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A generalization of Cartan space.

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A space such that the area of a domain on a hypersurface $x^i=x^i(u^\alpha)$, $\alpha=1,2,\cdots,n-1$, is given by the $(n-1)$-ple integral

$$\int_{(n-1)} F(x^i, \partial x^i/\partial u^\alpha)\, du^1 \cdots du^{n-1}$$

is called a Cartan space.. E. Cartan [1] has shown that this space may be regarded as a manifold of the hyperplane elements $(x^i, \partial x^i/\partial u^\alpha)$. Thereafter L. Berwald has treated the geometry at large of this space, and T. Okubo and the present auther have extended this geometry to the higher order $(n-1)$-ple integrals of some special forms. In this paper the auther will establish the geometry of a space in which the area of a domain on a K-dimensional surface $x^i=x^i(u^\alpha)$, $i=1,2,\cdots,n$; $\alpha=1,\cdots,K$, is given by the K-ple integral

$$\int_K F(u^\alpha, x^i, \partial x^i/\partial u^\alpha, \partial^2 x^i/\partial u^\alpha\, \partial u^\beta)\, du^1 \cdots du^K .$$

It is convenient to regard the space in question as a manifold of the K-dimensional surface elements of the third order and u^α, $(\alpha=1,\cdots, K)$ which we shall denote by $F_n^{(3)}$. Namely, the manifold $F_n^{(3)}$ consists of all system of values of $u^\alpha, x^i, \partial x^i/\partial u^\alpha, \partial^2 x^i/\partial u^\alpha \partial u^\beta, \partial^3 x^i/\partial u^\alpha \partial u^\beta \partial u^\gamma$.

Throughout this paper we shall use the notations

$$X_a^i=\frac{\partial x^i}{\partial \bar{x}^a}, \quad X_i^a=\frac{\partial \bar{x}^a}{\partial x^i}, \quad X_{a(2)}^i=\frac{\partial^2 x^i}{\partial \bar{x}^{a_1}\,\partial \bar{x}^{a_2}}, \quad X_{i(2)}^a=\frac{\partial^2 \bar{x}^a}{\partial x^{i_1}\,\partial x^{i_2}}, \cdots,$$

$$U_\lambda^\alpha=\frac{\partial u^\alpha}{\partial \bar{u}^\lambda}, \quad U_\alpha^\lambda=\frac{\partial \bar{u}^\lambda}{\partial u^\alpha}, \quad U_{\lambda(2)}^\alpha=\frac{\partial^2 u^\alpha}{\partial \bar{u}^{\lambda_1}\,\partial \bar{u}^{\lambda_2}}, \quad U_{\alpha(2)}^\lambda=\frac{\partial^2 \bar{u}^\lambda}{\partial u^{\alpha_1}\,\partial u^{\alpha_2}}, \cdots,$$

$$U_{\lambda(s)}^{\alpha(s)}=U_{\lambda_1}^{\alpha_1} U_{\lambda_2}^{\alpha_2} \cdots U_{\lambda_s}^{\alpha_s}, \qquad U_{\alpha(s)}^{\lambda(s)}=U_{\alpha_1}^{\lambda_1} U_{\alpha_2}^{\lambda_2} \cdots U_{\alpha_s}^{\lambda_s}.$$

which are evaluated for the transformations

$$\bar{x}^a=\bar{x}^a(x^i)\,, \qquad a=1,2,\cdots,n\,; \qquad i=1,2,\cdots,n,$$

$$\bar{u}^\lambda=\bar{u}^\lambda(u^\alpha)\,, \qquad \lambda=1,2,\cdots,K\,; \qquad \alpha=1,2,\cdots,K\,.$$

Moreover we shall use the notations

$$p^i_\alpha=\frac{\partial x^i}{\partial u^\alpha}\,, \quad p^i_{\alpha(2)}=p^i_{\alpha_1\alpha_2}=\frac{\partial^2 x^i}{\partial u^{\alpha_1}\partial u^{\alpha_2}}\,,\cdots, \quad F_{;\alpha}=\frac{\partial F}{\partial u^\alpha}\,, \quad F_{;i}=\frac{\partial F}{\partial x^i}$$

and

$$F^{\alpha(s)}_{;\,i}=\frac{\bar{\partial}}{\bar{\partial} p^i_{\alpha(s)}}F=\frac{l_1!\,l_2!\cdots l_t!}{s!}\,\frac{\partial}{\partial p^i_{\alpha(s)}}F$$

when the indices $\alpha_1, \alpha_2,\cdots,\alpha_s$ consist of $l_1, l_2,\cdots$ and l_t same indices.

§ 1. Fundamental tensor of $F^{(3)}_n$.

Suppose that we have the K-ple integral

(1) $$\int_K F(u^\alpha, x^i, p^i_\alpha, p^i_{\alpha\beta})\, du^1 \cdots du^K$$

which is invariant under the transformation-group of coordinates and parameters

(2) $$\bar{x}^a=\bar{x}^a(x^i), \qquad \bar{u}^\lambda=\bar{u}^\lambda(u^\alpha)\,.$$

It is easily seen that the function F is transformed under any parameter transformation as follows

(3) $$\bar{F}(u^\lambda, x^i, p^i_\lambda, p^i_{\lambda\mu})=F(u^\alpha, x^i, p^i_\alpha, p^i_{\alpha\beta})\,\Delta, \qquad (\Delta=|\,U^\alpha_\lambda\,|)$$

which behaves as a scalar under any coordinate transformation; that is

(4) $$\bar{F}(u^\lambda, x^a, p^a_\lambda, p^a_{\lambda\mu})=F(u^\alpha, x^i, p^i_\alpha, p^i_{\alpha\beta})\,\Delta\,.$$

It is well known that the identity (3) holds good when and only when the following identities are satisfied by the function $F(u^\alpha, x^i, p^i_\alpha, p^i_{\alpha\beta})$,

(5a) $$F_{;\,\alpha}=0 \qquad (\alpha=1,2,\cdots,K)\,,$$

(5b) $$F_{;\,i}^{\alpha\beta}\, p^i_\gamma=0 \qquad (\gamma=1,2,\cdots,K)\,,$$

(5c) $$2F_{;\,i}^{\alpha\beta}\, p^i_{\beta\gamma}+F_{;\,i}^{\alpha}\, p^i_\gamma=\delta^\alpha_\gamma\, F \qquad (\gamma=1,2,\cdots,K).$$

From (5b) we have the identities

$$\frac{\partial^2 F}{\partial p^{[i_1}_{\alpha_1\ \beta_1}\ \partial p^{[j_1}_{\alpha_2\ \beta_2}}\ \frac{\partial^2 F}{\partial p^{i_2}_{\alpha_3\ \beta_3}\ \partial p^{j_2}_{\alpha_4\ \beta_4}}\cdots\cdots\frac{\partial^2 F}{\partial p^{i_{n-K}]}_{\alpha_{N-1}\ \beta_{N-1}}\ \partial p^{j_{n-K}]}_{\alpha_N\ \beta_N}}\ p^i_\gamma=0$$

and

$$\frac{\partial^2 F}{\partial p^{[i_1}_{\alpha_1\ \beta_1}\ \partial p^{[j_1}_{\alpha_2\ \beta_2}}\ \frac{\partial^2 F}{\partial p^{[i_2}_{\alpha_3\ \beta_3}\ \partial p^{j_2}_{\alpha_4\ \beta_4}}\cdots\cdots\frac{\partial^2 F}{\partial p^{i_{n-K}]}_{\alpha_{N-1}\ \beta_{N-1}}\ \partial p^{j_{n-K}]}_{\alpha_N\ \beta_N}}\ p^j_\gamma=0$$

putting $N=2(n-K)$, so that there is a quantity $\rho^{\alpha_1\alpha_2\cdots\alpha_N,\ \beta_1\beta_2\cdots\beta_N}$ such that

(6) $$\frac{\partial^2 F}{\partial p^{[i_1}_{(\alpha_1\ (\beta_1}\ \partial p^{[j_1}_{\alpha_2\ \beta_2}}\ \frac{\partial^2 F}{\partial p^{i_2}_{\alpha_3\ \beta_3}\ \partial p^{j_2}_{\alpha_4\ \beta_4}}\cdots\cdots\frac{\partial^2 F}{\partial p^{i_{n-K}]}_{\alpha_{N-1}\ \beta_{N-1}}\ \partial p^{j_{n-K}]}_{\alpha_N)\ \beta_N)}}$$

$$=\varepsilon_{i_1\cdots i_{n-K}i'_1\cdots i'_K}\,\varepsilon_{j_1\cdots j_{n-K}j'_1\cdots j'_K}\,p^{[i'_1}_1\cdots p^{i'_K]}_K\,p^{[j'_1}_1\cdots p^{j'_K]}_K$$

$$\times\ \rho^{\alpha_1\alpha_2\cdots\alpha_N,\ \beta_1\beta_2\cdots\beta_N},$$

where we have put

$$\varepsilon_{i_1 i_2\cdots i_{n-K}i'_1\cdots i'_K}=n!\ \delta^1_{[i_1}\cdots\delta^{n-K}_{i_{n-K}}\ \delta^{n-K+1}_{i'_1}\cdots\delta^{n}_{i'_K]}\ .$$

It is seen from (4) and (6) that $\rho^{\alpha_1\alpha_2\cdots\alpha_N,\ \beta_1\beta_2\cdots\beta_N}$ behaves under the transformation (2) in the manner

$$\rho^{\lambda_1\lambda_2\cdots\lambda_N,\ \mu_1\mu_2\cdots\mu_N}=\Delta^{n-K-2}\,D^2\,\rho^{\alpha_1\alpha_2\cdots\alpha_N,\ \beta_1\beta_2\cdots\beta_N}\,U^{\lambda(N)}_{\alpha(N)}\,U^{\mu(N)}_{\beta(N)},$$

putting $D=|\,X^i_a\,|$.

Suppose now that the $\binom{K+N-1}{N}$-rowed determinant $\rho=|\,\rho^{\alpha_1\cdots\alpha_N,\ \beta_1\cdots\beta_N}|$ does not vanish. Since $|\,U^{\lambda(N)}_{\alpha(N)}\,|=|\,U^\lambda_\alpha\,|^{\binom{K+N-1}{N-1}}$[6], we have the u-tensor

$$P^{\alpha_1\cdots\alpha_N,\ \beta_1\cdots\beta_N}=\rho^{-x}\,F^{-y}\,\rho^{\alpha_1\cdots\alpha_N,\ \beta_1\cdots\beta_N}$$

putting $x=1/\binom{N+K-1}{N}$, $\quad y=-2\,N/K$.

Let $P_{\alpha_1\cdots\alpha_N,\ \gamma_1\cdots\gamma_N}$ be the inverse system of $P^{\alpha_1\cdots\alpha_N,\ \beta_1\cdots\beta_N}$, that is $P^{\alpha_1\cdots\alpha_N,\ \beta_1\cdots\beta_N}P_{\alpha_1\cdots\alpha_N,\ \gamma_1\cdots\gamma_N}=\delta^{(\beta_1}_{\gamma_1}\cdots\delta^{\beta_N)}_{\gamma_N}$, and put $P^{\alpha_1\cdots\alpha_N,\ \beta_1\cdots\beta_N}P_{\alpha_1\cdots\alpha_N,\ \gamma_1\cdots\gamma_N/\gamma}$ $=Q^{\beta_1\cdots\beta_N}_{\gamma_1\cdots\gamma_N;\ \gamma}$, then the quantity $Q^{\beta_1\beta_2\cdots\beta_N}_{\gamma_1\gamma_2\cdots\gamma_N;\ \gamma}$ is transformed by (2) as follows

$$\bar{Q}^{\mu_1\mu_2\cdots\mu_N}_{\nu_1\nu_2\cdots\nu_N;\ \nu}=Q^{\beta_1\beta_2\cdots\beta_N}_{\gamma_1\gamma_2\cdots\gamma_N;\ \gamma}\ U^{\mu(N)}_{\beta(N)}\,U^{\gamma(N)}_{\nu(N)}\,U^{\gamma}_{\nu}$$

$$+\bar{P}^{\lambda_1\lambda_2\cdots\lambda_N,\ \mu_1\mu_2\cdots\mu_N} P_{\alpha_1\alpha_2\cdots\alpha_N,\ \gamma_1\gamma_2\cdots\gamma_N} U^{\alpha(N)}_{\lambda(N)/\nu} U^{\gamma(N)}_{\nu(N)}$$

$$+\bar{P}^{\lambda_1\lambda_2\cdots\lambda_N,\ \mu_1\mu_2\cdots\mu_N} P_{\alpha_1\alpha_2\cdots\alpha_N,\ \gamma_1\gamma_2\cdots\gamma_N} U^{\alpha(N)}_{\lambda(N)} U^{\gamma(N)}_{\nu(N)/\nu},$$

where the symbol $/\nu$ indicates the total differentiation with respect to $\bar{u}^\nu$. Putting $\mu_1=\nu_1, \mu_2=\nu_2,\cdots, \mu_{N-1}=\nu_{N-1}$ and contracting these indices one gets

$$\text{(7)}\qquad \bar{Q}^{\mu_1\mu_2\cdots\mu_{N-1}\lambda}_{\mu_1\mu_2\cdots\mu_{N-1}\mu\,;\,\nu} = Q^{\beta_1\beta_2\cdots\beta_{N-1}\alpha}_{\beta_1\beta_2\cdots\beta_{N-1}\beta\,;\,\gamma} U^\lambda_\alpha U^\beta_\mu U^\gamma_\nu$$

$$+N\bar{P}^{\lambda_1\lambda_2\cdots\lambda_{N-1}\rho,\ \mu_1\mu_2\cdots\mu_{N-1}\lambda} \bar{P}_{\lambda_1\lambda_2\cdots\lambda_{N-1}\omega,\ \mu_1\mu_2\cdots\mu_{N-1}\mu} U^\omega_\alpha U^\alpha_{\rho\nu}$$

$$+U^{\mu_1\mu_2\cdots\mu_{N-1}}_{\beta_1\beta_2\cdots\beta_{N-1}} U^\lambda_{\beta_N} (U^{\beta_1}_{\mu_1} U^{\beta_2}_{\mu_2}\cdots U^{\beta_{N-1}}_{\mu_{N-1}} U^{\beta_N}_\mu)_{/\nu}\,.$$

If $L^\alpha_{\beta\gamma}$ is a coefficient of the linear connection of u-tensor, it is transformed in the following way:

$$\text{(8)}\qquad \bar{L}^\omega_{\rho\nu} = L^\alpha_{\beta\gamma} U^\omega_\alpha U^\beta_\rho U^\gamma_\nu + U^\omega_\alpha U^\alpha_{\rho\nu}\,.$$

Eliminating $U^\omega_\alpha U^\alpha_{\rho\nu}$ from (7) and (8) we have

$$\bar{Q}^{\mu_1\mu_2\cdots\mu_{N-1}\lambda}_{\mu_1\mu_2\cdots\mu_{N-1}\mu\,;\,\nu} - \bar{K}^{\rho\lambda}_{\omega\mu} \bar{L}^\omega_{\rho\nu} = Q^{\beta_1\beta_2\cdots\beta_{N-1}\alpha}_{\beta_1\beta_2\cdots\beta_{N-1}\beta\,;\,\nu} U^\lambda_\alpha U^\beta_\mu U^\gamma_\nu$$

$$-K^{\varepsilon\alpha}_{\delta\beta} L^\delta_{\varepsilon\gamma} U^\lambda_\alpha U^\beta_\mu U^\gamma_\nu + \delta^{(\mu_1}_{\mu_1} \delta^{\mu_2}_{\mu_2}\cdots\delta^{\mu_{N-1}}_{\mu_{N-1}} U^{\lambda)}_\alpha U^\alpha_{\mu\nu}$$

$$+\sum_{t=1}^{m-1} \delta^{(\mu_1}_{\mu_1}\cdots\delta^{\mu_{t-1}}_{\mu_{t-1}} U^{\mu_t}_{\beta_t} U^{|\beta_t|}_{\mu_t\nu} \delta^{\mu_{t+1}}_{\mu_{t+1}}\cdots\delta^{\lambda)}_\mu,$$

where we have put

$$\bar{K}^{\rho\lambda}_{\omega\mu} = N\bar{P}^{\lambda_1\lambda_2\cdots\lambda_{N-1}\rho,\ \mu_1\mu_2\cdots\mu_{N-1}\lambda} \bar{P}_{\lambda_1\lambda_2\cdots\lambda_{N-1}\omega,\ \mu_1\mu_2\cdots\mu_{N-1}\mu}\,.$$

Moreover one gets

$$\text{(9)}\qquad \bar{Q}^\lambda_{\mu\nu} - \bar{K}^{\rho\lambda}_{\omega\mu} \bar{L}^\omega_{\rho\nu} = (Q^\alpha_{\beta\gamma} - K^{\varepsilon\alpha}_{\delta\beta} L^\delta_{\varepsilon\gamma}) U^\lambda_\alpha U^\beta_\mu U^\gamma_\nu + pU^\lambda_\alpha U^\alpha_{\mu\nu} + q\,\delta^\lambda_\mu\,\partial_\nu \log \varDelta,$$

putting $\bar{Q}^\lambda_{\mu\nu} = \bar{Q}^{\mu_1\cdots\mu_{N-1}\lambda}_{\mu_1\cdots\mu_{N-1}\mu\,;\,\nu}$, where p, q are suitable rational numbers. Putting $\lambda=\mu$ and contracting this indix we have

$$\text{(10)}\qquad \bar{Q}_\nu - \bar{K}^\rho_\omega \bar{L}^\omega_{\rho\nu} = (Q_\gamma - K^\varepsilon_\delta L^\delta_{\varepsilon\gamma}) U^\gamma_\nu + (p+qn-q)\,\partial_\nu \log \varDelta\,,$$

where we have put $\bar{Q}^\lambda_{\lambda\nu} = \bar{Q}_\nu$ and $\bar{K}^\rho_\omega = \bar{K}^{\rho\lambda}_{\omega\lambda} = \dfrac{n(n+1)\cdots(n+N-2)}{N!}\,\delta^\rho_\omega$.

Eliminating $\partial_\nu \log \Delta$ from (9) and (10) one obtains

$$(p+qn-q)(\bar{Q}^\lambda_{\mu\nu}-\bar{K}^{\rho\lambda}_{\omega\mu}\bar{L}^\omega_{\rho\nu})-q(\bar{Q}_\nu-\bar{K}^\rho_\omega\bar{L}^\omega_{\rho\nu})\delta^\lambda_\mu$$
$$=\{(p+qn-q)(Q^\alpha_{\beta\gamma}-K^{\varepsilon\alpha}_{\delta\beta}L^\delta_{\varepsilon\gamma})-q(Q_\gamma-K^\varepsilon_\delta L^\delta_{\varepsilon\gamma})\delta^\alpha_\beta\}U^\lambda_\alpha U^\beta_\mu U^\gamma_\nu$$
$$+p(p+qn-q)U^\lambda_\alpha U^\alpha_{\mu\nu},$$

so that the law of transformation of the quantity

$$\frac{1}{p}(\bar{Q}^\lambda_{\mu\nu}-\bar{K}^{\rho\lambda}_{\omega\mu}\bar{L}^\omega_{\rho\nu})-\frac{q}{p(p+qn-q)}(\bar{Q}_\nu-\bar{K}^\rho_\omega\bar{L}^\omega_{\rho\nu})\delta^\lambda_\mu$$

is the same as the law of transformation of $\bar{L}^\lambda_{\mu\nu}$. Hence we may put

$$\text{(11)}\quad \frac{1}{p}(\bar{Q}^\lambda_{\mu\nu}-\bar{K}^{\rho\lambda}_{\omega\mu}\bar{L}^\omega_{\rho\nu})$$
$$-\frac{q}{p(p+qn-q)}\left(\bar{Q}_\nu-\frac{n(n+1)\cdots(N+n-2)}{N!}\bar{L}_\nu\right)\delta^\lambda_\mu=\bar{L}^\lambda_{\mu\nu},$$

from which it follows

$$\text{(12)}\quad \frac{1}{p}\left\{\bar{Q}_\nu-\frac{n(n+1)\cdots(N+n-2)}{N!}\bar{L}_\nu\right\}$$
$$-\frac{q}{p(p+qn-q)}\left\{K\bar{Q}_\nu-\binom{N+n-2}{N}\bar{L}_\nu\right\}=\bar{L}_\nu,$$

putting $\bar{L}_\nu=\bar{L}^\lambda_{\lambda\nu}$.

If we denote by $\bar{N}^{\rho\lambda}_{\omega,\mu}$ the quantity $\frac{1}{p}\bar{K}^{\rho\lambda}_{\omega\mu}+\delta^\lambda_\omega\delta^\rho_\mu$, and assume that the K^2-rowed determinant $|\bar{N}^{\rho\lambda}_{\omega,\mu}|$ is different from zero, then we can uniquely determine the quantities $\bar{L}^\lambda_{\mu\nu}$ and $\bar{L}_\nu$ from (11) and (12). Moreover we put $L^\lambda_{(\mu\nu)}=G^\lambda_{\mu\nu}$ which is transformed in the manner

$$\bar{G}^\lambda_{\mu\nu}=G^\alpha_{\beta\gamma}U^\lambda_\alpha U^\beta_\mu U^\gamma_\nu-U^\beta_\mu U^\gamma_\nu U^\lambda_{\beta\gamma}.$$

We have now the quantities

$$E^\alpha_i=2\left(\frac{\partial F}{\partial p^i_{\alpha\beta}}\right)_{/\beta}-\frac{\partial F}{\partial p^i_\alpha}\qquad (i=1,2,\cdots,n;\ \alpha=1,2,\cdots,K),$$

which are known as the components of the Synge vectors. These are not intrinsic under arbitrary parameter transformation. After some calculation we see that the Synge vector E^{α}_{i} is transformed by the transformation-group (2) in the manner

$$E^{\lambda}_{a} = \Delta\, E^{\alpha}_{i}\, X^{i}_{a}\, U^{\lambda}_{\alpha} + \Delta\, \frac{\partial F}{\partial p^{i}_{\alpha\beta}}\, X^{i}_{a}\, U^{\lambda}_{\alpha\beta}\,.$$

By the above equation and the law of transformation of $G^{\alpha}_{\beta\gamma}$ we can see that the vectors

$$\mathfrak{E}^{\alpha}_{i} = -\frac{1}{F}\left\{E^{\alpha}_{i} + \frac{\partial F}{\partial p^{i}_{\beta\gamma}}\, G^{\alpha}_{\beta\gamma}\right\} \qquad (\alpha = 1, 2, \cdots, K)$$

are intrinsic, that is

$$\mathfrak{E}^{\lambda}_{a} = \mathfrak{E}^{\alpha}_{i}\, X^{i}_{a}\, U^{\lambda}_{\alpha}\,.$$

By virtue of (5b) and (5c), it is known that there are the relations

$$\mathfrak{E}^{\alpha}_{i}\, p^{i}_{\gamma} = \delta^{\alpha}_{\gamma} \qquad (\alpha, \gamma = 1, 2, \cdots, K)\,. \tag{13}$$

We see that the rank of the $\binom{K+1}{2} n$—rowed determinant $| F^{\alpha(2)}_{;\,i\,;}{}^{\beta(2)}_{j} |$ is not greater than $\binom{K+1}{2} (n-K)$, because of $F^{\alpha(2)}_{;\,i\,;}{}^{\beta(2)}_{j}\, p^{j}_{\gamma} = 0$ $(\gamma = 1, 2, \cdots, K)$. Suppose now that the determinant $| F^{\alpha(2)}_{;\,i\,;}{}^{\beta(2)}_{j} |$ is of rank $\binom{K+1}{2} (n-K)$, then we have one and only one system of the intrinsic quantities $B^{\kappa}_{\gamma(2)}{}^{j}_{\beta(2)}$ satisfying the equations

$$B^{\kappa}_{\gamma(2)}{}^{j}_{\beta(2)} = B^{j}_{\beta(2)}{}^{\kappa}_{\gamma(2)}\,,$$

$$B^{\kappa}_{\gamma(2)}{}^{j}_{\beta(2)}\, F^{\alpha(2)}_{;\,i\,;}{}^{\beta(2)}_{j} = F(\delta^{\kappa}_{i} - p^{\kappa}_{\alpha}\, \mathfrak{E}^{\alpha}_{i})\, \delta^{\alpha(2)}_{\gamma(2)}\,,$$

$$B^{\kappa}_{\gamma(2)}{}^{j}_{\beta(2)}\, \mathfrak{E}^{\alpha}_{j} = 0 \qquad (\alpha = 1, 2, \cdots, K)\,,$$

because of $(\delta^{\kappa}_{i} - p^{\kappa}_{\alpha}\, \mathfrak{E}^{\alpha}_{i})\, p^{i}_{\gamma} = 0$ $(\gamma = 1, 2, \cdots, K)$. The quantities $B^{\kappa}_{\gamma(2)}{}^{j}_{\beta(2)}$ thus determined are quantities in $F^{(3)}_{n}$.

If we put $B^{i}_{\alpha(2)}{}^{j}_{\beta(2)\,;}{}^{\alpha(3)}_{i}{}_{;}{}^{\beta(3)}_{j} = g^{\alpha_3\beta_3}$, these are components of a symmetric u-tensor of the second degree. Suppose that the determinant $| g^{\alpha\beta} |$ be different from zero and $g_{\alpha\beta}$ be the inverse system of $g^{\alpha\beta}$. If we put $g_{\alpha\beta}\, \mathfrak{E}^{\alpha}_{i}\, \mathfrak{E}^{\beta}_{j} + \dfrac{1}{F}\, F^{\alpha_1}_{;\,i}{}^{\beta_1}{}^{\alpha_2}_{;\,j}{}^{\beta_2}\, g_{(\alpha_1(\beta_1}\, g_{\alpha_2)\beta_2)} = g_{ij}$ for which the relation $g_{ij}\, p^{i}_{\alpha}\, p^{j}_{\beta} = g_{\alpha\beta}$ holds good and assume that $g = | g_{ij} | \neq 0$, then we have the conjugate system g^{ij} such that $g^{ij}\, g_{i\kappa} = \delta^{j}_{\kappa}$. The quantities g^{ij} and g_{ij} thus defined are components of a contravariant tensor and a covariant tensor in $F_n^{(3)}$ respectively. We shall adopt g_{ij} and g^{ij} as the fundamental tensors of the space $F_n^{(3)}$ and raise the subscripts by means of g^{ij} and lower the superscripts by means of g_{ij}. Moreover we shall raise the greek subscripts by means of $g^{\alpha\beta}$ and lower the greek superscripts by means of $g_{\alpha\beta}$.

§ 2. Covariant differential of the vector in $F_n^{(3)}$.

After some calculation it is known that the quantity

$$\Lambda^{\kappa}_{i\gamma}=\frac{1}{K+1}B^{\kappa}_{\alpha\gamma}{}^{j}_{\beta(2)}\{F_{;}{}^{\alpha}_{i;}{}^{\beta(2)}_{j}+G^{\alpha}_{\gamma_1\gamma_2}F_{;}{}^{\gamma(2)}_{i\,;}{}^{\beta(2)}_{j}\}$$

is transformed by (2) as follows:

$$\Lambda^{c}_{a\nu}=\Lambda^{\kappa}_{i\gamma}X^{i}_{a}X^{c}_{K}U^{\gamma}_{\nu}+(\delta^{\kappa}_{i}-p^{\kappa}_{\alpha}\mathfrak{C}^{\alpha}_{i})X^{c}_{K}X^{i}_{ab}p^{b}_{\nu}.$$

It is easily seen that the quantity

$$v^{\beta}_{j/\alpha}-\Lambda^{i}_{j\alpha}v^{\beta}_{i}+G^{\beta}_{\alpha\gamma}v^{\gamma}_{j}$$

is intrinsic when v^{β}_{j} is an intrinsic quantity which satisfies the relation $v^{\beta}_{j}p^{j}_{\gamma}=0$ $(\gamma=1,2,\cdots,K)$.

As a consequence of the above result the quantity

$$H^{\alpha(3)}_{i}{}_{\beta}=\frac{1}{g}\{g_{;}{}^{\alpha(3)}_{i}{}_{/\beta}-\Lambda^{j}_{i\beta}g_{;}{}^{\alpha(3)}_{j}-2\Lambda^{j}_{j\beta}g_{;}{}^{\alpha(3)}_{i}+3G^{(\alpha_1}_{\beta\gamma}g_{;}{}^{|\gamma|}_{i}{}^{\alpha_2\alpha_3)}\}$$

is intrinsic. Moreover it is seen that the above expression is linear with respect to the highest derivatives $p^{j}_{\beta(4)}$. Accordingly, if we put $g^{\beta(\alpha_4}H_{i}{}^{\alpha(3))}_{\beta}=H^{\alpha(4)}_{i}$, this is written in the form

(14) $$H^{\alpha(4)}_{i}=H^{\alpha(4)}_{i}{}^{\beta(4)}_{j}p^{j}_{\beta(4)}+P^{\alpha(4)}_{i}(x^{\kappa},p^{\kappa}_{\gamma(1)},p^{\kappa}_{\gamma(2)},p^{\kappa}_{\gamma(3)}),$$

where the second term of the right hand members is the function of x^{κ}, $p^{\kappa}_{\gamma(1)}$, $p^{\kappa}_{\gamma(2)}$, $p^{\kappa}_{\gamma(3)}$. Since $H^{\alpha(4)}_{i}{}^{\beta(4)}_{j}p^{j}_{\gamma}=0$ $(\gamma=1,2,\cdots,K)$, the $\binom{K+3}{4}n$— rowed determinant $|H^{\alpha(4)}_{i}{}^{\beta(4)}_{j}|$ has the highest rank $\binom{K+3}{4}(n-K)$. If we assume that $|H^{\alpha(4)}_{i}{}^{\beta(4)}_{j}|$ is of rank $\binom{K+3}{4}(n-K)$, then from (14) we can derive another intrinsic system

$$T^{\kappa}_{\gamma(4)}=(\delta^{\kappa}_{j}-p^{\kappa}_{\alpha}\mathfrak{E}^{\alpha}_{j})\,p^{j}_{\gamma(4)}+\overline{P}^{\kappa}_{\gamma(4)}(x^{i},p^{i}_{\alpha(1)},p^{i}_{\alpha(2)},p^{i}_{\alpha(3)})\,.$$

In order to determine the coefficients of connection of $F^{(3)}_{n}$, we shall assume that

$$(15)\qquad T^{\kappa}_{\gamma(4)}=0\,,$$

and denote by $D_{\gamma}f$ the total derivative of f with respect to u^{γ} under the condition (15). Then we can easily see that $D_{\gamma}\mathfrak{E}^{\alpha}_{i}$ is a function of only the surface elements of the third order.

Let us consider the quantity

$$\Gamma^{i}_{j\alpha}=\Lambda^{i}_{j\alpha}+p^{i}_{\beta}(D_{\alpha}\mathfrak{E}^{\beta}_{j}+G^{\beta}_{\gamma\alpha}\mathfrak{E}^{\gamma}_{j})\ [5]$$

which is a quantity in $F^{(3)}_{n}$. After some calculation we see that the law of transformation of $\Gamma^{i}_{j\alpha}$ is

$$(16)\qquad \Gamma^{a}_{b\lambda}=X^{a}_{i}X^{j}_{b}U^{\alpha}_{\lambda}\Gamma^{i}_{j\alpha}-X^{a}_{ij}X^{i}_{b}p^{j}_{\alpha}U^{\alpha}_{\lambda}\,.$$

We shall now determine the coefficient of connection using the quantity $\Gamma^{i}_{j\alpha}$. Let us consider the operation $P^{\beta(l)}$ which is defined as follows

$$P^{\beta(l)}(T^{J})=\sum_{t=l}^{3}\binom{t}{l}\,T^{J}{}_{;}{}^{\beta(l)}{}_{\kappa}^{\alpha(t-l)}(dp^{\kappa}_{\alpha(t-l)}-p^{\kappa}_{\alpha(t-l)\gamma}du^{\gamma})\,.$$

Then we have the following theorem.

THEOREM 1. *When T^{J} behaves so as $T^{A}=\mathfrak{A}^{A}_{J}T^{J}$ under the transformation-group* (2), *$P^{\beta(l)}(T^{J})$ are transformed under* (2) *in the manners*

$$(17)\qquad P^{\mu(s)}(T^{A})=\mathfrak{A}^{A}_{J}\sum_{l=s}^{3}P^{\beta(l)}(T^{J})'A^{\mu(s)}_{\beta(l)}\qquad (s=1,2,3)\,,$$

where $\mathfrak{A}^{A}_{J}$ are functions of only X^{a}_{i}, $X^{a}_{i(2)},\cdots$, U^{λ}_{α}, $U^{\lambda}_{\alpha(2)},\cdots$, and $'A^{\mu(s)}_{\beta(l)}$ are the same as the coeffiicients of the transformation of $p^{i}_{\beta(l)}$ under any parameter transformation, that is

$$p^{i}_{\beta(l)}=\sum_{s=1}^{l}p^{i}_{\mu(s)}\,'A^{\mu(s)}_{\beta(l)}\,.$$

On the other hand the quantities $K^{\gamma(t)}_{\beta(l)}$ $(t,l=1,2,3\,;\ t\leqq l)$ which are defined as follows

$$K^{\gamma}_{\beta}=\delta^{\gamma}_{\beta},\quad K^{\gamma}_{\beta(2)}=G^{\gamma}_{\beta_1\beta_2},\quad K^{\gamma}_{\beta(3)}=D_{(\beta_3}G^{\gamma}_{\beta_1\beta_2)}+G^{\gamma}_{\alpha(\beta_3}G^{\alpha}_{\beta_1\beta_2)}\,,$$

$$K^{\gamma(2)}_{\beta(2)}=\delta^{\gamma_1}_{(\beta_1}\delta^{\gamma_2}_{\beta_2)},\quad K^{\gamma(2)}_{\beta(3)}=3G^{\gamma_1}_{(\beta_1\beta_2}\delta^{\gamma_2}_{\beta_3)},\quad K^{\gamma(3)}_{\beta(3)}=\delta^{\gamma_1}_{(\beta_1}\delta^{\gamma_2}_{\beta_2}\delta^{\gamma_3}_{\beta_3)}\,,$$

are transformed in the following manner

$$(18)\qquad K^{\nu(t)}_{\mu(s)}=U^{\nu(t)}_{\gamma(t)}\sum_{l=t}^{s}K^{\gamma(t)}_{\beta(l)}A^{\beta(l)}_{\mu(s)}\qquad (s=1,2,3),$$

where $A^{\beta(l)}_{\mu(s)}$ is the inverse system of $'A^{\mu(s)}_{\beta(l)}$, that is

$$\sum_{s=l}^{t}A^{\beta(l)}_{\mu(s)}\,'A^{\mu(s)}_{\beta(t)}=\delta^{\beta(l)}_{\beta(t)}.$$

From (17) and (18) one gets the following theorem.

THEOREM 2. *Under the transformation-group* (2) *the quantity*

$$\sum_{l=1}^{3}P^{\beta(l)}(T^{J})\,K^{\gamma}_{\beta(l)}=\sum_{l=1}^{3}\sum_{t=l}^{3}\binom{t}{l}\,T^{J}{}_{;}{}^{\beta(l)}{}_{\kappa}^{\alpha(t-l)}\,K^{\gamma}_{\beta(l)}\,(dp^{\kappa}_{\alpha(t-l)}-p^{\kappa}_{\alpha(t-l)\gamma}\,du^{\gamma})$$

is transformed in the manner

$$\sum_{s=1}^{3}P^{\mu(s)}(T^{A})\,K^{\nu}_{\mu(s)}=\mathfrak{A}^{A}_{J}\,U^{\nu}_{\gamma}\sum_{l=1}^{3}P^{\beta(l)}(T^{J})\,K^{\gamma}_{\beta(l)}.$$

By the above theorem and (16) it follows that the quantity

$$\Gamma^{i}_{j}=\frac{1}{K}\sum_{l=1}^{3}P^{\beta(l)}(\Gamma'^{i}_{j\alpha})\,K^{\alpha}_{\beta(l)}+\Gamma'^{i}_{j\alpha}\,du^{\alpha}$$

behaves under (2) as follows:

$$\Gamma^{a}_{b}=\Gamma^{i}_{j}X^{a}_{i}X^{j}_{b}+X^{a}_{i}X^{i}_{bc}\,dx^{c},$$

so that we can define the covariant differential of a vector v^{i} in the manner

$$\delta v^{i}=dv^{i}+\Gamma^{i}_{j}v^{j}$$

or

$$(19)\qquad \delta v^{i}=dv^{i}+\sum_{s=0}^{2}C^{i}_{j}{}^{\beta(s)}_{\kappa}\,v^{i}\,dp^{\kappa}_{\beta(s)}+C^{i}_{j\gamma}\,v^{i}\,du^{\gamma},$$

putting

$$C^{i}_{j\kappa}=\frac{1}{K}\{\Gamma'^{i}_{j\alpha;}{}^{\alpha}_{\kappa}+G^{\alpha}_{\beta_1\beta_2}\Gamma'^{i}_{j\alpha;}{}^{\beta(2)}_{\kappa}+D_{(\beta_1}G^{\alpha}_{\beta_2\beta_3)}\Gamma'^{i}_{j\alpha;}{}^{\beta(3)}_{\kappa}$$
$$+G^{\alpha}_{\gamma(\beta_1}G^{\gamma}_{\beta_2\beta_3)}\Gamma'^{i}_{j\alpha;}{}^{\beta(3)}_{\kappa}\},$$

$$C^{i}_{j}{}^{\beta(1)}_{\kappa}=\frac{1}{K}\{2\Gamma'^{i}_{j\alpha;}{}^{\beta_1\alpha}_{\kappa}+3G^{\alpha}_{\beta_2\beta_3}\Gamma'^{i}_{j\alpha;}{}^{\beta_1\beta_2\beta_3}_{\kappa}\},$$

$$C^{i}_{j}{}^{\beta(2)}_{\kappa}=\frac{3}{K}\Gamma'^{i}_{j\alpha;}{}^{\beta(2)\alpha}_{\kappa},\qquad C^{i}_{j\gamma}=\frac{1}{K}\Gamma'^{i}_{j\gamma}-\sum_{s=0}^{2}C^{i}_{j}{}^{\beta(s)}_{\kappa}p^{\kappa}_{\beta(s)\gamma}.$$

§ 3. Intrinsic Pfaff's forms.

In order to define the covariant derivatives of the vector we shall derive the intrinsic Pfaff's forms. By the theorem 2 we see that the expressions

$$\frac{1}{g}\sum_{l=s}^{3} P^{\alpha(l)}(g;{}^{\beta(3)}_{j})\, K^{\gamma(s)}_{\alpha(l)} = \frac{1}{g}\sum_{l=s}^{3}\sum_{t=l}^{3}\binom{t}{l}\, g;{}^{\beta(3)}_{j};{}^{\alpha(l)\,\delta(t-l)}_{i}\, K^{\gamma(s)}_{\alpha(l)}$$

$$\times\{dp^{i}_{\delta(t-l)} - p^{i}_{\delta(t-l)\gamma}\, du^{\gamma}\} \qquad (s=1,2,3)$$

are intrinsic. Putting $t-l=r$ we have

$$\frac{1}{g}\sum_{r=0}^{3-s}\sum_{l=s}^{3-r}\binom{l+r}{l}\, g;{}^{\beta(3)}_{j};{}^{\alpha(l)\delta(r)}_{i}\, K^{\gamma(s)}_{\alpha(l)}\{dp^{i}_{\delta(r)} - p^{i}_{\delta(r)\alpha}\, du^{\alpha}\}$$

or

(20) $$\frac{1}{g}\binom{3}{s}\, g;{}^{\beta(3)}_{j};{}^{\alpha(s)\,\delta(3-s)}_{i}\,\delta^{\gamma(s)}_{\alpha(s)}\, dp^{i}_{\delta(3-s)} + \frac{1}{g}\sum_{r=0}^{2-s} M^{\beta(3)}_{j}{}^{\gamma(s)\alpha(r)}_{i}\, dp^{i}_{\alpha(r)}$$

$$+\frac{1}{g} M^{\beta(3)}_{j}{}^{\gamma(s)}_{\alpha}\, du^{\alpha} \qquad (s=1,2,3),$$

where we have put

$$M^{\beta(3)}_{j}{}^{\gamma(s)\alpha(r)}_{i} = \sum_{l=s}^{3-r}\binom{l+r}{l}\, g;{}^{\beta(3)}_{i};{}^{\delta(l)\alpha(r)}_{i}\, K^{\gamma(s)}_{\delta(l)},$$

$$M^{\beta(3)}_{j}{}^{\gamma(s)}_{\alpha} = -\sum_{r=0}^{3-s}\sum_{l=s}^{3-r}\binom{l+r}{l}\, g;{}^{\beta(3)}_{i};{}^{\alpha(l)\delta(r)}_{j}\, K^{\gamma(s)}_{\alpha(l)}\, p^{i}_{\delta(r)\alpha}.$$

Suppose now that the $\binom{K+2}{3}n$— rowed determinant $|\, g;{}^{\beta(3)}_{j};{}^{\alpha(3)}_{i}\,|$ is of rank $\binom{K+2}{3}(n-K)$, then we can derive from (20) the intrinsic Pfaff's forms

(21) $$\omega^{i}_{\alpha(3-s)} = (\delta^{i}_{j} - p^{i}_{\alpha}\,\mathfrak{E}^{\alpha}_{j})\, dp^{j}_{\alpha(3-s)} + \sum_{r=0}^{2-s} P^{i}_{\alpha(3-s)}{}^{\beta(r)}_{j}\, dp^{j}_{\beta(r)}$$

$$+ Q^{i}_{\alpha(3-s)\gamma}\, du^{\gamma} \qquad (s=1,2,3).$$

Moreover we can derive from the intrinsic differential of $g;{}^{\alpha(3)}_{i}$ defined by

$$\delta\, g;{}^{\alpha(3)}_{i} = \frac{1}{g}\{dg;{}^{\alpha(3)}_{i} - \Gamma^{j}_{i}\, g;{}^{\alpha(3)}_{j} - 2\Gamma^{j}_{j}\, g;{}^{\alpha(3)}_{i} + 3G^{(\alpha_1}_{\beta\gamma}\, g;{}^{\alpha_2\alpha_3)\beta}_{i}\, du^{\gamma}\}$$

another intrinsic Pfaff's form

$$\omega^{i}_{\alpha(3)}=(\delta^{i}_{j}-p^{i}_{\alpha}\,\mathfrak{C}^{\alpha}_{j})\,dp^{j}_{\alpha(3)}+\sum_{r=0}^{2}P^{i}_{\alpha(3)}{}^{\beta(r)}_{j}\,dp^{j}_{\beta(r)}+Q^{i}_{\alpha(3)\gamma}\,du^{\gamma}\,. \tag{22}$$

§ 4. Covariant derivatives.

When we put

$$\delta v^{i}=\nabla_{\gamma}v^{i}\,du^{\gamma}+\sum_{r=0}^{3}\nabla^{\alpha(r)}_{j}v^{i}\,\omega^{j}_{\alpha(r)},$$

it follows from (19), (21) and (22)

$$\nabla^{\alpha(3)}_{\kappa}v^{i}=\frac{\partial v^{i}}{\partial p^{\kappa}_{\alpha(3)}},$$

$$\nabla^{\alpha(2)}_{\kappa}v^{i}=\frac{\partial v^{i}}{\partial p^{\kappa}_{\alpha(2)}}+C^{i}_{j}{}^{\alpha(2)}_{\kappa}v^{i}-P^{j}_{\beta(3)}{}^{\alpha(2)}_{\kappa}\nabla^{\beta(3)}_{j}v^{i}\,,$$

$$\nabla^{\alpha(1)}_{\kappa}v^{i}=\frac{\partial v^{i}}{\partial p^{\kappa}_{\alpha(1)}}+C^{i}_{j}{}^{\alpha(1)}_{\kappa}v^{i}-P^{j}_{\beta(3)}{}^{\alpha(1)}_{\kappa}\nabla^{\beta(3)}_{j}v^{i}-P^{j}_{\beta(2)}{}^{\alpha(1)}_{\kappa}\nabla^{\beta(2)}_{j}v^{i}\,,$$

$$\nabla_{\kappa}v^{i}=\frac{\partial v^{i}}{\partial x^{\kappa}}+C^{i}_{j\kappa}v^{j}-P^{j}_{\beta(3)\kappa}\nabla^{\beta(3)}_{j}v^{i}-P^{j}_{\beta(2)\kappa}\nabla^{\beta(2)}_{j}v^{i}-P^{j}_{\beta(1)\kappa}\nabla^{\beta(1)}_{j}v^{i},$$

$$\nabla_{\gamma}v^{i}=\frac{\partial v^{i}}{\partial u^{\gamma}}+C^{i}_{j\gamma}v^{j}-\sum_{s=0}^{3}Q^{\kappa}_{\alpha(s)\gamma}\nabla^{\alpha(s)}_{\kappa}v^{i}\,,$$

under the conditions

$$\nabla^{\alpha(s)}_{K}v^{i}\,p^{\kappa}_{\gamma}=0\qquad(\gamma=1,2,\cdots,K)\,.$$

The quantities $\nabla^{\alpha(s)}_{\kappa}v^{i}$, $(s=0,1,2,3)$ and $\nabla_{\gamma}v^{i}$ thus defined are covariant derivatives of a vector v^{i} in $F^{(3)}_{n}$.

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