\mathfrak{M}$ implies $\varphi(x)\in\mathfrak{M}$. This shows that any maximal Ψ_X-filter $\mathfrak{M}'\supset\mathfrak{M}$ belongs to the same class $[\mathfrak{M}_1]$. Let $[\mathfrak{M}_1]\to x_1\in X$, and N be a nbd of x_1 in the space X. There exists a Ψ_X-set M such that $N\supset M\in\mathfrak{M}$. This shows that $\mathfrak{F}$ converges to x_1 in the space Z. Let $[\mathfrak{M}_1]$ be a non convergent class, then $z_1=[\mathfrak{M}_1]$ is a point of Z_0. If $g(\frown_i \varphi_i(x_i))$ is a nbd of z_1, then we have $\frown_i \varphi_i(x_i)\in\mathfrak{M}$. This shows that $\mathfrak{F}$ converges to z_1 in the space Z. The rest follows easily from the theorem 6. (4) is evident by (3).

PROPOSITION 13. *The space Z obtained by completion of X has the following properties:*

(1) *Z is an additive space (i.e. with the additive closure operator) if X is an additive space.*

(2) *Z is a T-space if X is a T-space and Ψ_X is open and basic.*

(3) *Z is a T_0-space if X is a T_0-space and Ψ_X is open and basic.*

3.3 **Uniqueness of completion.** The completion considered above is characterized by

THEOREM 7. *Let Ψ_X be a basic* GU *of a space X. The space Z obtained by the completion of X with respect to ψ_X has the following properties (described for S). Conversely, a space S with these properties is mapped on Z by a homeomorphism leaving each point of X fixed.*

(1) *X is a subspace of S.*

(2) *If p_1, p_2 are distinct points of $S-X$, then $\bar{p}_1\subset S-X$ and $\bar{p}_1\neq p_2$.*

(3) *There exists a basic* GU *Ψ_S, whose contraction to X is Ψ_X.*

(4) *$\{U;\ U=\frown_i \psi_i(x_i)\ni p,\ \psi_i\in\Psi_S,\ x_i\in X,\ i=1,\cdots,n\}$ is a nbd basis of $p\in S-X$.*

(5) *Any non-convergent maximal Ψ_X-filter $\mathfrak{M}$ in X has a point $p\in S-X$ such that $\psi(x)\ni p$ if and only if $\psi(x)\frown X\in\mathfrak{M}$ for any $\psi\in\Psi_S$, $x\in X$, and conversely, for any point $p\in S-X$ there exists a non-convergent maximal Ψ-filter $\mathfrak{M}$ for which p satisfies the above condition.*

PROOF. We have proved already that Z has these properties (1)-(5). Let S be another space with these properties (1)-(5). For a non-convergent maximal Ψ_X-filter $\mathfrak{M}$, there exists a point $p=f(\mathfrak{M})$ such that $\psi(x)\ni p$ if and only if $\psi(x)\frown X\in\mathfrak{M}$. Let $p_i=f(\mathfrak{M}_i)$, $i=1,2$. If $p_1\neq p_2$, then there exists a point $q\in S-X$ such that $q\in\bar{p}_i$, $\notin\bar{p}_j$, $(i,j)=(1,2)$. By (4), we have a nbd $\psi(x)$ of q which contains p_i but not p_j, that is

$\psi(x)\frown X \in \mathfrak{M}_i$, $\notin \mathfrak{M}_j$. This shows $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$ by (3). If $p_1=p_2=p$, then $\psi(x)\frown X \in \mathfrak{M}_1$ if and only if $\psi(x)\frown X \in \mathfrak{M}_2$. This shows $\mathfrak{M}_1 \approx \mathfrak{M}_2$. Therefore we can define a mapping f of Z into S by putting $f(x)=x$ for $x \in X$, and $f(z)=f(\mathfrak{M})$ for $z=[\mathfrak{M}] \in Z-X$. Clearly f is a one-to-one mapping of Z onto S and we have $fg(\psi(x)\frown X)=\psi(x)$. Therefore f is a topological mapping, since $z \in g((\frown_i \psi_i(x_i))\frown X)$ if and only if $f(z) \in fg((\frown_i \psi_i(x_i))\frown X)=\frown_i \psi_i(x_i)$.

Example. Let R be a space of real numbers with the usual topology, and X be a subspace of R con:tituted by all rational numbers. $\Psi_X =\{\varphi_\varepsilon\,;\,\varepsilon>0\}$ be an open basic GU (named ε-GU), where φ_ε is defined by $\varphi_\varepsilon(x)=\{x'\,;\,|x'-x|<\varepsilon\}$ for any positive real number ε. $\Psi'_X=\{\varphi_r;\,r$ is rational$\}$ is a sub-collection of Ψ_X. Clearly Ψ'_X is an open basic GU of X, and is equivalent to Ψ_X. A filter $\mathfrak{M}$ is a Ψ_X-filter if and only if $\mathfrak{M}$ is a Cauchy filter in the usual sence. Therefore to each maximal Ψ_X-filter $\mathfrak{M}$ corresponds a point λ of R, to which $\mathfrak{M}$ converges in R. If λ is rational, then $\mathfrak{M}$ converges to λ in X, if λ is irrational, then $[\mathfrak{M}]$ is non-convergent. Let $\mathfrak{M} \to \lambda$ in R. Then we have $\varphi_r(x) \in \mathfrak{M}$ if and only if $|x-\lambda|<r$ for any $\varphi_r \in \Psi'_X$. Therefore the space obtained by the completion of X with respect to Ψ'_X is topologically equivalent to R. On the other hand, a maximal Ψ_X-filter $\mathfrak{M}$, which corresponds to an irrational number λ, contains the subset (λ, ∞) or $(-\infty, \lambda)$ of X. We denote the first case by $\mathfrak{M} \to \lambda^+$, the second case by $\mathfrak{M} \to \lambda^-$. If $\mathfrak{M}_1 \to \lambda^+$ and $\mathfrak{M}_2 \to \lambda^+$, then clearly $\mathfrak{M}_1 \approx \mathfrak{M}_2$ with respect to Ψ_X. But if $\mathfrak{M}_1 \to \lambda^+$ and $\mathfrak{M}_2 \to \lambda^-$, then we have $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$ and $\mathfrak{M}_1$ does not converge to $\lambda^-=[\mathfrak{M}_2]$ in the space Z obtained by the completion of X with respect to Ψ_X. This example shows that equivalent GU's of a space do not necessarily lead to equivalent spaces by the completion.

§ 4 Compactification.

4.1 Special filter. If a space X is Ψ_X-t-bounded, then the space Z obtained by the completion of X is a compact space. For example, we obtain the "trivial compactification" of X, taking the trivial GU as Ψ_X. But for purpose of compactification, it happens that the completion considered in § 3 adds some superfluous points, even if Ψ_X is basic. So it seems more adequate to restrict with some conditions the classes of equivalent filters taken as new points. We assume that Ψ_X is basic and X is Ψ_X-t-bounded throughout this section.

A maximal Ψ-filter $\mathfrak{M}$ is called a *special filter*, if any maximal filter $\mathfrak{M}' < \mathfrak{M}$ is equivalent to $\mathfrak{M}$. A point $z=[\mathfrak{M}]$ of Z_0 is called a *special point*, if $\mathfrak{M}$ is a special filter.

PROPOSITION 14. *If a space X is totally bounded with respect to a basic* GU Ψ_X, *then we have:*

(1) *$\mathfrak{M}$ is a special filter if and only if there exist for any $\varphi(x) \in \mathfrak{M}$ a finite number of elements φ_i of Ψ_X and points x_i of X such that $\varphi(x) \supset \bigcap_i (X-\varphi_i(x_i)) \in \mathfrak{M}$.*

(2) *If $z_i=[\mathfrak{M}_i] \in Z_0$, $i=1,2$, then $\mathfrak{M}_1 < \mathfrak{M}_2$ is equivalent to $z_1 \in \bar{z}_2$.*

(3) *For any point z of Z_0, $z=\bar{z}$ if and only if z is a special point.*

(4) *For any point z of Z_0, there exists a special point z_1 such that $z_1 \in \bar{z}$.*

PROOF. (1): If the condition is not fulfilled for a certain $\varphi_0(x_0) \in \mathfrak{M}$, then any finite number of sets $X-\varphi_i(x_i) \in \mathfrak{M}$, $i=1,\cdots,n$ and $X-\varphi_0(x_0)$ have a non vacuous intersection. Thus there exists a maximal filter $\mathfrak{M}_1$ containing all of these sets. As $\varphi'(x') \notin \mathfrak{M}$ implies $\varphi'(x') \notin \mathfrak{M}_1$, so we have $\mathfrak{M}_1 < \mathfrak{M}$. But clearly $\mathfrak{M}_1 \not\approx \mathfrak{M}$. Thus the condition is necessary. The condition is also sufficient, for $\mathfrak{M}_1 < \mathfrak{M}$, $\varphi(x) \in \mathfrak{M}$ imply $\varphi(x) \in \mathfrak{M}_1$.

(2): This is proved easily.

(3): Let $z=[\mathfrak{M}]$ and $z=\bar{z}$, and $\mathfrak{M}_1 < \mathfrak{M}$ be a maximal filter. Then $\mathfrak{M}_1$ is a Ψ_X-filter for X is Ψ_X-t-bounded. If $[\mathfrak{M}_1] \to x$, then $[\mathfrak{M}] \to x$ in contradiction with $z \in Z_0$. Thus $z_1=[\mathfrak{M}_1] \in Z_0$, and $z_1 \in \bar{z}=z$ implies $\mathfrak{M}_1 \approx \mathfrak{M}$ by (2). Therefore the condition is necessary. The condition is clearly sufficient by (2).

(4): By the proposition 10, we can prove easily that $\bar{\bar{z}} \subset Z_0$. Therefore $\bar{\bar{z}}=\bar{z}$ since the subspace Z_0 is a T-space. Let $\mathfrak{F} \ni \bar{z}$ be a maximal family of closed sets with the finite intersection property. The compactness of Z shows that the intersection M of all sets of $\mathfrak{F}$ contains a point z_1. Clearly $z_1 \in \bar{z}_1$, $\bar{z}_1 \in \mathfrak{F}$ and $M \supset \bar{z}_1$. If $z_2 \in \bar{z}_1$ then $\bar{z}_2$ belongs to $\mathfrak{F}$, and $z_1=z_2$ follows from $\bar{z}_2 \ni z_1$, $\bar{z}_1 \ni z_2$. This shows that z_1 is special.

PROPOSITION 15. *Let R be a set-union of X and the set of all special points. Then we have:*

(1) *The subspace R of X is compact.*

(2) *The subspace $R-X$ is a T_1-space.*

(3) *If X is compact, then $R=X$.*

PROOF. For any maximal filter $\mathfrak{N}$ in R, $\mathfrak{F}=\{H; H\subset Z, H\frown R\in\mathfrak{N}\}$ is a maximal filter in Z converging to a point z. If $z\in R$, then clearly $\mathfrak{N}\rightarrow z$ in R. If $z\in Z-R$, then there exists a point $z_1\in\bar{z}$ of R, and $\psi(z_1)$ is a nbd of z for any ψ, since $\psi(z_1)\ni z$. Thus $\psi(z_1)\in\mathfrak{F}$, and this shows that $\mathfrak{F}$ converges to z_1. Therefore $\mathfrak{N}$ converges to a point in the space R. The rest is evident.

4.2 Uniqueness of compactification. The compactification considered above is characterized by

THEOREM 8. *Let X be Ψ_X-t-bounded and Ψ_X be basic. The space R obtained by the compactification of X with respect to Ψ_X has the following properties (described for S). Conversely, a space S with these properties is mapped on R by a homeomorphism leaving each point of X invariant.*

(1) *X is a subspace of S.*

(2) *S is a compact space.*

(3) *$\bar{p}\subset S-X$ for any point p of $S-X$.*

(4) *If a special filter $\mathfrak{M}$ (with respect to Ψ_X) in X converges to a point p of $S-X$, then p is the unique limit point of $\mathfrak{M}$.*

(5) *For any point p of $S-X$ there exists a special filter $\mathfrak{M}$ in X which converges to p.*

(6) *There exists a basic* GU *Ψ_S of S, which satisfies the conditions:*

a) *Ψ_X is the contraction of Ψ_S.*

b) *If $\psi(x)\ni p$, $\psi\in\Psi_S$, $x\in X$, $p\in S-X$, then $\psi(x)$ is a nbd of p.*

c) *If p, p' are distinct points of $S-X$, there exist ψ and x such that $\psi\in\Psi_S$, $x\in X$ and $\psi(x)\ni p$, $\not\ni p'$.*

d) *$\bigcup_i \psi_i(x_i)\supset X$ implies $\bigcup_i \psi_i(x_i)=S$ for any finite number of ψ_i and x_i such that $\psi_i\in\Psi_S$, $x_i\in X$, $i=1,\cdots,n$.*

e) *$\psi(x)\frown(S-X)=S-X-(\overline{X-\psi(x)})$ for any $\psi\in\Psi_S$ and $x\in X$.*

PROOF. We prove first that R has these properties (1)–(6). (1)–(3) and (5) are evident. Let Ψ_R be the contraction of Ψ_Z to R, then a)–d) are evident. and e) follows from the proposition 11. Let $\mathfrak{M}$ be a special filter in X and $z_1=[\mathfrak{M}_1]\in R-X$. If $\mathfrak{M}\not\approx\mathfrak{M}_1$ then there exists $\varphi(x')$ such that $\varphi(x')=\psi(x')\frown X$ and $\varphi(x')\in\mathfrak{M}_1$, $X-\varphi(x')\in\mathfrak{M}$. This shows that $\mathfrak{M}\rightarrow z_1\in R-X$ implies $[\mathfrak{M}]=z_1$. Thus (4) is proved. Next, we shall construct a one-to-one mapping $f: R\rightarrow S$. Let S be a space with the properties (1)-(6). If $[\mathfrak{M}]=z \in R-X$, then $\mathfrak{M}$ is a special filter. As S is a compact space, there exists a limit point p of $\mathfrak{M}$. Clearly $p \in S-X$, thus p is determined uniquely by $\mathfrak{M}$. We denote this point p by $f(\mathfrak{M})$. Let $p_i=f(\mathfrak{M}_i)$ and $p_1 \neq p_2$. Then there exists a certain $\psi(x)$ such that $\psi(x) \ni p_1, \not\ni p_2$. As $\varphi(x)=\psi(x) \frown X \in \mathfrak{M}_1$, we have $\psi_i(x_i)$, $i=1,\cdots,n$ such that $\varphi_i(x_i)=\psi_i(x_i) \frown X$ and $\varphi(x) \supset \frown_i (X-\varphi_i(x_i)) \in \mathfrak{M}_1$. Then $\psi(x) \smile \psi_1 \smile \cdots \smile \psi_n(x_n)=S$ by the condition d). Therefore $\psi_i(x_i) \ni p_2$ for a certain i, and this shows $\mathfrak{M}_1 \not\approx \mathfrak{M}_2$. Let $f(\mathfrak{M}_1)=f(\mathfrak{M}_2)=p$. If $\varphi(x) \in \mathfrak{M}_1$, then there exist $\varphi_i(x_i)$, $i=1,\cdots,n$ such that $\varphi_i(x_i)=\psi_i(x_i) \frown X$ and $\varphi(x) \supset \frown_i (X-\varphi_i(x_i)) \in \mathfrak{M}_1$. Clearly, $X-\varphi_i(x_i) \ni p$ implies $\psi_i(x_i) \not\ni p$, so we have $\varphi(x) \in \mathfrak{M}_2$ similarly as above. This shows $\mathfrak{M}_1 < \mathfrak{M}_2$. Thus we have $\mathfrak{M}_1 \approx \mathfrak{M}_2$ by symmetry. Therefore we can define a one-to one mapping of R into S by putting $f(x)=x$ for $x \in X$ and $f(z)=f(\mathfrak{M})$ for $z=[\mathfrak{M}] \in R-X$. By the condition (5), f maps R onto S. Finally, we shall prove the continuity of f and f^{-1}. It is evident that f is a topological mapping if we have $\psi(x)=fg^*(X \frown \psi(x))$ for any $\psi \in \Psi_S$ and $x \in X$, where $g^*(M)=g(M) \frown R$ to any subset M of X. For any point x' of X, we have clearly $x' \in \psi(x)$ if and only if $x' \in fg^*(X \frown \psi(x))$. Let $p=f(z)$, $z=[\mathfrak{M}] \in R-X$. If $p \in \psi(x)$, then $z \in g^*(X \frown \psi(x))$. If $z \in g^*(X \frown \psi(x))$, then we have as above $\psi_i(x_i)$ $i=1,\cdots,n$ such that $\psi(x) \smile \psi_1(x_1) \smile \cdots \smile \psi_n(x_n)=S$ and $p \notin \psi_1(x_1) \smile \cdots \smile \psi_n(x_n)$. Thus we have $\psi(x)=fg^*(X \frown \psi(x))$ and the proof is completed.

As a special case of the theorem 8, we have

THEOREM 9.[4] *Let X be a space with open nbd basis, and $\mathfrak{G} \ni X$ a basis of open sets. Then there exists a space S with the following properties. Moreover such a space is unique up to homeomorphism.*

(1) *S is a compact space with open nbd basis.*

(2) *X is a subspace of S.*

(3) *Each point of $S-X$ is closed.*

(4) *$\mathfrak{U}=\{U;\ U=S-\overline{X-G},\ G \in \mathfrak{G}\}$ is a basis of open sets of S.*

(5) *$\smile_i G_i=X$ implies $\smile_i (S-\overline{X-G_i})=S$ for any finite number of sets G_i, $i=1,\cdots,u$ of $\mathfrak{G}$.*

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4) Cf. N. A. Shanin: On special extension of topological spaces, Doklady URSS. 38. K. Morita: On the simple extension of a space with respect to uniformity. II, Proc. Japan Acad. 27.

PROOF. For any finite number of sets G_i, $i=1,\cdots,n$ of $\mathfrak{G}$ such that $\bigcup_i G_i=X$, we can define a correspondence $\varphi : x \rightarrow \varphi(x)=G_i \ni x$. Let Ψ_X be the collection of all such correspondences. Then clearly Ψ_X is an open GU of X, and X is Ψ_X-t-bounded. Similarly, we can define an open basic GU Ψ_S of S using $\mathfrak{U}$ instead of $\mathfrak{G}$. We can check easily that the space S with this Ψ_S satisfies the conditions (1)–(6) of the theorem 8.

§ 5 C-extension.

5.1 **Cauchy filter.** Let X be again a general space and Ψ any GU of X. By the completion considered in § 3, we obtain a complete space Z. But Z is not necessarily the "minimal" complete space containing X, and equivalent GU's do not lead to Z's topologically equivalent with each other. For this reason, we shall consider in this section another way to extend the space by means of Cauchy filters.

We denote by $S(M, \varphi)$ the set-union $\cup\{\varphi(x): \varphi(x)\frown M \neq \phi\}$ for any subset M of X, and by $\mathfrak{M}^*$ the family of sets $\{M' ; M' \supset S(M, \varphi)$, $M \in \mathfrak{M}, \cdot M \in \Psi\}$ for any filter $\mathfrak{M}$ in X. A filter $\mathfrak{M}$ is called a *Cauchy filter*, if for any $\varphi \in \Psi$ there exists a point x such that $\varphi(x) \in \mathfrak{M}^*$. Two Cauchy filters $\mathfrak{M}_1$, $\mathfrak{M}_2$ are called a *Cauchy pair* and denote by $\mathfrak{M}_1 \sim \mathfrak{M}_2$, if $\mathfrak{M}_1^*=\mathfrak{M}_2^*$.

PROPOSITION 16. *Let $\mathfrak{M}_1$ and $\mathfrak{M}_2$ be Cauchy filters. Then $\mathfrak{M}_1 \sim \mathfrak{M}_2$ if and only if for any set $M_1 \in \mathfrak{M}$ and $\varphi_1 \in \Psi$ there exist a set $M_2 \in \mathfrak{M}_2$ and $\varphi_2 \in \Psi$ such that $S(M_2, \varphi_2) \subset S(M_1, \varphi_1)$. Therefore $\mathfrak{M}_1 \subset \mathfrak{M}_2$ implies $\mathfrak{M}_1 \sim \mathfrak{M}_2$.*

PROOF. Clearly the condition is necessary. If the condition is fulfilled, then $\mathfrak{M}_1^* \subset \mathfrak{M}_2^*$ is evident. Let $M \in \mathfrak{M}_2$ and $\varphi \in \Psi$. Then there exist a point x_0, a set $M_1 \in \mathfrak{M}_1$ and $\varphi_1 \in \Psi$ such that $\varphi(x_0) \supset S(M_1, \varphi_1)$. For this pair of M_1 and φ_1, we have a set $M_2 \in \mathfrak{M}_2$ and $\varphi_2 \in \Psi$ such that $S(M_2, \varphi_2) \subset S(M_1, \varphi_1)$. Thus $\varphi(x_0)\frown M \neq \phi$ follows from $\varphi(x_0) \supset S(M_2, \varphi_2) \in \mathfrak{M}_2$. This shows $S(M, \varphi) \supset \varphi(x_0) \supset S(M_1, \varphi_1)$. Therefore $\mathfrak{M}_1^* \supset \mathfrak{M}_2^*$.

The following properties are evident.

PROPOSITION 17. *If $\Psi \sim \Psi'$ then a filter $\mathfrak{M}$ is a Cauchy filter with respect to Ψ if and only if $\mathfrak{M}$ is a Cauchy filter with respect to Ψ', and two Cauchy filters are a Cauchy pair with respect to Ψ if and only if they are a Cauchy pair with respect to Ψ'.*

Remark. A Cauchy filter is clearly a Ψ-filter, but a Ψ-filter is not necessarily a Cauchy filter. $\mathfrak{M}_1 \sim \mathfrak{M}_2$ and $\mathfrak{M}_1 \to x$ does not necessarily imply $\mathfrak{M}_2 \to x$, even if Ψ is an open basic GU. A uniformly continuous image of a Cauchy filter is not necessarily a Cauchy filter. We shall show this by an example. Let X be the interval $[0, 1]$ of rational numbers. For any point $x \in X$ and any nbd N of x, we define φ' putting $\varphi'(x) = N$ if $x' = x$; $\varphi'(x') = [0, 2/3)$ if $1/2 \geqq x' \neq x$; $\varphi'(x') = (1/3, 1]$ if $1/2 < x' \neq x$. Let us denote by Ψ' the collection of all such φ' taking x and N in every possible way, and by Ψ the ε-GU of X. (see example in 3.3) Clearly Ψ' is an open basic GU and $\Psi' < \Psi$. Any filter containing the interval $[0, 1/3]$ is a Cauchy filter, and any two such filters are a Cauchy pair with respect to Ψ'. The filter containing 1/2 as an element is a convergent filter, but not a Cauchy filter with respect to Ψ'. The identity mapping is clearly $\Psi \cdot \Psi'$-u-continuous.

5.2 **Space Y.** We consider $\mathfrak{M}^*$ obtained from a maximal filter $\mathfrak{M}$ of a space X with Ψ_X as a point y, $\mathfrak{M}$ being a Cauchy filter having no limit point. We denote by Y_0 the set of all these points, and by Y the set-union of X and Y_0.

PROPOSITION 18. *Let* $X_1^* = X_1 + \{y\,; y \in Y_0, y = \mathfrak{M}^* \ni X_1\}$. *Then we have:* (1) $X_1^* \cap X = X_1$, $X^* = Y$, $\phi^* = \phi$,

(2) $X_1 \subset X_2$ *implies* $X_1^* \subset X_2^*$,

(3) $\bigcap_i X_i^* = \phi$, *if and only if* $\cap X_i = \phi$, *for any finite number of subsets* X_i of X.

Now, we introduce a topology into the set Y by defining a nbd system as follows: A nbd of a point x of X is a such subset U of Y that $U \supset N^*$ for a certain nbd N of x in the space X. A nbd of a point $y = \mathfrak{M}^*$ of Y is such subset U of Y that $U \supset \varphi(x)^* \ni y$ for a certain point $x \in X$ and $\varphi \in \Psi_X$.

PROPOSITION 19. *The space Y has the following properties:*

(1) *X is a subspace of Y.*

(2) $\overline{X_1} \supset X_1^*$, *where the bar indicates the closure operation in Y.*

(3) $X_1^* \cap Y_0 = Y_0 - \overline{X - X_1}$.

(4) *If G is an open subset of the space X, then* $G^* = Y - \overline{X - G}$ *and G^* is an open subset of the space Y.*

PROOF. (1) is evident. (2): Let $y = \mathfrak{M}^*$ be a point of $X_1^* \cap Y_0$, then any nbd $\varphi(x)^*$ of y intersects with X_1, since $X_1 \in \mathfrak{M}^*$. (3): Let $y = \mathfrak{M}^*$ be a point of $X_1^* \cap Y_0$. Then there exist a set $M \in \mathfrak{M}$ and $\varphi \in \Psi_X$ such that $X_1 \supset S(M, \varphi)$, and there exists a point x such that $\varphi(x) \in \mathfrak{M}^*$. Clearly $\varphi(x)^*$ is a nbd of y, and $(X-X_1) \frown \varphi(x)^* = \phi$. This shows $X_1^* \frown Y_0 \subset Y_0 - \overline{X-X_1}$. The inverse inclusion follows from the fact that there exists a nbd $\varphi(x)^*$ of y for any point $y \in Y_0 - \overline{X-X_1}$, such that $(X-X_1) \frown \varphi(x)^* = \phi$. (4): As $G = X - \overline{X-G}$, we have $G^* = G \smile (G^* \frown Y_0) = Y - \overline{X-G}$. Clearly any point of G^* is an inner point of G^*.

PROPOSITION 20. *If $y \in X_1^* \frown Y_0$ for a subset X_1 of X, then there exists $\varphi \in \Psi_X$ such that $\varphi(x)^* \ni y$ implies $\varphi(x)^* \subset X_1$ for any point x of X. Therefore X_1^* is relatively open in Y_0.*

PROPOSITION 21. *If $y = \mathfrak{M}^*$ is a point of Y_0, then $y = \bar{y} = \frown\{\overline{M}; M \in \mathfrak{M}\}$.*

PROOF. As $\mathfrak{M}$ is a non-convergent maximal filter, we have $\frown\{\overline{M}; M \in \mathfrak{M}\} \subset Y_0$. Let $y_1 = \mathfrak{M}_1^*$ be another point of Y_0. Then there exist a set $M \in \mathfrak{M}$ and $\varphi \in \Psi_X$ such that $S(M, \varphi) \notin \mathfrak{M}_1^*$ by the proposition 16. As $M \frown \varphi(x')^* = \phi$ for a nbd $\varphi(x')^*$ of y_1, we have $\overline{M} \not\ni y_1$. On the other hand, we have clearly $y \in \frown\{\overline{M}; M \in \mathfrak{M}\}$. Therefore $y = \frown\{\overline{M}; M \in \mathfrak{M}\}$. If $x \in \bar{y} \frown X$ and $M \in \mathfrak{M}$, then for any nbd N of x in the space X, we have $M \frown N \neq \phi$. This shows that $x \in \bar{y} \frown X$ implies $x \in \frown\{\overline{M}; M \in \mathfrak{M}\}$ in contradiction with the above result. Thus $\bar{y} \subset Y_0$. Therefore $\bar{y} = y$, since the subspace Y_0 has an open nbd basis.

5.3 **Characterization of space Y.** Now we define a GU of Y as follows: For any element φ of Ψ_X, let ψ be a correspondence assigning to each point $x \in X$ a nbd $\psi(x) = \varphi(x)^*$ of x, and to each point $y = \mathfrak{M}^* \in Y_0$ a nbd $\psi(y) = \varphi(x')^*$ of y. Then the element φ can be regarded as the contraction of ψ to X, so it is also denoted by ψ_X. The collection of all correspondences such as ψ constitutes clearly a GU of Y, and Ψ_X is a contraction of this GU to X. We denote by Ψ_Y this GU of Y. The space Y with the GU Ψ_Y constructed above is called a *space obtained by the C-extension* of the space X with respect to the GU Ψ_X of X.

We say that a GU Ψ_X of X *agrees* with the topology at x, if for any nbd N of x there exists $\varphi \in \Psi_X$ such that $N \supset S(x, \varphi)$, and that Ψ_X agrees with the topology in X, if Ψ_X agrees with the topology at each point of X. By the proposition 20, Ψ_Y agrees with the topology of Y at each point of Y_0. Thus we have

PROPOSITION 22. (1) *Ψ_Y is a* GU *of Y, which agrees with the* *topology at each point of* Y_0.

(2) *If* Ψ_X *is a basic* GU *of* X, *then* Ψ_Y *is a basic* GU *of* Y.

(3) *If* Ψ_X *agrees with the topology in* X, *then* Ψ_Y *agrees also with the topology in* Y.

PROPOSITION 23. *If* Ψ_X *is a basic* GU *of a space* X, *then the space* Y *with* Ψ_Y *obtained by C-extension of* X *with respect to* Ψ_X *has the following properties (described for a space* R *with* Ψ_R*):*

a) X *is a subspace of* R.

b) *Each point of* $R-X$ *is closed.*

c) Ψ_R *is a basic* GU *of* R, *which agrees with the topology of* R *at each point of* $R-X$, *and whose contraction* Ψ'_X *to* X *is equivalent with* Ψ_X.

d) $\{\psi(p)\,;\ p\in R\}=\{\psi(x)\,;\ x\in X\}$ *for any* $\psi\in\Psi_R$.

e) *If* $p\in\psi(p')-X, \psi\in\Psi_R$, *then* $\psi(p')$ *is a nbd of* p.

f) $\psi(p)\frown X\supset\psi'(p')\frown X$ *if and only if* $\psi(p)\supset\psi'(p')$ *for any* $\psi, \psi'\in\Psi_R$, $p, p'\in R$.

g) *For any Cauchy filter* $\mathfrak{M}\ni X$, *there exists a point* p *such that* $p\in\frown\{\overline{M}: M\in\mathfrak{M}\}$ *(the bar indicates the closure operation of* R).

h) *For any point* p *of* $R-X$, *there exists a Cauchy filter* $\mathfrak{M}\ni X$ *such that* $p\in\frown\{\overline{M}\,;\, M\in\mathfrak{M}\}$.

PROOF. All these properties but g) are proved already. The property g) follows from the next lemma 1.

LEMMA 1. *Let* R *and* Ψ_R *satisfy the conditions* a)–f). *Then a filter* $\mathfrak{M}\ni X$ *is a Cauchy filter with respect to* Ψ_R *if and only if* $\mathfrak{M}'=\{M'\,;\ M'=M\frown X, M\in\mathfrak{M}\}$ *is a Cauchy filter in* X *with respect* Ψ_X.

PROOF. By the proposition 17, we may assume that $\Psi'_X=\Psi_X$. Let $\mathfrak{M}$ be a Cauchy filter with respect to Ψ_R, and $\varphi=\psi_X, \psi\in\Psi_R$. Then there exist $\psi'\in\psi_R$, $M\in\mathfrak{M}$, $p\in R$ such that $\psi(p)\supset S(M,\psi')$ (in R). If $\psi(p)=\psi(x)$ and $\psi'_X=\varphi'$, then we have $\varphi(x)=\psi(p)\frown X\supset S(M,\psi')\frown X\supset S(M\frown X_1,\varphi')$. Thus the condition is necessary. Conversely let $\mathfrak{M}$ be a Cauchy filter with respect to Ψ_X and $\psi\in\Psi_R$. Then there exist $\psi'\in\Psi_R$, $M'\in\mathfrak{M}'$, $x\in X$ such that $\varphi(x)\supset S(M',\varphi')$ (in X) for $\varphi=\psi_X$ $\varphi'=\psi'_X$. Thus we have $\psi(x)\supset S(M',\psi')$ (in R). Therefore the condition is also sufficient.

For simplicity, we shall write from now on $\mathfrak{M}$ for $\mathfrak{M}'$. Thus we may consider, by the lemma 1, a Cauchy filter $\mathfrak{M}\ni X$ in R also a

Cauchy filter in X and conversely.

LEMMA 2. *Let R and Ψ_R satisfy the conditions* a)–f). *Then* $\psi(p) = (\psi(p) \cap X) \cup (R - X - \overline{X - \psi(p)})$ *for any* $\psi \in \Psi_R$ *and* $p \in R$.

LEMMA 3. *Let R and Ψ_R satisfy the conditions* a)–f), *and* $\mathfrak{M} \ni X$ *be a Canchy filter converging to a point* $p \in R - X$. *Then for any* $\psi \in \Psi_R$ *there exist a set M of $\mathfrak{M}$ and an element ψ' of Ψ_R such that* $\psi(p) \supset S(M, \psi')$. *Therefore* $\psi(p) \cap X \in \mathfrak{M}^*$ *for any* $\psi \in \Psi_R$.

PROOF. By the condition c), there exists an element ψ_1 of Ψ_R such that $\psi(p) \supset S(p, \psi_1)$, and there exist also $\psi' \in \Psi_R$, $M \in \mathfrak{M}$ and $p' \in R$ for this ψ_1 such that $\psi_1(p') \supset S(M, \psi')$. Therefore $\psi(p) \supset \psi_1(p') \supset S(M, \psi')$. Thus for any $\varphi' \in \Psi_X$ such that $\varphi'(x) \subset \psi'_X(x)$ to each point $x \in X$, we have $\psi(p) \cap X \supset S(M, \varphi')$. This shows $\psi(p) \cap X \in \mathfrak{M}^*$.

LEMMA 4. *Let R and Ψ_R satisfy the conditions* a)–f), *and* $y_i = \mathfrak{M}_i^* \in Y_0$, $\mathfrak{M}_i \to p_i$, $i = 1, 2$. *Then* $p_i \in R - X$, *and we have* $\mathfrak{M}_1 \sim \mathfrak{M}_2$ *if and only if* $p_1 = p_2$.

PROOF. By the proposition 17, we may assume that $\Psi_X = \Psi'_X$. Clearly $p_i \in S - X$ since $\mathfrak{M}_i$ is a non-convergent maximal filter in X. Let $\mathfrak{M}_1 \sim \mathfrak{M}_2$. Then for any $\psi \in \Psi_R$, we have $\psi(p_1) \cap X \in \mathfrak{M}_1^* = \mathfrak{M}_2^*$ by lemma 3. Thus there exists $\psi' \in \Psi_R$ such that $\psi(p_1) \cap X \supset \psi'(p_2) \cap X$. This shows $p_1 = p_2$ by the conditions b), c) and f). Conversely let $p_1 = p_2 = p$ and $M_1 \in \mathfrak{M}_1$, $\varphi = \psi_X$, $\psi \in \Psi_R$. Then there exist a set M_2 of $\mathfrak{M}_2$ and $\psi' \in \Psi_R$ such that $\psi(p) \supset S(M_2, \psi')$ by the lemma 3. If $\varphi' = \psi'_X$, then $M_2 \cap \varphi'(x) \neq \phi$ implies $\psi(p) \supset \psi'(x) \supset \varphi'(x)$. On the other hand $\psi(p) \cap M_1 \neq \phi$ since $\psi(p) \cap X \in \mathfrak{M}_1^*$. Thus $S(M_1, \varphi) \supset \psi(p) \cap X$. Therefore we have $S(M_1, \varphi) \supset S(M_2, \varphi')$ and this implies $\mathfrak{M}_1 \sim \mathfrak{M}_2$ by the proposition 16.

THEOREM 10. *If Ψ_X is a basic* GU *of a space X, then the C-extension Y with Ψ_Y of X with respect to Ψ_X is characterized as a space R with the properties* a)–h) *of the proposition* 23. *That is, such a space R is mapped on Y by a topological mapping leaving each point of X invariant. Moreover a space R with the properties* a)–g) *contains a subspace which is mapped topologically onto Y.*

PROOF. Suppose a space R has the properties a)–g). Let us define a mapping f: $Y \to R$ as follows: For a point x of X we put $f(x) = x$, and for a point $y = \mathfrak{M}^*$ of Y_0, we take a point $p \in S - X$ such that $\mathfrak{M} \to p$ and put $f(y) = p$. Such a point p is determined uniquely by y. By lemma 4, this mapping f is a one-to-one mapping defined on Y.

To prove that the mapping f is a topological mapping of Y onto $f(Y)$, we may assume that $f(Y)=R$. We denote by K^* the subset $(K\frown X)^*$ of Y for any subset K of R. We shall show first that $f(\psi(p)^*)=\psi(p)$ for any $\psi\in\Psi_R$ and $p\in R$. For any point x of X, we have clearly $x\in f(\psi(p)^*)$ if and only if $x\in\psi(p)$. Let $p_1\in R-X$ and $p_1=f(y_1)$, $y_1=\mathfrak{M}_1^*$. If $p_1\in f(\psi(p)^*)$, then there exist a set $M\in\mathfrak{M}_1$ and $\varphi'=\psi'_X$, $\psi'\in\Psi_R$ such that $\psi(p)\frown X\supset S(M,\varphi')$ since $\psi(p)\frown X\in\mathfrak{M}_1^*$. On the other hand, we have $\psi'(p_1)\frown X\in\mathfrak{M}_1^*$ by the lemma 3. Therefore $\psi(p)\supset\psi'(p_1)\ni p_1$ follows from $\psi(p)\frown X\supset\psi'(p_1)\frown X$. Thus we have $\psi(p)\supset f(\psi(p)^*)$. If $p_1\in\psi(p)$, then there exists $\psi'\in\Psi_R$ such that $\psi(p)\supset\psi'(p_1)$. Therefore $\psi(p)^*\ni y_1$ since $\psi(p)\frown X\supset\psi'(p_1)\frown X\in\mathfrak{M}_1^*$. Thus we have $\psi(p)\subset f(\psi(p)^*)$. The continuity of the mapping f^{-1} is proved easily using the relation $f^{-1}(\psi(x))=\psi(x)^*$ proved above. Finally we shall show that f is $\Psi_Y\cdot\Psi_R$-u-continuous. Let $\psi\in\Psi_R$ and $\varphi=\psi_X$. For any $y\in Y_0$ there exists a point $x=x(y)$ such that $\psi(f(y))=\psi(x)$. As $\varphi(x)^*=f^{-1}\psi(f(y))$ contains y, the set $\varphi(x)^*$ is a nbd of y. Thus $\Psi_Y>f^{-1}(\Psi_R)$ and f is $\Psi_Y\cdot\Psi_R$-u-continuous by proposition 5.

COROLLARY. *If Ψ_X and Ψ'_X are two equivalent basic* GU's *of a space X, then we obtain the unimorphic spaces by the C-extensions of X with respect to Ψ_X and Ψ'_X.*

LEMMA 5. *Let X be a space with a basic* GU *Ψ_X satisfying the following conditions:*

(1) *X is Ψ_X-t-bounded.*

(2) *If $\bigcup_i\varphi_i(x_i)=X$ for any finite number of $\varphi_i\in\Psi_X$, $i=1,\cdots,n$, then there exists an element φ of Ψ_X such that $\{\varphi(x);\ x\in X\}=\{\varphi_i(x_i);\ i=1,\cdots,n\}$.*

Then the C-extension of X with respect to Ψ_X is a compact space.

PROOF, We can prove easily that the C-extensioned space Y of X is Ψ_Y-t-bounded. Let $\mathfrak{F}$ be a non-convergent maximal Ψ_Y-filter in Y. Then for any $y\in Y$ there exists $\psi\in\Psi_X$ such that $\psi(y)\notin\mathfrak{F}$. Let $\psi(y)=\varphi(x)^*$ and $M_y=X-\varphi(x)$. If $\bigcap_i M_{y_i}=\phi$ for a certain finite number of $M_{y_i}=X-\varphi_i(x_i)$ $i=1,\cdots,n$, then $\bigcup_i\varphi_i(x_i)=X$ and there exists $\varphi\in\Psi_X$ such that $\{\varphi(x);\ x\in X\}=\{\varphi_i(x_i);\ i=1,\cdots,n\}$. But this is impossible for $\mathfrak{F}$ is a Ψ_Y-filter. Thus the family $\{M_y;\ y\in Y\}$ has the finite intersection property. Let $\mathfrak{M}\supset\{M_y;\ y\in Y\}$ be a maximal family of sets expressed as $M=X-\varphi(x)$ with the finite intersection property. We can easily prove that $\mathfrak{M}$ is a Cauchy filter with respect to Ψ_X, and

$\frown\{\bar{M};\ M \in \mathfrak{M}\} \subset Y_0$. If $\mathfrak{M}_1 \supset \mathfrak{M}$ is any maximal filter in X, then $\mathfrak{M}_1$ is a Cauchy filter and $y_1 = \mathfrak{M}_1^* \in Y_0$. But $\bar{M}_{y_1} \not\ni y_1$ and $M_{y_1} \in \mathfrak{M} \subset \mathfrak{M}_1$ in contradiction with the proposition 21. Therefore any maximal Ψ_Y-filter has a limit point. Thus the space Y is compact by theorem 6.

THEOREM 11. *If a space X with a basic* GU Ψ_X *satisfies the condition of the lemma* 5, *then the C-extension of X coincides with the compactification considered in* § 4.

PROOF. If $\varphi_0(x_0) \in \mathfrak{M}$ for a special filter $\mathfrak{M}$, then there exist a finite number of $\varphi_i(x_i)$, $i=1,\cdots,n$, such that $\{\varphi(x_0) \supset \frown_i (X-\varphi_i(x_i)) = M \in \mathfrak{M}$. Let φ be an element of Ψ_X such that $\{\varphi(x);\ x \in X\} = \{\varphi_0(x_0), \varphi_1(x_1), \cdots, \varphi_n(x_n)\}$. Then we have $\varphi_0(x_0) \supset S(M, \varphi)$. This shows that any special filter in X is a Cauchy filter, and Ψ_R of R obtained by compactification agrees with the topology of R at $p \in R-X$. Thus we can check easily that all conditions of the proposition 23 are satisfied in the space R.

§ 6 Uniformity.

6.1 **Definition.** An open GU Ψ of a space X is called a *uniformity* if Ψ agrees with the Ψ-topology of X. A uniformity Ψ with the following condition is called a *T-uniformity*:

(A) For any $\varphi_1, \varphi_2 \in \Psi$, there exists $\varphi \in \Psi$ such that to each point $x \in X$ we can find two point x_1, x_2 with $\varphi(x) \subset \varphi_1(x_1) \frown \varphi_2(x_2)$.

A uniformity Ψ is called *regular* or *completely regular*, according as Ψ satisfies the condition (B) or (C):

(B) For any $\varphi \in \Psi$, there exists $\varphi_1 \in \Psi$ such that to each point $x \in X$ we can find a point $x' \in X$ and $\varphi' \in \Psi$ with $\varphi(x') \supset S(\varphi_1(x), \varphi')$.

(C) For any $\varphi \in \Psi$, there exist $\varphi_1, \varphi_2 \in \Psi$ such that to each point $x \in X$ we can find a point x' with $\varphi(x') \supset S(\varphi_1(x), \varphi_2)$.

Now we have the following propositions.

PROPOSITION 24. (1) *A* GU Ψ *of X is a basic uniformity, if and only if Ψ is open and agrees with the topology of X.*

(2) *A filter $\mathfrak{M}$ in X is a Cauchy filter with respect to a regular uniformity Ψ of X, if and only if $\mathfrak{M}$ is a Ψ-filter.*

PROPOSITION 25. *Let Ψ be any* GU *of X, and $\mathfrak{M}_1, \mathfrak{M}_2$ be two Cauchy filters. The filter $\mathfrak{M}_1 \wedge \mathfrak{M}_2$, which is composed of the sets belonging to $\mathfrak{M}_1$ and $\mathfrak{M}_2$ simultaneously, is a Ψ-filter if $\mathfrak{M}_1 \sim \mathfrak{M}_2$, and we have $\mathfrak{M}_1 \sim \mathfrak{M}_2$ if $\mathfrak{M}_1 \wedge \mathfrak{M}_2$ is a Cauchy filter. Therefore, when Ψ is a regular* *uniformity,* $\mathfrak{M}_1 \sim \mathfrak{M}_2$ *if and only if* $\mathfrak{M}_1 \wedge \mathfrak{M}_2$ *is a Cauchy filter.*

PROPOSITION 26. *Let* Ψ *be a regular uniformity, and* $\mathfrak{M}_x$ *be a filter containing* x *as an element. Then a Cauchy filter* $\mathfrak{M}$ Ψ*-converges to the point* x*, if and only if* $\mathfrak{M} \sim \mathfrak{M}_x$.

PROPOSITION 27. *Let* Ψ *be a basic uniformity of a space* X*. Then* $\mathfrak{M} \rightarrow x$ *implies* $\mathfrak{M} \rightarrow x$ *for any Cauchy filter* $\mathfrak{M}$*. Therefore a Cauchy filter* $\mathfrak{M}$ *has at most one limit point if each point of* X *is closed.*

PROOF. First three propositions 24–26 are proved without any difficulty. We shall prove the last proposition. For any nbd N of x, there exists an element φ of Ψ such that $N \supset S(x, \varphi)$ by the proposition 24. For this φ, we have a set M of $\mathfrak{M}$, a point x' and $\varphi' \in \Psi$ such that $\varphi(x') \supset S(M, \varphi')$. Thus we have $\varphi(x') \supset \varphi'(x) \ni x$, since $\mathfrak{M} \rightarrow x$ and $\varphi'(x)$ is a nbd of x. Therefore, we have $N \supset \varphi(x') \supset M \in \mathfrak{M}$, and this shows that $\mathfrak{M}$ converges to the point x. Let x_1 be a limit point of $\mathfrak{M}$ and $\bar{x}_1 = x_1$. Then we have $N \supset S(M, \varphi') \supset \bar{M} \ni x_1$, and hence $x \in \bar{x}_1$. This shows that the point x is the unique limit point of $\mathfrak{M}$, when each point of X is closed.

6.2 Extension with respect to a uniformity. Now, let us consider the relation between the topology of X and the topology of Y obtained by the C-extension of X. For this purpose, we shall prove first the following theorem.

THEOREM 11.[5] (1) *A space* X *has a basic uniformity, if and only if a closure of any set is closed and* $\bar{x} \subset U$ *for each point* x *of any open set* U *in* X.

(2) *A space* X *has a basic* T*-uniformity, if and only if* X *is a* **T**-*space and* $\bar{x} \subset U$ *for each point* x *of any open set* U *in* X.

(3) *A space* X *has a basic regular uniformity, if and only if for any nbd* N *of any potnt* x*, there exists an open set* G *such that* $x \in G \subset \bar{G} \subset N$.

(4) *A space* X *has a basic completely regular uniformity, if and only if for any abd* N *of any point* x_1 *there exists a non negative real-valued continuous function* f *defined on* X *such that* $f(x_1)=0$ *and* $f(x)=1$ *for each point* $x \notin N$.

PROOF. (1), (2) and (3) are proved without any difficulty, for the

5) Cf. K. Morita: On the simple extension of a space with respect to a uniformity. I. Proc. Japan Acad. 27 (1951).

maximal open GU of X gives the desired uniformity. (4): Clearly the condition is sufficient. Let N be a nbd of a point x_1, and $N \supset S(x_1, \varphi_0)$. We can find φ_n; $n=1, 2, \cdots$, such that $\varphi_{n-1}(x') \supset S(\varphi'_n(x), \varphi_n)$ for each point x with a certain point x' and $\varphi'_n \in \Psi$. Then using the open covering $\mathfrak{U}_n = \{S(x, \varphi_n); x \in X\}$ we can define a desired function in the well known way.

THEOREM 12. *If Ψ_X is a uniformity [T-uniformity, regular or completely regular uniformity], then the* GU Ψ_Y *of C-extension Y is a uniformity [T-uniformity, regular or completely regular uniformity respectively].*

PROOF. Clearly Ψ_Y is an open GU of the space Y agreeing with the Ψ_Y-topology of Y by the proposition 22, since Ψ_X agrees with the Ψ_X-topology of X. Thus Ψ_Y is a uniformity by the proposition 24. The rest is proved directly from the definitions in 6.1.

By the theorems 11 and 12, we can formulate the relations between topologies and its C-extensions as follows:

THEOREM 13.[6] *If a space X and its basic* GU Ψ_X *have the properties in the first two columns of the following table, then the C-extension Y of X with respect to Ψ_X has the properties in the third column mentioned in the corresponding place.*

	space X	basic GU Ψ_X	C-extension
1)	there exists an open nbd basis, each point is closed	uniformity	there exists an open nbd basis, each point is closed
2)	T_1-space	T-uniformity	T_1-space
3)	regular space	regular T-uniformity	regular space
4)	completely regular space	completely regular T-uniformity	completely regular space

PROOF. It is sufficient to prove that each point x of X is closed in the space Y. Let $y=\mathfrak{M}^*$ be a point of $\bar{x} \frown Y_0$. Then for any $\varphi \in \Psi_X$ there exists $\varphi' \in \Psi_X$ such that $\varphi(x) \supset S(x, \varphi')$. If $\varphi'(x') \in \mathfrak{M}^*$, then $\varphi(x) \supset \varphi'(x') \in \mathfrak{M}^*$ since $\varphi'(x')^*$ is a nbd of y in Y. This shows $\mathfrak{M} \rightarrow x$ in contradiction with $y \in Y_0$.

THEOREM 14. *If Ψ_X is a regnlar uniformity of a space X, then the C-extension Y of X is Ψ_Y-complete.*

PROOF. Let $\mathfrak{F}$ be a Ψ_Y-filter in Y, and $\mathfrak{M}$ be the family of all subsets

6) Cf. K. Morita: loc. cit.

of X containing an intersection of a certain finite number of subsets M of X with $M^* \in \mathfrak{F}$. $\mathfrak{M}$ is clearly a Ψ_X-filter. Let $\mathfrak{M} \to x_0$ then $\mathfrak{M}$ Ψ_X-converges to the point x_0 by the proposition 27. For any $\psi \in \psi_Y$, there exist $\psi', \psi'' \in \Psi_Y$ such that $\psi(x_0) \supset S(\psi'(x_0), \psi'')$. If $\psi''(y) = \psi''(x) \in \mathfrak{F}$, then $\varphi''(x) \in \mathfrak{M}$ for $\varphi'' = \psi''_X$ and $\psi(x_0) \supset \psi''(x)$. Thus x_0 is a Ψ_Y-limit point of $\mathfrak{F}$ in Y. Let $\frown\{\bar{M}\,;\, M \in \mathfrak{M}\} \subset Y - X$. Then there exists a point $y_0 = \mathfrak{M}_0^* \in Y - X$ such that $\mathfrak{M} \subset \mathfrak{M}_0$. Let $\varphi = \psi_X$ and $\psi(y_0) = \varphi(x_0)^*$. Then we can find a set $M \in \mathfrak{M}_0$ and $\psi' \in \Psi_Y$ such that $\varphi(x_0) \supset S(M, \varphi')$ for $\varphi' = \psi'_X$. If $\psi'(y') = \varphi'(x')^* \in \mathfrak{F}$, then $M \frown \varphi'(x') \neq \phi$ since $\varphi'(x') \in \mathfrak{M}$. Thus $\psi(y_0) \supset \psi'(y') \in \mathfrak{F}$ follows from $\varphi(x_0) \supset \varphi'(x')$. This shows that y_0 is a Ψ_Y-limit point of $\mathfrak{F}$.

THEOREM 15. *Let Ψ_X be a regular uniformity of a space X. If a subset X_1 of X is Ψ_X-t-bounded, then X_1^* is Ψ_Y-t-bounded. Therefore X_1^* is conditionally compact if Ψ_X is basic and X_1 is a Ψ_X-t-bounded set.*

PROOF. For any $\varphi = \psi_X$, $\psi \in \Psi_Y$, there exists $\varphi_1 \in \Psi_X$ satisfying the condition (B) of 6.1 for φ. Let $\{x'_1, \cdots, x'_n\}$ be a finite set such that $X_1 \subset \smile_i \varphi_1(x'_i)$. Then we have for each $i = 1, \cdots, n$, a point x_i and $\varphi'_i \in \Psi_X$ such that $\varphi(x_i) \supset S(\varphi_1(x'_i), \varphi'_i)$. Clearly, we have $X_1 \subset \smile_i \psi(x_i)$. Let $y = \mathfrak{M}^*$ be any point of $X_1^* - X$. Then for each $i = 1, \cdots, n$, there exists a point x''_i such that $\varphi'_i(x''_i) \in \mathfrak{M}^*$. The set $M = \frown_i \varphi'_i(x''_i)$ intersects with X_1. Thus for a certain $j = 1, \cdots, n$, we have $M \frown \varphi_1(x'_j) \neq \phi$ and $\varphi(x_j) \supset S(\varphi_1(x'_j), \varphi'_j) \supset \varphi'_j(x''_j) \in \mathfrak{M}^*$. Therefore we have $X_i^* \subset \smile_i \psi(x_i)$.

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