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On the automorphisms of a real semi-simple Lie algebra.

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Let $\mathfrak{G}$ be a real semi-simple Lie algebra and G the group of automorphisms of $\mathfrak{G}$. As is well-known, the connected component G^0 of the identity of G coincides with the group of inner automorphisms of $\mathfrak{G}$. The object of this paper is to investigate the structure of the factor group G/G^0; i.e. the type of outer automorphisms of $\mathfrak{G}$ modulo the inner automorphisms.

E. Cartan [2][1] has treated our subject in 1927, and established intimate relations between the group G and the group of isometric transformations in a certain type of Riemannian space. However, the problem being algebraic in its character, a more direct treatment might be desirable. Now, Cartan [1] had earlier dealt with the same subject by a more algebraic method for the case where the Lie algebra $\mathfrak{G}$ is complex semi-simple. F. Gantmacher [4] [5] attempted to rearrange Cartan's results concerning complex semi-simple Lie algebras on the basis of the structure theory founded by H. Weyl [9], and then applied so obtained results to the classification of real simple Lie algebras which had been also achieved by Cartan with specific devices. It is to be noted that recently I. Satake [8] has given an algebraic proof to a theorem of Cartan on which Gantmacher had yet to depend.[2] In following this algebraic direction, we shall give in this paper some results analogous to the case of complex semi-simple Lie algebras.

In § 1, we reduce our problem to that concerning a maximal compact subgroup of the group G. This reduction was done by Cartan [3], but we shall perform it by means of a method suggested by G. D. Mostow [7]. Next, to investigate the structure of this maximal compact

1) Numbers in brackets refer to Bibliography at the end of this paper.

2) See Footnote 9).

subgroup, we study in § 2 the structure of the Lie algebra $\mathfrak{G}$ in connection with its complex form $\tilde{\mathfrak{G}}$ making use of the results of Gantmacher and Satake cited above. We then obtain in § 3 our main results: The group G/G^0 is described by means of the groups of "rotations" $\mathfrak{T}$ and $\mathfrak{S}$ which are determined by the situation of $\mathfrak{G}$ in $\tilde{\mathfrak{G}}$ expressed in terms of the root system of $\tilde{\mathfrak{G}}$. In § 4, we consider the real forms of the complex simple Lie algebras of type A_n as representatives of real simple Lie algebras, and apply our results to determine the groups G/G^0 in this case. This furnishes a new proof to the results established by Cartan [2] and by N. Jacobson [6].

§ 1. Structure of the group of automorphisms.

In the following, we denote by R the real number field and by C the complex number field.

Let $\mathfrak{G}$ be a real semi-simple n-dimensional Lie algebra and $\tilde{\mathfrak{G}}$ its complex form, i. e., the complex semi-simple Lie algebra obtained from $\mathfrak{G}$ by extending the coefficient field R to C. By a theorem of E. Cartan[3], we may consider $\mathfrak{G}$ as situated in $\tilde{\mathfrak{G}}$ in the following manner: Let $\mathfrak{G}_u$ be a unitary restriction of $\tilde{\mathfrak{G}}$ and take an involutive automorphism S of $\mathfrak{G}_u$. We decompose $\mathfrak{G}_u$ into the eigenspaces of S and obtain

(1) $$\mathfrak{G}_u=\mathfrak{G}_1+\mathfrak{G}_{-1},$$

where

(2) $$\mathfrak{G}_1=\{x+Sx\,;\ x\in\mathfrak{G}_u\},\qquad \mathfrak{G}_{-1}=\{x-Sx\,;\ x\in\mathfrak{G}_u\}.$$

Then, $\mathfrak{G}$ is given by the real linear space in $\tilde{\mathfrak{G}}$ of the following form;

(3) $$\mathfrak{G}=\mathfrak{G}_1+\sqrt{-1}\,\mathfrak{G}_{-1},$$

where $\sqrt{-1}\,\mathfrak{G}_{-1}=\{\sqrt{-1}\,x\,;\ x\in\mathfrak{G}_{-1}\}$.

Now the fundamental quadratic form of $\tilde{\mathfrak{G}}$;

(4) $$\Phi(x)=\text{trace of }(\text{ad }x)^2$$

where ad x is the adjoint mapping in $\tilde{\mathfrak{G}}$ of an element x of $\tilde{\mathfrak{G}}$, is negative definite on $\mathfrak{G}_u$. Therefore we can choose a suitable basis

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3) See e. g. Gantmacher [5] p. 226 Th. 6, or Mostow [7] Th. 1.

(5) $$e_1, e_2, \cdots\cdots, e_n$$

of $\tilde{\mathfrak{G}}$ contained in $\mathfrak{G}_u$ so that the form (4) is represented as

(6) $$\Phi(x) = -\{(x^1)^2 + (x^2)^2 + \cdots + (x^n)^2\}$$

for $x = x^1 e_1 + x^2 e_2 + \cdots + x^n e_n$. The involutive automorphism S leaving invariant this form, S is an orthogonal transformation with respect to the basis (5). Therefore, after applying a certain orthogonal transformation to (5) if necessary, we may assume that the first r vectors $e_1, \cdots, e_r$ span $\mathfrak{G}_1$ and the latter $n-r$ vectors $e_{r+1}, \cdots, e_n$ span $\mathfrak{G}_{-1}$ over R.

Let G, $\tilde{G}$ and G_u be the groups of all automorphisms of $\mathfrak{G}$, $\tilde{\mathfrak{G}}$ and $\mathfrak{G}_u$ respectively. Since an automorphism of $\mathfrak{G}$ or of $\mathfrak{G}_u$ is uniquely extensible to an automorphism of $\tilde{\mathfrak{G}}$, G and G_u may be considered as subgroups of $\tilde{G}$. G and G_u are then composed of all automorphisms of $\tilde{\mathfrak{G}}$ which leave invariant $\mathfrak{G}$ and $\mathfrak{G}_u$ respectively. As is well-known, $\tilde{G}$ is a Lie group whose Lie algebra is $\{\text{ad}\ x;\ x \in \tilde{\mathfrak{G}}\}$ and is isomorphic to $\tilde{\mathfrak{G}}$. Its closed subgroups G and G_u are those which correspond to the real subalgebras $\mathfrak{G}$ and $\mathfrak{G}_u$ respectively of $\tilde{\mathfrak{G}}$. Now, representing the automorphisms by matrices with respect to the basis (5), we may regard the group $\tilde{G}$ and so its subgroups G and G_u as linear groups. A non-singular matrix $A = (a_{ij})_{i,j=1}^{n}$, $a_{ij} \in C$, belongs to $\tilde{G}$ if and only if it satisfies all the relations

(7) $$A[e_i, e_j] = [Ae_i, Ae_j] \qquad (1 \leqq i, j \leqq n),$$

which are obviously expressed by algebraic equations with n^2 variables a_{ij}. We remark that the coefficients of these equations are given by the structure constants for the basis (5), so that they are real numbers. As the groups $\tilde{G}$, G and G_u are regarded as linear groups, their Lie algebras $\tilde{\mathfrak{G}}$, $\mathfrak{G}$ and $\mathfrak{G}_u$ are representable as linear Lie algebras, and they coincide with $\{X;\ x \in \tilde{\mathfrak{G}}\}$, $\{X;\ x \in \mathfrak{G}\}$ and $\{X;\ x \in \mathfrak{G}_u\}$ respectively where X is the matrix representing ad x with respect to the basis (5). Since automorphisms of $\tilde{\mathfrak{G}}$ leave invariant the form (6), $\tilde{G}$ consists of complex orthogonal matrices. Further, it is clear that G_u coincides with the subgroup composed of all real matrices in $\tilde{G}$. As regards G, since $\mathfrak{G}$ is spanned over R by the elements

$$e_1, \cdots, e_r,\ \sqrt{-1}\ e_{r+1}, \cdots,\ \sqrt{-1}\ e_n,$$

G consists of all matrices $A=(a_{ij})$ in $\tilde{G}$ satisfying the following conditions

(8) $\bar{a}_{ij}=a_{ij}$, for $1\leqq i, j\leqq r$ or $r+1\leqq i, j\leqq n$,

$\bar{a}_{ij}=-a_{ij}$, otherwise.

We define a subgroup K of $\tilde{G}$ as follows.

$$K=G\cap G_u.$$

Then K is the group of automorphisms of $\tilde{\mathfrak{J}}$ which leave invariant both $\mathfrak{G}$ and $\mathfrak{G}_u$. The Lie algebra of K is isomorphic to $\mathfrak{G}\cap\mathfrak{G}_u=\mathfrak{G}_1$. In our matricial representation, K consists of all real matrices in $\tilde{\mathfrak{J}}$ which satisfy (8), that is, matrices $A=(a_{ij})$, $a_{ij}\in R$, with the following property:

(9) $a_{ij}=0$, for $1\leqq i\leqq r$ and $r+1\leqq j\leqq n$, or

$r+1\leqq i\leqq n$ and $1\leqq j\leqq r$.

LEMMA 1. *An automorphism A in G_u belongs to K, if and only if A satisfies either one of the following conditions:*

(10) $A x\in\mathfrak{G}_1$ *for* $x\in\mathfrak{G}_1$,

(11) $AS=SA$.

PROOF. The necessity of (10) is obvious. If (10) is satisfied, then as A is an orthogonal transformation, $\mathfrak{G}_{-1}$ is also left invariant by A. So A leaves invariant $\mathfrak{G}$ as well as $\mathfrak{G}_u$. The condition (11) is equivalent to (9), since S is represented by the matrix (s_{ij}) where $s_{ij}=\delta_{i,j}$ for $1\leqq i\leqq r$ and $s_{ij}=-\delta_{i,j}$ for $r+1\leqq i\leqq n$, q. e. d.

Now, from what is mentioned above, we see that G is a pseudo-algebraic subgroup of the general linear group $GL(n, C)$ in the sense of C. Chevalley [3], that is, a subgroup whose matrices $A=(a_{ij})$ are defined by a certain number of algebraic equations for the $2n^2$ variables a'_{ij}, a''_{ij} $(1\leqq i, j\leqq n)$ where a'_{ij}, a''_{ij} are respectively real and imaginary parts of a_{ij}. In our case, these algebraic equations are given by (7) and (8). The following lemma concerning some type of pseudo-algebraic subgroups may be proved just as in the proof of a lemma on the same type of algebraic subgroups due to Chevalley ([3] p. 201).

LEMMA 2. *Let G be a pseudo-algebraic subgroup of $GL(n, C)$ and suppose that for a matrix $A=(a_{ij})$ of G the adjoint martix ${}^t\bar{A}=(\bar{a}_{ji})$*

also belongs to G. Then, for the unique expression of a matrix A in G in the form

$$A = A_h A_u \tag{12}$$

where A_h is a positive definite hermitian matrix and A_u a unitary matrix, the matrices A_h and A_u belong to G. Moreover, when we represent the matrix A_h uniquely in the form

$$A_h = \exp A'_h \tag{13}$$

where A'_h is an hermitian matrix, the matrix A'_h belongs to the Lie algebra $\mathfrak{G}$ of G.

In our case the group G satisfies the assumption of this lemma. To see this, as a matrix $A=(a_{ij})$ in G is complex orthogonal, i. e., $(a_{ji})=(a_{ij})^{-1}$, we need only to show that the complex conjugate matrix $\bar{A}=(\bar{a}_{ij})$ of A also belongs to G. This follows from the fact that $\bar{A}$ satisfies the relations (7) and (8) as well as A by a remark after (7). Thus the conclusion of Lemma 2 is valid for our group G.

We see first that unitary matrices in G are real orthogonal matrices and they form the subgroup K. Next, let H be the set of positive definite hermitian matrices contained in G. As G is a closed subgroup of $GL(n, C)$, its inner topology coincides with the relative topology induced from $GL(n, C)$, K is a compact subgroup and H is a closed subset in $GL(n, C)$. By Lemma 2, we obtain a one-to-one mapping from G onto the product space $H \times K$ which maps an element A of G with decomposition (12) to the pair (A_h, A_u). It is easily seen that this mapping is a homeomorphism. Next we show that H is homeomorphic to a real vector space. For this purpose, we consider the linear subspace $\mathfrak{G}_h$ in $\mathfrak{G}$ which consists of all hermitian matrices contained in $\mathfrak{G}$. We see easily from (3) that $\mathfrak{G}_h$ is really in accord with $\sqrt{-1}\,\mathfrak{G}_{-1}$. The second half of Lemma 2 asserts that (13) gives a one-to-one correspondence between $\mathfrak{G}_h$ and H. When we regard $\mathfrak{G}_h$ as a real vector space, this mapping, exp, is a homeomorphism. For, it is obtained by restricting the domain of definition to $\mathfrak{G}_h$ of the well-known homeomorphism, exp, from the set of all hermitian matrices onto the set of all positive definite hermitian matrices. Besides we see from this that K is a maximal compact subgroup of $\mathfrak{G}$.

Thus we have proved the following theorem, which does not depend on the matricial representation of G if $\exp \operatorname{ad} x$, $x \in G$, is understood to be the point $x(1)$ of the one-parameter subgroup $x(t)$ with the tangential vector ad x at the identity of G.

THEOREM 1. *Let* $\mathfrak{G}$ *be a real semi-simple Lie algebra and* $\tilde{\mathfrak{G}}$ *its complex form. We take a unitary restriction* $\mathfrak{G}_u$ *of* $\tilde{\mathfrak{G}}$ *and suppose* $\mathfrak{G}$ *relate to* $\mathfrak{G}_u$ *as in the formulae* (1), (2) *and* (3) *by means of an involutive automorphism* S *of* $\mathfrak{G}_u$. *Let* G *and* G_u *be the groups of automorphisms of* $\mathfrak{G}$ *and* $\mathfrak{G}_u$ *respectively which are extended to automorphisms of* $\tilde{\mathfrak{G}}$. *The Lie algebra of* G *is isomorphic to* $\mathfrak{G}$. *We set*

$$K = G \cap G_u$$

$$H = \{\exp \operatorname{ad} x\, ;\ x \in \sqrt{-1}\ \mathfrak{G}_{-1}\}\,.$$

Then K *is a maximal compact subgroup of* G *with the Lie algebra isomorphic to* $\mathfrak{G}_1$ *and* H *is homeomorphic to a real vector space. Every element* g *of* G *is uniquely and continuously represented in the form*

$$g = h\,k\,, \qquad h \in H\,, \qquad k \in K\,,$$

so that

$$G = H\,K$$

and G *is homeomorphic to the product space of* H *and* K.

We denote by G^0 and K^0 the connected components of the identity in the groups G and K respectively. G^0 is the so-called adjoint group of $\mathfrak{G}$ composed of all inner automorphisms of $\mathfrak{G}$. Then from this theorem we may deduce the following corollaries successively.

COROLLARY 1.[4] *With the same meanings as the decomposition of* G *in Theoerm* 1, *there holds*

$$G^0 = H\,K^0\,.$$

COROLLARY 2. $G/G^0 \cong K/K^0$ (*which is a finite group* !).

§ 2. Structure of the Lie algebra.

In the first half of this paragraph, we summarize for the sake of later use and in order to settle our notations some known results about structure and automorphisms of a complex semi-simple Lie algebra.

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4) This was first proved by Cartan [2] p. 368. Simpler proofs were given by Mostow [7] and by Y. Matsushima (Shijô Sugaku Danw. 2-5 (1947) p. 123 (in Japanese)). Also K. Iwasawa gave its somewhat sharpend form (Ann. Math. 50 (1949), p. 525).

Thereafter we shall clarify in more detail the situation of a real semi-simple Lie algebra $\mathfrak{G}$ in its complex form $\tilde{\mathfrak{G}}$ and determine the structure of the subalgebra $\mathfrak{G}_1$, which will be used in the next paragraph.

In a real or complex Lie algebra an element x_0 is said to be *regular*, if the multiplicity of 0-eigenvalues of ad x takes its minimum at x_0, and a subalgebra which coincides with the 0-eigenspace for the adjoint mapping of a regular element is called a *Cartan subalgebra.* Let $\tilde{\mathfrak{G}}$ be a complex semi-simple Lie algebra.[5] There exists in $\tilde{\mathfrak{G}}$ a *canonical basis*

(14) $$h_1, \cdots, h_l, e_\alpha, e_{-\alpha}, e_\beta, e_{-\beta}, \cdots ,$$

which is a complex basis with the following properties: $h_1, \cdots, h_l$ span a Cartan subalgebra $\tilde{\mathfrak{H}}$ of $\tilde{\mathfrak{G}}$. The commutations between the elements (14) are as follows.

(15) $$\begin{cases} [h_i, h_j]=0 & (1\leqq i, j\leqq l), \\ [h_i, e_\alpha]=\alpha_i e_\alpha , & \alpha_i \in R , \\ [e_\alpha, e_{-\alpha}]=-h_\alpha , & \\ \qquad \text{where } h_\alpha=\alpha^1 h_1+\cdots+\alpha^l h_l, & \alpha^i \in R , \\ [e_\alpha, e_\beta]=N_{\alpha,\beta}\, e_{\alpha+\beta}, & N_{\alpha,\beta} \in R . \end{cases}$$

The real vectors $\alpha=(\alpha_1, \cdots, \alpha_l)$ are called *roots* or *roots in contravariant form* and they form the *root system* $\sum$. $N_{\alpha,\beta} \neq 0$ if and only if $\alpha+\beta$ is again a root and $N_{\alpha,\beta}=N_{-\alpha,-\beta}$. Furthermore, $\Phi(e_\alpha, e_{-\alpha})=-1$, where $\Phi(\cdot, \cdot)$ is the polar form of the quadratic form (4), the so-called fundamental bilinear form of $\tilde{\mathfrak{G}}$.

The unitary restriction $\mathfrak{G}_u$ of $\tilde{\mathfrak{G}}$ with respect to this canonical basis (14) is the linear subspace spanned over R by the elements

$$\sqrt{-1}\, h_1,\cdots, \sqrt{-1}\, h_l, \quad e_\alpha+e_{-\alpha} , \quad \sqrt{-1}\,(e_\alpha-e_{-\alpha}) ,\cdots .$$

Now we denote by $\mathfrak{H}$ the real linear subspace of $\tilde{\mathfrak{G}}$ composed of the elements of the form

(16) $$h_\lambda=\lambda^1 h_1+\cdots+\lambda^l h_l , \qquad \lambda^i \in R ,$$

and call $\mathfrak{H}$ the *real part* of the Cartan subalgebra $\tilde{\mathfrak{H}}$. $\mathfrak{H}$ is characterized as the set of all elements h in $\tilde{\mathfrak{H}}$ for which all the eigenvalues of ad h

5) For the details about its structure mentioned in below, see Weyl [9].

are real. $\sqrt{-1}\,\mathfrak{H}=\{\sqrt{-1}\,h;\ h\in\mathfrak{H}\}$ is then a Cartan subalgebra of $\mathfrak{G}_u$. On the other hand, the fundamental quadratic form Φ of $\tilde{\mathfrak{G}}$ is positive definite on $\mathfrak{H}$, so that $\mathfrak{H}$ may be regarded as a real euclidean space with the inner product $(\cdot,\cdot)$ defined by $\Phi(\cdot,\cdot)$. We identify a real vector $\lambda=(\lambda^1,\cdots,\lambda^l)$ with the vector h_λ in (16). It is known that the vector α, h_α in (15), is the covariant form of the root in the contravariant form $\alpha:\alpha(\lambda)=\alpha_1\lambda^1+\cdots+\alpha_l\lambda^l=(\alpha,\lambda)$, and that these vectors $h_\alpha(\alpha\in\sum)$ span $\mathfrak{H}$ over R. Considering in covariant forms, the root system $\sum$ has the following properties:

(i) For $\alpha\in\sum$, $-\alpha\in\sum$ and no other multiples of α belong to $\sum$.

(ii) For $\alpha,\beta\in\sum$, $\frac{2(\alpha,\beta)}{(\alpha,\alpha)}$ is an integer and $\beta-\frac{2(\alpha,\beta)}{(\beta,\beta)}\alpha\in\sum$.

From these properties we may prove the following

LEMMA 3.[6] *Let $\alpha,\beta\in\sum$, $\alpha\neq\pm\beta$. Then $\beta-\big(\mathrm{sign}\,(\alpha,\beta)\big)\alpha\in\sum$.*

Moreover, it is known[6] that the root system $\sum$ has l linearly independent roots

$$\alpha^{(1)},\cdots,\alpha^{(l)} \tag{17}$$

with the following property: any root α is uniquely representable in the form

$$\alpha=p_1\alpha^{(1)}+\cdots+p_l\alpha^{(l)}, \tag{18}$$

where $p_1,\cdots,p_l$ are integers which are either all $\geqq 0$ or all $\leqq 0$. We call this system of roots a *fundamental basis* in $\sum$ and denote a fixed one by $\sum^0$. A root α is called *positive* or *negative* according to p_i are $\geqq 0$ or $\leqq 0$ in the expression (18) of α.

LEMMA 4. *Every positive root α can be connected from a root in $\sum^0$ by a chain of roots;*

$$\alpha^{(i)},\ \alpha^{(i)}+\alpha^{(j)},\cdots,\ \alpha-\alpha^{(k)},\ \alpha,$$

where each term is obtained by adding some root in $\sum^0$ to the foregoing term.

PROOF. We show first that $\alpha-\alpha^{(k)}$ is again a root for some root $\alpha^{(k)}$ in $\sum^0$ if α is a positive root not in $\sum^0$. Suppose this be false. Then for all $\alpha^{(k)}$ in $\sum^0$ $(\alpha,\alpha^{(k)})\leqq 0$ by Lemma 3 and, when α is expressed in the form (18),

6) See e.g. Satake [8] Lemma 1 and Prop. 3.

$$(\alpha, \alpha)=p_1(\alpha, \alpha^{(1)})+\cdots+p_l(\alpha, \alpha^{(l)})\leqq 0,$$

which is a contradiction. Now we see easily from the property of fundamental basis that $\alpha-\alpha^{(k)}$ is also a positive root. The lemma may be then proved by the induction on the integers $p_1+\cdots+p_l$ in (18) for positive roots α.

Next we recall some known results[7] concerning automorphisms of $\tilde{\mathfrak{G}}$ which leave invariant the Cartan subalgebra $\tilde{\mathfrak{H}}$. Such an automorphism A leaves invariant the real part $\mathfrak{H}$ of $\tilde{\mathfrak{H}}$, as is seen from the characterization of $\mathfrak{H}$ mentioned before. Then A induces in $\mathfrak{H}$ a *rotation* τ, that is, an orthognal transformation which permutes the roots among themselves. In this case, we have

$$(19) \qquad Ae_\alpha=\kappa_\alpha e_{\tau(\alpha)}, \qquad \kappa_\alpha \in C,$$

and the numbers κ_α satisfy the following relations:

$$(20) \qquad \kappa_\alpha \kappa_{-\alpha}=1,$$

$$(21) \qquad \kappa_{\alpha+\beta}=\kappa_\alpha \kappa_\beta \frac{N_{\tau(\alpha),\tau(\beta)}}{N_{\alpha,\beta}}, \qquad \text{if } \alpha+\beta \text{ is a root.}$$

In case A leaves invariant the unitary restriction $\mathfrak{G}_u$, too,

$$(22) \qquad \bar{\kappa}_\alpha=\kappa_{-\alpha},$$

and, together with (20),

$$(23) \qquad |\kappa_\alpha|=1.$$

Conversely, every rotation is induced by such an automorphism, where we may assume further that this automorphism leaves invariant $\mathfrak{G}_u$ as well as $\tilde{\mathfrak{H}}$. Let $\tilde{\mathfrak{T}}$ be the group of all rotations in $\mathfrak{H}$. $\tilde{\mathfrak{T}}$ is a finite group. We consider in $\mathfrak{H}$ the $(l-1)$-dimensional hyperplane E_α through the origin and orthogonal to a root α. The reflection σ_α with respect to E_α;

$$(24) \qquad \sigma_\alpha(h)=h-\frac{2\alpha(h)}{\alpha(h_\alpha)}h_\alpha \qquad (h \in \mathfrak{H}),$$

is surely a rotation by the property (ii) of the root system. The rotations of this form generate a normal subgroup $\tilde{\mathfrak{S}}$ of $\tilde{\mathfrak{T}}$ whose elements

7) See Cartan [1], or Gantmacher [4] Chapter 3.

we call ***inner rotations.*** Then a rotation is induced by an inner automorphism of $\tilde{\mathfrak{G}}$ and even of $\mathfrak{G}_u$ if and only if it is an inner rotation. Furthermore the connected components of $\mathfrak{H}-\bigcup_\alpha E_\alpha$ which are convex domains restricted by just l hyperplanes are obviously permuted among themselves under rotations and transitively even under inner rotations. In more detail, each one of these components is a fundamental domain of $\tilde{\mathfrak{S}}$. In view of this fact, we call a rotation which transforms a component, say Π, into itself a *particular rotation*, or precisely, a *particular rotation with respect to* Π.[8] All particular rotations with respect to a fixed component Π form a group $\tilde{\mathfrak{P}}$ and we have

$$\tilde{\mathfrak{T}}=\tilde{\mathfrak{S}}\tilde{\mathfrak{P}}, \qquad \tilde{\mathfrak{S}}\cap\tilde{\mathfrak{P}}=\{1\}. \tag{25}$$[9]

In the sequel, we take up the special component Π_0 defined by

$$\Pi_0=\{h;\ \alpha^{(i)}(h)>0, \quad \text{for all } \alpha^{(i)}\in\textstyle\sum^0\}, \tag{26}$$

and consider particular rotations with respect to Π_0. Such particular rotations are characterized as the orthogonal transformations in $\mathfrak{H}$ which permute the roots in the fundamental basis $\sum^0$ among themselves. Therefore they transform positive or negative roots to positive or negative roots respectively. By the way, we see that any particular rotation is conjugated to a particular rotation with respect to Π_0 by an inner rotation.

After these preparations, we return to a real semi-simple Lie algebra $\mathfrak{G}$. Henceforward we use the notations in the previous paragraph, and the canonical basis (14) etc. considered above are supposed to be those of the complex form $\tilde{\mathfrak{G}}$ of $\mathfrak{G}$. By theorems of Gantmacher[10] we may add the following assumptions on the involutive automorphism S of $\mathfrak{G}_u$ which determine the situation of $\mathfrak{G}$ in $\tilde{\mathfrak{G}}$: *S leaves invariant the Cartan subalgebra $\tilde{\mathfrak{H}}$ of $\tilde{\mathfrak{G}}$ and induces in $\mathfrak{H}$ a particular rotation ρ with respect to Π_0.* For brevity, we write λ^* for $\rho(\lambda)$ and h^* for $\rho(h)$. We may set

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8) This definition is apparently different but is equivalent with the original one given by Gantmacher [4] p. 132, which may be seen by Th. 26 in [4] p. 136.

9) This formula is given by Satake [8] p. 292, and furnishes a simple proof of Th. 24 in Gantmacher [4] p. 134.

10) See Gantmacher [4] p, 139 Th. 29, [5] p. 222 Th. 3 and p. 229 Th. 8.

(27) $$S e_\alpha = \nu_\alpha e_{\alpha^*}, \qquad \nu_\alpha \in C.$$

Then, corresponding to (20)–(23), the following relations hold.

(28) $$\nu_\alpha \nu_{-\alpha} = 1,$$

(29) $$\nu_{\alpha+\beta} = \nu_\alpha \nu_\beta \frac{N_{\alpha^*,\beta^*}}{N_{\alpha,\beta}}, \qquad \text{if } \alpha+\beta \text{ is a root,}$$

(30) $$\bar{\nu}_\alpha = \nu_{-\alpha},$$

(31) $$|\nu_\alpha| = 1.$$

Moreover, since S is involutive, we have $\alpha^{**} = \alpha$ and

(32) $$\nu_\alpha \nu_{\alpha^*} = 1.$$

Especially, we have $\nu_\alpha = \pm 1$ if $\alpha = \alpha^*$. Therefore, we can devide all roots into the three parts;

(33) $$\begin{aligned} &\sum_1 = \{\alpha;\ \alpha = \alpha^*,\ \nu_\alpha = 1\}, \\ &\sum_2 = \{\beta;\ \beta = \beta^*,\ \nu_\beta = -1\}, \quad \text{and} \\ &\sum_3 = \{\xi;\ \xi \neq \xi^*\}. \end{aligned}$$

We shall indicate henceforward by the letters α, β and ξ the roots in $\sum_1$, $\sum_2$ and $\sum_3$ respectively without explicit notices. (Sometimes, $\alpha, \beta, \cdots$ are yet used to denote general roots with no confusion). Since ρ is a particular rotation with respect to Π_0, it permutes the roots in the fundamental basis $\sum^0$ among themselves, so that we may write these roots in the following way instead of (17).

(34) $$\alpha_1, \cdots, \alpha_p, \beta_1, \cdots, \beta_q, \xi_1, \xi_1^*, \cdots, \xi_r, \xi_r^*.$$ [11]

LEMMA 5. *$\xi - \xi^*$ is not a root.*

PROOF. We represent ξ by (34) in the form

$$\xi = p_1 \alpha_1 + \cdots + p_p \alpha_p + q_1 \beta_1 + \cdots + q_q \beta_q + r_1 \xi_1 + r_2 \xi_1^* + \cdots + r_{2r-1} \xi_r + r_{2r} \xi_r^*.$$

Then,

$$\xi^* = p_1 \alpha_1 + \cdots + p_p \alpha_p + q_1 \beta_1 + \cdots + q_q \beta_q + r_2 \xi_1 + r_1 \xi_1^* + \cdots + r_{2r} \xi_r + r_{2r-1} \xi_r^*,$$

and

$$\xi - \xi^* = (r_1 - r_2)\xi_1 + (r_2 - r_1)\xi_1^* + \cdots + (r_{2r-1} - r_{2r})\xi_r + (r_{2r} - r_{2r-1})\xi_r^*.$$

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11) Hereafter suffices for roots do not indicate their co- or contravariant components.

If $\xi-\xi^*$ is a root, this is the expression (19) for $\xi-\xi^*$. Therefore the coefficients are all $\geqq 0$ or all $\leqq 0$. Then $r_{2k-1}=r_{2k}$ $(1\leqq k\leqq r)$ and $\xi=\xi^*$, which is a contradiction.

LEMMA 6. *If* $(\xi,\xi^*)\neq 0$, *then* $\xi+\xi^*$ *is a root and it belongs to* $\sum_2$.

PROOF. As ρ is a particular rotation, $\xi^*\neq-\xi$. Then by Lemma 3 either $\xi+\xi^*$ or $\xi-\xi^*$ is a root. The latter case is excluded by Lemma 5. The second assertion; $\nu_{\xi+\xi^*}=-1$ is easily verified by (29) and (32).

We shall see later that the converse of this lemma is true (p. 125).

Now we shall investigate the structure of the subalgebra $\mathfrak{G}_1$. For convenience, we call the complex linear subspace in $\tilde{\mathfrak{G}}$ which is spanned over C by a real linear subspace $\mathfrak{G}_R$ the *complexification* of $\mathfrak{G}_R$ and denote by $\tilde{\mathfrak{G}}_R$. Then the complexification $\tilde{\mathfrak{G}}_1$ of $\mathfrak{G}_1$ is the 1-eigenspace of the automorphism S extended over $\tilde{\mathfrak{G}}$. On the other hand, as S is assumed to leave invariant $\tilde{\mathfrak{H}}$ and so $\mathfrak{H}$, we have

$$\tilde{\mathfrak{H}}=\tilde{\mathfrak{H}}_1+\tilde{\mathfrak{H}}_{-1},\qquad \text{where}\qquad \tilde{\mathfrak{H}}_1=\tilde{\mathfrak{G}}_1\cap\tilde{\mathfrak{H}}\qquad \tilde{\mathfrak{H}}_{-1}=\tilde{\mathfrak{G}}_{-1}\cap\tilde{\mathfrak{H}},$$

$$\mathfrak{H}=\mathfrak{H}_1+\mathfrak{H}_{-1},\qquad \text{where}\qquad \mathfrak{H}_1=\tilde{\mathfrak{G}}_1\cap\mathfrak{H}\qquad \mathfrak{H}_{-1}=\tilde{\mathfrak{G}}_{-1}\cap\mathfrak{H}.$$

$\tilde{\mathfrak{H}}_1$ and $\tilde{\mathfrak{H}}_{-1}$ coincide with the complexifications of $\mathfrak{H}_1$ and $\mathfrak{H}_{-1}$ respectively. Now a theorem of Gantmacher[12] asserts that $\tilde{\mathfrak{H}}_1$ contains a regular element of $\tilde{\mathfrak{G}}$. From this theorem we can deduce the following lemma which plays an essential rôle in the sequel.

LEMMA 7. $\sqrt{-1}\,\mathfrak{H}_1$ *contains a regular element* x_0 *of* $\tilde{\mathfrak{G}}$ *and it is a regular element of* $\tilde{\mathfrak{G}}_1$.

In fact, returning to the definition of a regular element, we see easily the first half of this lemma from the fact that the coefficients of Killing's equation of an element x in $\tilde{\mathfrak{G}}$ with respect to the basis (5) are polynomials of the parameters x^i of x with real coefficients. The second half is proved by the fact that $\operatorname{ad} x$ for $x\in\tilde{\mathfrak{G}}_1$ reduces $\tilde{\mathfrak{G}}_1$.

$\mathfrak{G}_1$ is the Lie algebra of the compact group K, or precisely, of K^0. Therefore it is the direct sum of its center $\mathfrak{Z}^0$ and its semi-simple commutator algebra $\mathfrak{G}_1'$, and a Cartan subalgebra of $\mathfrak{G}_1$ is commutative and is the direct sum of $\mathfrak{Z}^0$ and a Cartan subalgebra of $\mathfrak{G}_1'$. Hence, the Cartan subalgebra containing the regular element x_0 which is chosen

12) See Gantmacher [4] p. 132 Th. 23.

in $\sqrt{-1}\mathfrak{H}_1$ by Lemma 7 coincides with $\sqrt{-1}\mathfrak{H}_1$ and is the direct sum of $\sqrt{-1}\mathfrak{Z}$ and $\sqrt{-1}\mathfrak{H}_1'$ where we set $\mathfrak{Z}=\sqrt{-1}\mathfrak{Z}^0$ and $\mathfrak{H}_1'=\tilde{\mathfrak{G}}_1'\cap\mathfrak{H}$ so that $\mathfrak{H}_1=\mathfrak{Z}+\mathfrak{H}_1'$. Then $\tilde{\mathfrak{G}}_1$ is obviously the direct sum of the complexifications $\tilde{\mathfrak{Z}}$ and $\tilde{\mathfrak{G}}_1'$. $\tilde{\mathfrak{G}}_1'$ is semi-simple and coincides with the commutator algebra of $\tilde{\mathfrak{G}}_1$ and $\tilde{\mathfrak{H}}_1'$ is a Cartan subalgebra of $\tilde{\mathfrak{G}}_1'$.

We look for the root system of the complex semi-simple Lie algebra $\tilde{\mathfrak{G}}_1'$ with respect to the Cartan subalgebra $\tilde{\mathfrak{H}}_1'$. As (14) is a basis in $\tilde{\mathfrak{G}}$ and as $\tilde{\mathfrak{G}}_1$ is the set of elements of the form $x+Sx\,(x\in\tilde{\mathfrak{G}})$, we see by (27) and (33) that $\tilde{\mathfrak{G}}_1$ is spanned by the elements of the forms

$$\begin{aligned}&h_\alpha, \qquad e_\alpha,\\&h_\beta,\\&h_\xi+h_{\xi^*},\quad e_\xi+\nu_\xi e_{\xi^*}.\end{aligned}$$

Therefore $\tilde{\mathfrak{G}}_1'$ is spanned by the commutators of any two of these elements. By making use of (15), (28), (29), (30), (32) and Lemma 5, we can calculate these commutators and see that they are linear combinations of the following elements

$$h_\alpha, \qquad h_\xi+h_{\xi^*}, \tag{35}$$

$$e_\alpha, \qquad e_\xi+\nu_\xi e_{\xi^*}. \tag{36}$$

It is clear that the elements in (35) are contained in $\mathfrak{H}_1'$ and span the Cartan subalgebra $\tilde{\mathfrak{H}}_1'$ over C. Moreover, if $h_\lambda\in\mathfrak{H}_1'$, then $\lambda^*=\lambda$ and $(\xi,\lambda)=(\xi^*,\lambda)$, therefore,

$$\begin{aligned}&[h_\lambda, e_\alpha]=(\alpha,\lambda)e_\alpha,\\&[h_\lambda, e_\xi+\nu_\xi e_{\xi^*}]=(\xi,\lambda)(e_\xi+\nu_\xi e_{\xi^*}).\end{aligned} \tag{37}$$

These formulae imply that the elements in (36) are common eigenvectors for ad h where h are the elements in the Cartan subalgebra $\tilde{\mathfrak{H}}_1'$ of $\tilde{\mathfrak{G}}_1'$, and that the roots in contravariant form are of the forms α' or ξ' which are obtained by restricting the linear forms α or ξ respectively onto $\tilde{\mathfrak{H}}_1'$. It is easily seen that $\mathfrak{H}_1'$ coincides with the real part of $\tilde{\mathfrak{H}}_1'$. We set

$$e_{\alpha'}=e_\alpha \qquad e_{\xi'}=e_\xi+\nu_\xi e_{\xi^*}.$$

Then, by (15), (32) and Lemma 5,

$$(38) \qquad \begin{aligned}&[e_{\alpha'}, e_{-\alpha'}]=-h_\alpha,\\&[e_{\xi'}, e_{-\xi'}]=-(h_\xi+h_{\xi*}).\end{aligned}$$

We denote by $\Phi'(\cdot, \cdot)$ the fundamental bilinear form of $\tilde{\mathfrak{G}}_1'$ and compute, as in Weyl [9], the left side of the identity

$$\Phi'\big(h'+e_{\gamma'}, [e_{-\gamma'}, h'+e_{\gamma'}]\big)=0$$

for $h'\in\mathfrak{H}_1'$ and $\gamma'=\alpha'$ or ξ'. Then, setting $-h_{\gamma'}=[e_{\gamma'}, e_{-\gamma'}]$ and $c_{\gamma'}=-\Phi'(e_{\gamma'}, e_{-\gamma'})$, we have

$$\gamma'(h)=\Phi'(h', h_{\gamma'})/c_{\gamma'}.$$

This shows that $c_{\gamma'}$ is real and that the covariant form of the root γ' is given by $h_{\gamma'}/c_{\gamma'}$.

Finally, we apply Lemma 6. Then, if $(\xi,\xi^*)\neq 0$, $\xi+\xi^*=\beta$ and $\xi'=\beta'/2$ since $\xi'=\xi^{*\prime}$.

The results thus obtained are formulated in the following

LEMMA 8. *$\tilde{\mathfrak{H}}_1'$ is a Cartan subalgebra of $\tilde{\mathfrak{G}}_1'$ and $\mathfrak{H}_1'$ coincides with its real part. The root system of $\tilde{\mathfrak{G}}_1'$ with respect to the Cartan subalgebra $\tilde{\mathfrak{H}}_1'$ is as follows.*

$$\begin{array}{lll}\alpha', & h_\alpha/c_{\alpha'}, & \\ \beta'/2, & h_\beta/c_{\xi'}, & \text{where } \beta=\xi+\xi^*,\ (\xi,\xi^*)\neq 0,\\ \xi', & (h_\xi+h_{\xi*})/c_{\xi'}, & \text{where } (\xi,\xi^*)=0.\end{array}$$

In this table, the left and right terms in each row are respectively the contra- and covariant forms of a root, and α, ξ run over $\sum_1$ and $\sum_2$ respectively.

For the later use, we add more one

LEMMA 9. *For $h_\lambda\in\mathfrak{Z}$, $\alpha(h_\lambda)=0$, $(\xi+\xi^*)(h_\lambda)=0$.*

PROOF.
$$\begin{aligned}\alpha(h_\lambda)&=(\alpha,\lambda)\\&=\Phi(h_\alpha, h_\lambda)\\&=\Phi\big(-[e_{\alpha'}, e_{-\alpha'}], h_\lambda\big) \qquad \big(\text{by } (38)\big)\\&=\Phi\big(e_{-\alpha'}, [e_{\alpha'}, h_\lambda]\big)=0.\end{aligned}$$

$$\begin{aligned}(\xi+\xi^*)(h_\lambda)&=(\xi+\xi^*,\lambda)\\&=\Phi(h_{\xi+\xi*}, h_\lambda)\\&=\Phi\big(-[e_{\xi'}, e_{-\xi'}], h_\lambda\big) \qquad \big(\text{by } (38)\big)\\&=\Phi\big(e_{-\xi'}, [e_{\xi'}, h_\lambda]\big)=0.\end{aligned}$$

§ 3. The group G/G^0.

We reserve the notations in the foregoing paragraphs.

LEMMA 10. *Let A be an automorphism (contained) in K. Then A leaves invariant $\mathfrak{H}_1$ if and only if A leaves invariant $\mathfrak{H}$.*

PROOF. If A leaves invariant $\mathfrak{H}$, then by Lemma 1 A does so $\mathfrak{H}_1=\mathfrak{H}\cap\mathfrak{G}_1$. Conversely, let A be an automorphism in K which leaves invariant $\mathfrak{H}_1$. Then, from $\tilde{\mathfrak{H}}\supset\sqrt{-1}\,\mathfrak{H}_1$ it follows that $A\tilde{\mathfrak{H}}$ is a Cartan subalgebra which contains $\sqrt{-1}\,\mathfrak{H}_1$. By Lemma 7, $\sqrt{-1}\,\mathfrak{H}_1$ contains a regular element of $\tilde{\mathfrak{G}}$. Therefore the Cartan subalgebras $A\tilde{\mathfrak{H}}$ and $\tilde{\mathfrak{H}}$ coincide as they are the 0-eigenspace of the adjoint mapping for this common regular element. Hence $A\mathfrak{H}=\mathfrak{H}$.

LEMMA 11. *Let A be an automorphism in K. Then there exists an automorphism U in K^0 such that UA leaves invariant $\mathfrak{H}$.*

PROOF. As we have already seen, $\sqrt{-1}\,\mathfrak{H}_1$ and so $\sqrt{-1}\,A\mathfrak{H}_1$ are Cartan subalgebras in the Lie algebra $\mathfrak{G}_1$ of the compact connected Lie groug K^0. As is well known, the maximal abelian subgroups in K^0 corresponding to $\sqrt{-1}\,\mathfrak{H}_1$ and $\sqrt{-1}\,A\,\mathfrak{H}_1$ are conjugate to one another. This implies easily the existence of U in K^0 such that $UA\,\mathfrak{H}_1=\mathfrak{H}_1$. Then by the previous lemma $UA\,\mathfrak{H}=\mathfrak{H}$, q. e. d.

We shall denote by K^* the subgroup of K composed of all automorphisms which leave invariant $\tilde{\mathfrak{H}}$ and set $K_0^*=K^*\cap K^0$. Then Lemma 11 asserts that

$$K=K^0\,K^*\,.$$

Therefore,

$$K/K^0\cong K^*/K_0^*\,. \tag{39}$$

On the other hand, an automorphism A in K^* leaves invariant $\tilde{\mathfrak{H}}$ and hence induces in $\mathfrak{H}$ a rotation. We prove

LEMMA 12. *If the rotation induced by an automorphism A in K^* is the identical transformation, then this automorphism is in K_0^*.*

PROOF. We prove this lemma in a more precise form: under the assumption of the lemma, the automorphism A belongs to the maximal abelian subgroup of K^0 corresponding to the Cartan subalgebra $\sqrt{-1}\,\mathfrak{H}_1$.

As was shown by Gantmacher,[13] such an automorphism of $\mathfrak{G}_u$

13) See Gantmacher [4] p. 128 Th. 19.

that fixes each element of $\tilde{\mathfrak{H}}$ is of the form $\exp(\mathrm{ad}\ \sqrt{-1}\,h)$ with some h in $\mathfrak{H}$. Therefore the automorphism A in our lemma has this form. Then, for any root α,

$$A\,e_\alpha=\exp\left(\alpha\,(\sqrt{-1}\,h)\right)\,e_\alpha\,. \tag{40}$$

Since A leaves invariant $\tilde{\mathfrak{G}}_1$ which is spanned by the elements (35) and (36),

$$A(e_\xi+\nu_\xi\,e_{\xi^*})=\exp\left(\xi(\sqrt{-1}\,h)\right)\,e_\xi+\nu_\xi\exp\left(\xi^*(\sqrt{-1}\,h)\right)\,e_{\xi^*}$$

must be a scalar multiple of $e_\xi+\nu_\xi\,e_{\xi^*}$. So we have

$$\exp\left(\xi(\sqrt{-1}\,h)\right)=\exp\left(\xi^*(\sqrt{-1}\,h)\right)\,. \tag{41}$$

We put $h_1=(h+h^*)/2$. Then h_1 is certainly contained in $\mathfrak{H}_1$. We show that $\exp\left(\mathrm{ad}\,(\sqrt{-1}\,h_1)\right)=\exp\left(\mathrm{ad}\,(\sqrt{-1}\,h)\right)$, which will prove our assertion. For a root α with $\alpha=\alpha^*$, $\alpha(h)=\alpha^*(h)=\alpha(h^*)$ and $\alpha(h_1)=\alpha(h)$. For a root ξ, we see from (41) that

$$\exp\left(\xi(\sqrt{-1}\,h_1)\right)=\exp\left(\xi(\sqrt{-1}\,h)\right)\,.$$

By (40), we have the desired result, q. e. d.

Now, let $\mathfrak{T}$ be the group of rotations in $\mathfrak{H}$ which are induced by automorphisms in K^* and $\mathfrak{S}$ its subgroup composed of rotations induced by automorphisms in K_0^*. Of course $\mathfrak{S}$ is contained in $\tilde{\mathfrak{S}}$. Lemma 12 implies that

$$K^*/K_0^*\cong\mathfrak{T}/\mathfrak{S}\,.$$

Combining this with Corollary 2 to Theorem 1 and (39), we have arrived at the following

THEOREM 2. $G/G^0\cong\mathfrak{T}/\mathfrak{S}$.

The remainder of this paragraph is devoted to characterize the groups $\mathfrak{T}$ and $\mathfrak{S}$ so as to make Theorem 2 significant.

THEOREM 3. *A rotation τ belongs to $\mathfrak{T}$ if and only if it satisfies the following conditions:*

$$\tau\,\rho=\rho\,\tau\,, \tag{42}$$

$$\tau(\alpha)\in\textstyle\sum_i\,,\quad \textit{for}\quad \alpha\in\sum_i\ (i=1,2,3)\,. \tag{43}$$

Here ρ denotes as in § 2 the particular rotation which is induced by *the involutive automorphism S mentioned in Theorem* 1 *and* $\sum_i$ $(i=1, 2, 3)$ *are defined in* (33)

REMARK. Under the assumption (42) for τ, (43) is equivalent to the following condition:

$$\tau(\alpha)\in\sum_1, \qquad \text{for} \qquad \alpha\in\sum_1, \tag{44}$$

and this is surely satisfied in case $\sum_1$ or $\sum_2$ is empty.

PROOF OF THEOREM 3. As was mentioned in § 2, a rotation τ may be considered to be induced by an automorphism A in G_u. By definition this automorphism belongs to K^*, if it belongs to K. According to Lemma 1 the latter condition is equivalent to

$$AS=SA. \tag{45}$$

When we consider this condition on $\mathfrak{H}$, we see immediately that (42) and (45) are equivalent on $\mathfrak{H}$. Outside of $\mathfrak{H}$, (45) holds if and only if

$$ASe_\alpha=SAe_\alpha$$

for all roots α. The numbers κ_α being as in (19), this relation is expressed by

$$\nu_\alpha\,\kappa_{\alpha^*}\,e_{\tau(\alpha^*)}=\kappa_\alpha\,\nu_{\tau(\alpha)}\,e_{\tau(\alpha)^*}.$$

Under the assumption (42), $\tau(\alpha^*)=\tau(\alpha)^*$ and therefore this relation is again equivalent to

$$\nu_\alpha\,\kappa_{\alpha^*}=\kappa_\alpha\,\nu_{\tau(\alpha)}. \tag{46}$$

Thus, in order that the automorphism A belongs to K^* it is necessary and sufficient that its inducing rotation τ satisfies the conditions (42) and the numbers κ_α for A satisfy the relation (46) for all roots α.

Now suppose that τ belongs to $\mathfrak{T}$. Then we may assume that A belongs to K^*. From the above considerations, we see that the property (42) is true for τ. Further, (46) implies $\nu_\alpha=\nu_{\tau(\alpha)}$ for any root α such that $\alpha=\alpha^*$, and this shows the condition (43) for τ.

To prove the converse, we first show that under the assumption (42) (46) is valid for all roots if it is valid for roots in the fundamental basis of the root system. In virtue of Lemma 4, this will be seen when we prove that if (46) is valid for a root α then so is for $-\alpha$, and that if (46) is valid for two roots α, β whose sum is again a root then so is for $\alpha+\beta$. We prove these assertions. If (46) is valid for a root α, then applying (20) and (28) we have

$$\nu_{-\alpha}\kappa_{-\alpha^*}=\frac{1}{\nu_\alpha\kappa_{\alpha^*}}=\frac{1}{\kappa_\alpha\nu_{\tau(\alpha)}}=\kappa_{-\alpha}\nu_{\tau(-\alpha)},$$

which is the formula (46) for $-\alpha$. For the second assertion, using (21) and (29) repeatedly, we have under our assumptions

$$\begin{aligned}\nu_{\alpha+\beta}\kappa_{(\alpha+\beta)^*}&=\nu_\alpha\nu_\beta\frac{N_{\alpha^*,\beta^*}}{N_{\alpha,\beta}}\kappa_{\alpha^*}\kappa_{\beta^*}\frac{N_{\tau(\alpha^*),\tau(\beta^*)}}{N_{\alpha^*,\beta^*}}\\&=\kappa_\alpha\nu_{\tau(\alpha)}\kappa_\beta\nu_{\tau(\beta)}\frac{N_{\tau(\alpha^*),\tau(\beta^*)}}{N_{\alpha,\beta}}\\&=\kappa_{\alpha+\beta}\nu_{\tau(\alpha)+\tau(\beta)}\frac{N_{\alpha,\beta}}{N_{\tau(\alpha),\tau(\beta)}}\frac{N_{\tau(\alpha),\tau(\beta)}}{N_{\tau(\alpha)^*,\tau(\beta)^*}}\frac{N_{\tau(\alpha)^*,\tau(\beta)^*}}{N_{\alpha,\beta}}\\&=\kappa_{\alpha+\beta}\nu_{\tau(\alpha+\beta)},\end{aligned}$$

which is the formula (46) for $\alpha+\beta$.

Now suppose that τ is a rotation with the properties (42) and (43). Let A be an arbitrary automorphism in G_u which induces τ in $\mathfrak{H}$. On account of (23), (31) and the linear independence of the roots in the fundamental basis (34), we can find an element h of $\mathfrak{H}$ satisfying the following equations.

$$\begin{aligned}&\exp\big(\sqrt{-1}\,\alpha_1(h)\big)=\cdots=\exp\big(\sqrt{-1}\,\alpha_p(h)\big)=\\&=\exp\big(\sqrt{-1}\,\beta_1(h)\big)=\cdots=\exp\big(\sqrt{-1}\,\beta_q(h)\big)=\\&=\exp\big(\sqrt{-1}\,\xi_1(h)\big)=\cdots=\exp\big(\sqrt{-1}\,\xi_r(h)\big)=1,\\&\exp\big(\sqrt{-1}\,\xi_k^*(h)\big)=\kappa_{\xi_k}\nu_{\tau(\xi_k)}/\nu_{\xi_k}\kappa_{\xi_k^*}\qquad(1\leqq k\leqq r).\end{aligned}$$

Consider the automorphism

$$A'=A\exp\big(\mathrm{ad}\,(\sqrt{-1}\,h)\big).$$

Then A' leaves invariant $\mathfrak{H}$ and induces the same rotation τ. Moreover, if we set

$$A'e_\alpha=\kappa'_\alpha e_{\tau(\alpha)},$$

we see from the above choice of h and from the definition of A' that the numbers κ'_α are related to κ_α for roots α in the fundamental basis in the following way.

$$\kappa'_{\alpha_i}=\kappa_{\alpha_i} \qquad (1\leqq i\leqq p),$$

$$\kappa'_{\beta_j}=\kappa_{\beta_j} \qquad (1\leqq j\leqq q),$$

$$\kappa'_{\xi_k}=\kappa_{\xi_k},$$

$$\nu_{\xi_k}\kappa'_{\xi_k^*}=\kappa_{\xi_k}\nu_{\tau(\xi_k)} \qquad (1\leqq k\leqq r).$$

From these equations, we shall show that the relation

$$(46)' \qquad \nu_\alpha\kappa'_{\alpha^*}=\kappa'_\alpha\nu_{\tau(\alpha)},$$

which is the condition (46) for the automorphism A', is satisfied by all roots α. By what is mentioned above it is sufficient to see this that (46)′ is valid for the roots α in the fundamental basis. If $\alpha=\alpha_i$, $(1\leqq i\leqq p)$ or $\alpha=\beta_j$ $(1\leqq j\leqq q)$, (46)′ is clearly true by the assumption (43). For $\alpha=\xi_k$ $(1\leqq k\leqq r)$ (46)′ is derived from the latter two of the above equations, and, taking the reciprocals of the both sides of this relation and applying (32) and (42), we find the relation (46)′ for $\alpha=\xi_k^*$ $(1\leqq k\leqq r)$. Thus, as was seen in the beginning of this proof, A' is an automorphism in K^* and so τ is a rotation in $\mathfrak{T}$. This completes the proof of Theorem 3.

Next we consider the subgroup $\mathfrak{S}$ which is composed of inner rotations of $\mathfrak{H}$ induced by automorphisms in K_0^*. By Lemma 10 such an automorphism reduces $\mathfrak{H}_1$ and $\mathfrak{H}_1'=\tilde{\mathfrak{G}}_1'\cap\mathfrak{H}$. Therefore its inducing rotation leaves invariant the subspace $\mathfrak{H}_1'$ of $\mathfrak{H}$.

LEMMA 13. *A rotation of $\mathfrak{H}$ contained in $\mathfrak{S}$ induces an inner rotation in $\mathfrak{H}_1'$, and conversely an inner rotation of $\mathfrak{H}_1'$ is induced by one and only one rotation of $\mathfrak{H}$ belonging to $\mathfrak{S}$. Here and henceforward $\mathfrak{H}_1'$ is regarded as the real part of the Cartan subalgebra $\tilde{\mathfrak{H}}_1'$ of $\tilde{\mathfrak{G}}_1'$ (by Lemma* 8) *and an inner rotation in it is also considered with respect to $\tilde{\mathfrak{G}}_1'$.*

PROOF. An element U of K^0 is by Lemma 1 an automorphism of $\mathfrak{G}_u$ which leaves invariant $\mathfrak{G}_1$, therefore it induces an automorphism U_1 of $\mathfrak{G}_1$. While, K^0 is the group with the Lie algebra $\mathfrak{G}_1$. It is easily seen that this automorphism U_1 coincides with the automorphism of $\mathfrak{G}_1$ which is induced from the inner automorphism of the group K^0 raised up by the element U of K^0. On the other hand, as was seen in § 2 $\mathfrak{G}_1$ is the direct product of its center $\mathfrak{Z}^0$ and the semi-simple commutator algebra $\mathfrak{G}_1'$. These subalgebras generate in the group K^0 closed subgroups Z^0 and $K^{0'}$ respectively. Z^0 is a central subgroup and

$$K^0 = Z^0 K^{0'} .$$

From these considerations we see first that every automorphism in K^0 leaves fixed each element of $\mathfrak{Z}^0$ and leaves invariant $\mathfrak{G}_1'$, and secondly that the automorphism U_1 for an automorphism U in K^0 may be seen in $\mathfrak{G}_1'$ to be induced by an inner automorphism of the group $K^{0'}$.

Now, suppose that σ is a rotation in $\mathfrak{H}$ which belongs to $\mathfrak{S}$ and is induced by an automorphism U in K. By what is mentioned above, as far as we concern with the behavior of U_1 in $\mathfrak{G}_1'$ we may assume that U is an automorphism in $K^{0'}$. Hence the restriction σ' of σ in $\mathfrak{H}_1'$ may be considered to be induced by an inner automorphism of the group $K^{0'}$. Therefore, σ' is an inner rotation of $\mathfrak{H}_1'$.

To prove the converse, let us remark that $\mathfrak{G}_1'$ is a unitary restriction of $\tilde{\mathfrak{G}}_1'$ with respect to (a canonical basis related to) the Cartan subalgebra $\tilde{\mathfrak{H}}_1'$ of $\tilde{\mathfrak{G}}_1'$. In fact, since $\mathfrak{G}_1'$ is a real form of $\tilde{\mathfrak{G}}_1'$ namely a real subalgebra whose complex form is $\tilde{\mathfrak{G}}_1'$ and since it is the Lie algebra of the compact group $K^{0'}$, $\mathfrak{G}_1'$ is a unitary restriction with respect to a Cartan subalgebra of $\tilde{\mathfrak{G}}_1'$.[14] By the conjugateness of maximal abelian subgroups in $K^{0'}$, we may easily see that this Cartan subalgebra is mapped onto $\tilde{\mathfrak{H}}_1'$ by an inner automorphism of $\mathfrak{G}_1'$, which implies the assertion. Let σ' be an inner rotation of $\mathfrak{H}_1'$. Then as was mentioned in § 2, we may assume that σ' is induced by an inner automorphism of the unitary restriction $\mathfrak{G}_1'$. Therefore we find an element U in the group $K^{0'}$ such that the inner automorphism of the group $K^{0'}$ raised up by this element U induces in the Lie algebra $\tilde{\mathfrak{G}}_1'$ an automorphism which leaves invariant $\tilde{\mathfrak{H}}_1'$ and induces in $\mathfrak{H}_1'$ the rotation σ'. Considering U as an automorphism of $\tilde{\mathfrak{G}}$ belonging to K^0, we see from $\mathfrak{H}_1 = \mathfrak{H}_1' + \mathfrak{Z}$ that U leaves invariant $\mathfrak{H}_1$ and so $\mathfrak{H}$ by Lemma 10: U belongs to K^*. It is then clear that σ' is the restriction to $\mathfrak{H}_1'$ of the inner rotation of $\mathfrak{H}$ belonging to $\mathfrak{S}$ induced by this automorphism U.

Finally, we show that a rotation σ in $\mathfrak{S}$ is uniquely determined by its restriction σ' into $\mathfrak{H}_1'$. To see this it is sufficient to prove that if

14) This is seen from Gantmacher [5] p. 226 Th. 5.

σ' is the identical transformation then so is σ. Suppose σ is induced by an automorphism U belonging to K_0^*. The assumption on σ' implies that U leaves fixed any element of the Cartan subalgebra $\sqrt{-1}\,\mathfrak{H}_1$ of $\mathfrak{G}_1$ and this in turn implies that U is commutative with the element of the maximal abelian subgroup in K^0 corresponding to $\sqrt{-1}\,\mathfrak{H}_1$. Then U belongs to this subgroup and therefore is of the form $\exp(\mathrm{ad}\sqrt{-1}\,h)$ with a suitable element h of $\mathfrak{H}_1$. Hence U fixes any element of $\mathfrak{H}$ and so σ is the identical transformation.

LEMMA 14. *Let σ be an inner rotation of $\mathfrak{H}$. Suppose that σ leaves invariant the subspace $\mathfrak{H}_1$ in which σ leaves fixed every element of $\mathfrak{Z}$ and induces in $\mathfrak{H}_1'$ an inner rotation σ'. Then σ belongs to $\mathfrak{S}$.*

PROOF. By the previous lemma, we can find an element U in K_0^* which induces in $\mathfrak{H}_1'$ the inner rotation σ'. Let τ be the inner rotation of $\mathfrak{H}$ induced by this automorphism U. We show that $\tau=\sigma$ which proves our lemma. Since σ is an inner rotation, we can find an element V of G_u^0 (the connected component of the identity in G_u) so that V leaves invariant $\tilde{\mathfrak{H}}$ and induces in $\mathfrak{H}$ the rotation σ. The element $U^{-1}V$ is obviously contained in G_u^0, leaves invariant $\mathfrak{H}$ and leaves fixed each element of $\mathfrak{H}_1$. Now, by Lemma 7, $\sqrt{-1}\,\mathfrak{H}_1$ contains a regular element x_0 of $\tilde{\mathfrak{G}}$ and naturally of $\mathfrak{G}_u$. For a sufficiently small $\varepsilon>0$, $\exp \mathrm{ad}\,\varepsilon x_0$ is a regular element of the compact group G_u^0, that is, an element whose centralizer just coincides with the maximal abelian subgroup T corresponding to the Cartan subalgebra $\sqrt{-1}\,\mathfrak{H}$ in the Lie algebra $\mathfrak{G}_u$. It follows from the fact that $U^{-1}V$ leaves fixed the element x_0 in $\mathfrak{G}_u$, $U^{-1}V$ commutes with the regular element $\exp \mathrm{ad}\,\varepsilon x_0$. Therefore $U^{-1}V$ is contained in T and hence U and V induce in $\mathfrak{H}$ the same rotation: $\tau=\sigma$, q. e. d.

We now proceed to characterize the group $\mathfrak{S}$. First we determine the inner rotations of $\mathfrak{H}_1'$. The group of all inner rotations in $\mathfrak{H}_1'$ is generated by reflections of $\mathfrak{H}_1'$ with respect to the hyperplanes through the origin and orthogonal to each root. By Lemma 8, these reflections are of the following forms: for $h'\in\mathfrak{H}_1'$

$$\sigma'_{\alpha'}(h')=h'-\frac{2\,\alpha'(h')}{\alpha'(h_\alpha/c_{\alpha'})}\,\frac{h_\alpha}{c_{\alpha'}} \tag{47}$$

$$=h'-\frac{2\,\alpha(h')}{\alpha(h_\alpha)}\,h_\alpha\,,$$

$$(48)\qquad \sigma'_{\beta'}(h')=h'-\frac{2\,\beta'(h')}{\beta'(h_\beta/c_{\xi'})}\frac{h_\beta}{c_{\xi'}}$$
$$=h'-\frac{2\beta(h')}{\beta(h_\beta)}h_\beta,\qquad \text{where } \beta=\xi+\xi^*,\ (\xi,\xi^*)\neq 0\,.$$

$$(49)\qquad \sigma'_{\xi'}(h')=h'-\frac{2\,\xi'(h')}{\xi'(h_\xi+h_{\xi*}/c_{\xi'})}\frac{h_\xi+h_{\xi*}}{c_{\xi'}}$$
$$=h'-\frac{2\,\xi(h')}{\xi(h_\xi)}(h_\xi+h_{\xi*}),\qquad \text{where } (\xi,\xi^*)=0\,.$$

By Lemma 13, each of these rotations in $\mathfrak{H}_1'$ is induced by one and only one rotation of $\mathfrak{H}$ belonging to $\mathfrak{S}$. Therefore, whenever we determine the rotations of $\mathfrak{H}$ belonging to $\mathfrak{S}$ and inducing the inner rotations (47), (48) and (49) in $\mathfrak{H}_1'$, we see that these rotations of $\mathfrak{H}$ are the generators of the group $\mathfrak{S}$. For this determination we apply Lemma 14. Then, we have only to research the inner rotations of $\mathfrak{H}$ which reduce $\mathfrak{H}_1$, fix each element of $\mathfrak{Z}$ and induce the rotations (47), (48) and (49) in $\mathfrak{H}_1'$. This is performed as follows.

ad (47): We consider the inner rotation of $\mathfrak{H}$ which is the reflection with respect to the hyperplane E_α:

$$(50)\qquad \sigma_\alpha(h)=h-\frac{2\,\alpha(h)}{\alpha(h_\alpha)}h_\alpha\,.$$

σ_α reduces $\mathfrak{H}_1$. For, if $h=h^*$,

$$\big(\sigma_\alpha(h)\big)^*=h^*-\frac{2\,\alpha(h)}{\alpha(h_\alpha)}h_\alpha^*$$
$$=h-\frac{2\,\alpha(h)}{\alpha(h_\alpha)}h_\alpha=\sigma_\alpha(h)\,.$$

σ_α leaves fixed each element of $\mathfrak{Z}$, as is seen from Lemma 9. Obviously σ_α coincides with $\sigma'_{\alpha'}$ in $\mathfrak{H}_1'$. Therefore it leaves invariant $\mathfrak{H}_1'$. Thus σ_α is the uniquely determined rotation in $\mathfrak{S}$ which induces in $\mathfrak{H}_1'$ the rotation $\sigma'_{\alpha'}$ in (47).

ad (48): We see just as for $\sigma'_{\alpha'}$ that the reflection with respect to the hyperplane E_β:

$$(51)\qquad \beta(h)=h-\frac{2\beta(h)}{\beta(h_\beta)}h_\beta$$

is the uniquely determined rotation in $\mathfrak{S}$ which induces in $\mathfrak{H}_1'$ the rotation $\sigma'_{\beta'}$ in (48).

ad (49): In this case, we consider two reflections:

$$\sigma_\xi(h)=h-\frac{2\,\xi(h)}{\xi(h_\xi)}\,h_\xi\,,$$

$$\sigma_{\xi*}(h)=h-\frac{2\,\xi^*(h)}{\xi^*(h_{\xi*})}\,h_{\xi*}\,.$$

The product of these rotations is, as $\xi^*(h_\xi)=0$ by assumption,

$$\sigma_{\xi*}\,\sigma_\xi(h)=h-\frac{2\,\xi(h)}{\xi(h_\xi)}\,h_\xi-\frac{2\,\xi^*(h)}{\xi^*(h_{\xi*})}\,h_{\xi*}\,. \tag{52}$$

Of course $\xi(h_\xi)=\xi^*(h_{\xi*})$. For an element h in $\mathfrak{H}_1$, $\xi(h)=\xi(h^*)=\xi^*(h)$ and (52) is written in the form

$$\sigma_{\xi*}\,\sigma_\xi(h)=h-\frac{2\,\xi(h)}{\xi(h_\xi)}\,(h_\xi+h_{\xi*})\,.$$

This implies first that $\sigma_{\xi*}\,\sigma_\xi$ leaves invariant $\mathfrak{H}_1$ and coincides with $\sigma'_{\xi'}$ in $\mathfrak{H}'_1$ and secondly that $\sigma_{\xi*}\,\sigma_\xi$ leaves fixed each element of $\mathfrak{Z}$ by Lemma 9 and by $2\,\xi(h)=(\xi^*+\xi)\,(h)$. Thus we see that (52) is the uniquely determined rotation in $\mathfrak{S}$ which induces the rotation $\sigma'_{\xi'}$ in $\mathfrak{H}'_1$.

Finally we prove the converse of Lemma 6, i. e., that $\xi+\xi^*$ is not a root if $(\xi,\xi^*)=0$. In proving that the rotation of $\mathfrak{H}$ in $\mathfrak{S}$ which induces the inner rotation $\sigma'_{\xi'}$ in $\mathfrak{H}'_1$ is $\sigma_{\xi*}\,\sigma_\xi$ we use the property $(\xi,\xi^*)=0$ only. Now suppose that $\xi+\xi^*$ is a root β. Then we see as for the second case in above that σ_β is also a rotation in $\mathfrak{S}$ which induces $\sigma'_{\xi'}$ in $\mathfrak{H}'_1$. By Lemma 13, we have $\sigma_\gamma=\sigma_{\xi*}\,\sigma_\xi$, which contradicts to the fact that σ_γ and $\sigma_{\xi*}\,\sigma_\xi$ are linear transformations in $\mathfrak{H}$ of determinants -1 and 1 respectively.

We have thus obtained the following

THEOREM 4. *The group $\mathfrak{S}$ is generated by the inner rotations*

$$\sigma_\alpha,$$
$$\sigma_\beta,\quad \textit{where}\quad \beta=\xi+\xi^*\quad \textit{and}\quad (\xi,\xi^*)\neq 0\,,$$
$$\sigma_{\xi*}\,\sigma_\xi,\quad \textit{where}\quad (\xi,\xi^*)=0\,.$$

Here α,β and ξ are roots classified by (33), $\xi^*=\rho(\xi)$ *and* σ_γ $(\gamma=\alpha,\beta$ or $\xi)$ *is the reflection with respect to the hyperplane E_γ through the origin and orthogonal to the root γ.*

Combining Theorems 2, 3 and 4, our purpose to determine the group G/G^0 is completely accomplished.

§ 4. Applications.

We shall apply our results obtained in § 3 to the actual determination of the groups G/G^0 for the real forms $\mathfrak{G}$ of the complex simple Lie algebras $\tilde{\mathfrak{G}}$ of type A_n.[15)]

By definition $\tilde{\mathfrak{G}}$ is the Lie algebra of the complex unimodular group $SL(n+1, C)$ of a certain degree $n+1$ and is composed of complex matrices of degree $n+1$ with zero trace. Let e_{ij} $(1 \leqq i, j \leqq n+1)$ be the matrix $(\delta_{i,k}\delta_{j,l})_{k,l=1}^{n+1}$ and set $h_i = e_{ii}$ $(1 \leqq i \leqq n+1)$. Then

$$h_1 - h_{n+1}, \cdots, h_n - h_{n+1}, e_{12}, e_{21}, \cdots, e_{ij}, e_{ji}, \cdots \qquad (1 \leqq i < j \leqq n+1). \tag{55}$$

form a basis of $\tilde{\mathfrak{G}}$ which has the properties of a canonical basis (14) defined in § 2 except the property $\Phi(e_\alpha, e_{-\alpha}) = -1$. As will be easily verified, the loss of the last property gives no affections in the following considerations. We may regard the Cartan subalgebra $\tilde{\mathfrak{H}}$ as in the $(n+1)$-dimensional complex vector space $\tilde{E}_{n+1}$ spanned by $h_1, \cdots, h_{n+1}$. Then the root with the eigenvector e_{ij} $(1 \leqq i, j \leqq n+1)$ has the following contravariant form.

$$\alpha_{ij}(h) = \lambda^i - \lambda^j \qquad \text{for} \tag{54}$$

$$h = \lambda^1 h_1 + \cdots + \lambda^{n+1} h_{n+1}, \quad \lambda^1 + \cdots + \lambda^{n+1} = 0.$$

$\sum = \{\alpha_{ij};\ 1 \leqq i, j \leqq n+1\}$ is the root system for $\tilde{\mathfrak{G}}$. As is calculated in Satake [8], a fundamental basis of $\sum$ is given by $\sum^0 = \{\alpha_{i\,i+1};\ 1 \leqq i \leqq n\}$.

On account of (54) the real part $\mathfrak{H}$ of the Cartan subalgebra $\tilde{\mathfrak{H}}$ may be identified with the linear subspace orthogonal to $h_1 + \cdots + h_{n+1}$ in the real euclidean space E_{n+1} spanned by $h_1, \cdots, h_{n+1}$. We shall indicate by an element of E_{n+1} the element of $\mathfrak{H}$ congruent to this element modulo the 1-dimensional subspace spanned by $h_1 + \cdots + h_{n+1}$, and by a linear transformation in E_{n+1} leaving invariant $\mathfrak{H}$ its inducing

15) For the different approaches to the same results, see Cartan [2] pp. 385-400 and Jacobson [6].

linear transformation in $\mathfrak{H}$. Then an inner rotation is realized by the linear transformation in E_{n+1};

$$\sigma_\pi(h_i)=h_{\pi(i)} \qquad (1\leqq i\leqq n+1),$$

where π is a permutation of the letters $1,\cdots,n+1$, and $\pi\rightarrow\sigma_\pi$ gives an isomorphism between the group $\tilde{\mathfrak{S}}$ and the symmetric group of degree $n+1$. It is clear that

(55) $$\sigma_\pi(\alpha_{ij})=\alpha_{\pi(i)\ \pi(j)} \qquad (1\leqq i,\ j\leqq n+1).$$

Let ρ_0 be the rotation defined by

$$\rho_0(h_i)=-h_{n+2-i} \qquad (1\leqq i\leqq n+1).$$

Then ρ_0 is the only one non-trivial particular rotation transforming the roots in the fundamental basis $\sum^0$ among themselves. From (25) in §2 we see that the group $\tilde{\mathfrak{T}}$ of all rotations is given by

(56) $$\tilde{\mathfrak{T}}=\tilde{\mathfrak{S}}+\rho_0\tilde{\mathfrak{S}}.$$

We define more one rotation τ_0 by the formula

$$\tau_0(h_i)=-h_i \qquad (1\leqq i\leqq n+1).$$

Then

(57) $$\tau_0(\alpha_{ij})=\alpha_{ji} \qquad (1\leqq i,\ j\leqq n+1),$$

and (56) is also written as

(58) $$\tilde{\mathfrak{T}}=\tilde{\mathfrak{S}}+\tau_0\tilde{\mathfrak{S}}.$$

Now, as was introduced in §§1 and 2, a real form $\mathfrak{G}$ of $\tilde{\mathfrak{G}}$ is obtained by making use of an involutive automorphism S of the unitary restriction $\mathfrak{G}_u$ of $\tilde{\mathfrak{G}}$ with respect to the basis (53). Following Gantmacher[16], we may assume moreover that S is of the form

(59) $$S=S_0\exp(\mathrm{ad}\ \pi\sqrt{-1}\,h_0),^{17)}$$

where S_0 is an automorphism of $\mathfrak{G}_u$ uniquely determined by an involutive particular rotation ρ in the following way and h_0 is an element from $\mathfrak{H}$ such that $\rho(h_0)=h_0$. S_0 leaves invariant $\tilde{\mathfrak{H}}$ and induces in $\mathfrak{H}$ the particular rotation ρ, and if we set

16) See footnote 10).

17) In the formulae (59) and (61) π denotes the usual ratio for a circle.

(60) $$S_0 e_\alpha = \mu_\alpha e_{\alpha^*}, \qquad \alpha^* = \rho(\alpha),$$

the numbers μ_α are equal to 1 for roots α in the fundamental basis $\sum^0$ and are determined for other roots α by the relations

$$\mu_\alpha \mu_{-\alpha} = 1,$$
$$\mu_{\alpha+\beta} = \frac{N_{\alpha^*, \beta^*}}{N_{\alpha, \beta}} \mu_\alpha \mu_\beta,$$

by making use of Lemma 4. From (59) and (60), it follows that the numbers ν_α defined in (27): $Se_\alpha = \nu_\alpha e_{\alpha^*}$ are given by

(61) $$\nu_\alpha = \mu_\alpha \exp\left(\pi\sqrt{-1}\,\alpha(h_0)\right).$$

We shall write μ_{ij} and ν_{ij} for the numbers μ_α and ν_α respectively with $\alpha = \alpha_{ij}$.

Thus we may say that a real form $\mathfrak{G}$ is determined by an involutive particular rotation ρ and an element h_0 with the property $\rho(h_0) = h_0$ in $\mathfrak{H}$. Now, by just the same argument as in Gantmacher [5][18] we may see that a real form $\mathfrak{G}$ of $\tilde{\mathfrak{G}}$ is, up to isomorphism, one of the following types (a), (b), (c) and (d), and to each case we shall determine the group G/G^0. According to Theorem 2 this determination is reduced to that of the groups $\mathfrak{T}$ and $\mathfrak{S}$, which is performed by means of Theorems 3 and 4. In the following, we set

$$\mathfrak{T}_1 = \mathfrak{T} \cap \tilde{\mathfrak{S}}.$$

(a) $\rho =$ the identical transformation, $h = h_{l+1} + \cdots + h_{n+1}$, $1 \leq l \leq \frac{n+1}{2}$.

$\mathfrak{G}$ is the Lie algebra of the group G^l_{n+1} of linear transformations in $n+1$ complex variables leaving invariant the hermitian form

$$x_1\bar{x}_1 + \cdots + x_l\bar{x}_l - x_{l+1}\bar{x}_{l+1} - \cdots - x_{n+1}\bar{x}_{n+1}.$$

In this case, obviously $\mu_\alpha = 1$ for all roots α and by (61)

$$\nu_{ij} = \begin{cases} 1, & \text{for } 1 \leq i, j \leq l \text{ or } l+1 \leq i, j \leq n+1, \\ -1, & \text{otherwise.} \end{cases}$$

Therefore

$\sum_1 = \{\alpha_{ij};\ 1 \leq i, j \leq l \text{ or } l+1 \leq i, j \leq n+1\}$, and
$\sum_2$ consists of all roots not in $\sum_1$.

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18) See Gantmacher [5] § 7 and § 17.

In order to apply Theorem 3 for the determination of the group $\mathfrak{T}$ we need only to research what rotations τ satisfy the condition (43), or its simpler form (44), since the condition (42) is trivial for our case. The rotation τ_0 belongs to $\mathfrak{T}$ by (57) and it follows from (58) that

$$\mathfrak{T}=\mathfrak{T}_1+\tau_0\,\mathfrak{T}_1\,. \tag{62}$$

As regards $\mathfrak{T}_1$, we see from (55) that an inner rotation σ_π belongs to $\mathfrak{T}_1$, if and only if the permutation π maps the subset $\{1, 2,\cdots, l\}$ onto itself or onto the subset $\{l+1,\cdots, n+1\}$. Of course the latter case occurs only when n is an odd integer $2f-1$ and $l=f$. On the other hand, by Theorem 4 $\mathfrak{S}$ is the group generated by the reflections $\sigma_\alpha(\alpha \in \sum_1)$. The permutations induced by these reflections are transpositions (i,j) where $1\leqq i,\ j\leqq l$ or $l+1\leqq i,\ j\leqq n+1$. Therefore $\mathfrak{S}$ consists of all rotations σ_π such that the permutations π map the subset $\{1,\cdots, l\}$ onto itself. Hence $\mathfrak{S}$ coincide with $\mathfrak{T}_1$ except when $n=2f-1$ and $l=f$. In this exceptional case $\mathfrak{S}$ is a subgroup of index 2 in $\mathfrak{T}_1$ and for example the inner rotation σ_{π_0} where $\pi_0=(1,f+1)(2,f+2)\cdots(f, n+1)$ does not belong to $\mathfrak{S}$ but to $\mathfrak{T}_1$. Combining (62), we have

$$\mathfrak{T}=\begin{cases}\mathfrak{S}+\tau_0\,\mathfrak{S}, & \text{if}\quad l\neq\dfrac{n+1}{2}\\ \mathfrak{S}+\sigma_{\pi_0}\mathfrak{S}+\tau_0\,\mathfrak{S}+\sigma_{\pi_0}\tau_0\,\mathfrak{S}, & \text{if}\quad n=2f-1 \text{ and } l=f.\end{cases} \tag{63}$$

Thus we conclude by Theorem 2 that the group G/G^0 *is an abelian group of order* 2 *or* 4 *according to each case in* (63). By the way, we note that τ_0 and σ_{π_0} are induced in a natural way from the following automorphisms A_* and A_0 respectively of the group $SL(n+1, C)$: For $(x_{ij}) \in SL(n+1, C)$,

$$A_*(x_{ij})=(x_{ji})^{-1}\,,$$

$$A_0(x_{ij})=(u_{ij})^{-1}(x_{ij})(u_{ij})\,,$$

where $u_{ij}=\delta_{i,\,j-f}$ if $1\leqq i\leqq f$ and $u_{ij}=\delta_{i,\,j+f}$ if $f+1\leqq i\leqq n+1$. Therefore, the *typical outer automorphisms* of $\mathfrak{G}$, namely those which are representatives of the cosets of G modulo G^0 except of the coset G^0, are the automorphisms corresponding to the automorphisms of the group G^l_{n+1} which are induced in this subgroup from A_* or from A_0, A_* and A_0A_* according to each case in (63).

For the following cases, we first compute the numbers μ_α in (60) for the particular rotation ρ_0 and obtain

$$\mu_{ij}=(-1)^{i+j+1} \qquad (1 \leqq i,\ j \leqq n+1). \tag{64}$$

For brevity we shall indicate by i^* the integer related to an integer $i\,(1 \leqq i \leqq n+1)$ by the equation $i+i^*=n+2$. ρ_0 is then given by

$$\rho_0(h_i)=-h_{i*} \qquad (1 \leqq i \leqq n+1),$$

and so

$$\rho_0(\alpha_{ij})=\alpha_{j*i*} \qquad (1 \leqq i,\ j \leqq n+1). \tag{65}$$

(b) $n=2f+1$; odd integer, $\rho=\rho_0$, $h=h_{f+1}+\cdots+h_{n+1}$. *$\mathfrak{G}$ is isomorphic to the Lie algebra of the real unimodular group $SL(n+1, R)$ of degree $n+1$.*

From (61) and (64) we find in this case

$$\nu_{i,j}=\begin{cases}(-1)^{i+j+1}, & \text{if } 1 \leqq i,\ j \leqq f \text{ or } f+1 \leqq i,\ j \leqq n+1, \\ (-1)^{i+j}, & \text{otherwise.}\end{cases}$$

Therefore, together with (65), we see that

$\sum_1$ is empty,

$\sum_2=\{\alpha_{ii*};\ 1 \leqq i \leqq n+1\}$, and

$\sum_3=\{\alpha_{ij};\ 1 \leqq i,\ j \leqq n+1,\ i \neq j^*\}$.

$\sum_3$ is empty for the case $n=1$.

We determine the group $\mathfrak{T}$ by Theorem 3. Using Remark after this theorem we see that $\mathfrak{T}$ consists of rotations τ satisfying the condition (42): $\tau\rho_0=\rho_0\tau$. Clearly τ_0 satisfies this condition and hence (62) is also valid for this case. For an inner rotation σ_π, $\sigma_\pi\rho_0(h_i)=-h_{\pi(i*)}$ and $\rho_0\sigma_\pi(h_i)=-h_{\pi(i)*}$ $(1 \leqq i \leqq n+1)$ and therefore σ_π belongs to $\mathfrak{T}_1$ if and only if

$$\pi(i^*)=\pi(i)^* \qquad \text{for} \quad i=1,\cdots,n+1.$$

Let X be the group of permutations which satisfy this condition. Then $\mathfrak{T}_1$ is isomorphic to X. Representing a permutation from X as a product of cyclic permutations, we can see easily that X is generated by the permutations of the following forms.

$$(i, i^*),\ (i,j)\,(i^*,j^*)\,(1 \leqq i,\ j \leqq n+1, \text{ and } i \neq j^*). \tag{66}$$

If $n=1$ the permutations of the second type do not appear in X and $X=(1,1^*)$. On the other hand, suppose $n\geqq 2$ and let $\xi=\alpha_{ij}$ $(i\neq j^*)$ be a root in $\sum_3$. Then $(\xi,\xi^*)=(\alpha_{ij},\alpha_{j^*i^*})=0$. According to Theorem 4 $\mathfrak{S}$ is now generated by the inner rotations $\sigma_{\xi^*}\sigma_{\xi}$ $(\xi\in\sum_3)$. When we set $\sigma_{\xi^*}\sigma_{\xi}=\sigma_{\pi_\xi}$, then we see that

$$\pi_\xi=(i,j)(i^*,j^*)\,.$$

Denote by Y the group of permutations generated by these permutations π_ξ $(\xi\in\sum_3)$, and in case $n=1$ set $Y=\{1\}$. Then, as is easily verified, Y is a normal subgroup of X and is isomorphic to $\mathfrak{S}$. Moreover, X/Y is of order 2. Indeed, from the relations

$$\{(i,j)(i^*,j^*)\}^{-1}(i,i^*)\{(i,j)(i^*,j^*)\}=(j,j^*)\quad(1\leqq i,\ j\leqq n+1)\,,$$

it follows that (i,i^*), $1\leqq i\leqq n+1$, are conjugated to each other by elements of Y and consequently, putting $\pi_1=(1,1^*)$,

$$X=Y+\pi_1 Y\,.$$

Therefore,

$$\mathfrak{T}_1=\mathfrak{S}+\sigma_{\pi_1}\mathfrak{S}\,.$$

Combining this with (62), we find

$$\mathfrak{T}=\mathfrak{S}+\sigma_{\pi_1}\mathfrak{S}+\tau_0\mathfrak{S}+\sigma_{\pi_1}\tau_0\mathfrak{S}\,.$$

Thus by Theorem 2 *the group* G/G^0 *is of order* 4. Moreover, typical outer automorphisms of $\mathfrak{G}$ are given by those which induce σ_{π_1}, τ_0 and $\sigma_{\pi_1}\tau_0$ in $\mathfrak{H}$. Passing from $\mathfrak{G}$ to the Lie algebra of $SL(n+1,R)$ through the exact form of isomorphism between them, we can see that the typical outer automorphisms of the latter Lie algebra are induced from the automorphisms A_1, A_* and A_1A_* of the group $SL(n+1,R)$, which are defined as follows: For $(r_{ij})\in SL(n+1,R)$,

$$A_*(r_{ij})=(r_{ji})^{-1}\,,$$
$$A_1(r_{ij})=(p_{ij})^{-1}(r_{ij})(p_{ij})\,,$$

where (p_{ij}) is a real orthogonal matrix of determinant -1.

(c) $n=2f-1$; odd integer, $\rho=\rho_0$, $h_0=0$. $\mathfrak{G}$ *is the Lie algebra of the quaternion unimodular group in f quaternion variables,* which is by definition the subgroup of $SL(n+1,C)$ composed of such matrices

(x_{ij}) that satisfy the condition $(v_{ij})^{-1}(x_{ij})(v_{ij})=(\bar{x}_{ij})$ where $v_{ij}=\delta_{i,\,j-f}$ if $1\leqq i\leqq f$ and $v_{ij}=-\delta_{i,\,j+f}$ if $f+1\leqq i\leqq n+1$.

In this case, by (61) and (64),

$$\nu_{ij}=(-1)^{i+j+1} \qquad (1\leqq i,\ j\leqq n+1)\,.$$

Therefore, together with (65), we see that

$$\textstyle\sum_1=\{\alpha_{ii*}\,;\ 1\leqq i\leqq n+1\}$$

$\sum_2$ is empty, and

$$\textstyle\sum_3=\{\alpha_{ij}\,;\ 1\leqq i,\ j\leqq n+1,\ i\neq j^*\}\,.$$

Quite analogous considerations as for the above case (b) imply that (62) holds also and $\mathfrak{T}_1$ is the same group as considered there. However, in this case the reflections $\sigma_\alpha(\alpha\in\sum_1)$ appear in the generators of the group $\mathfrak{S}$ in addition to the rotations $\sigma_{\xi*}\,\sigma_\xi(\xi\in\sum_3)$ so that σ_π with π in (66) are the generators of $\mathfrak{S}$. Therefore $\mathfrak{T}_1=\mathfrak{S}$ and by (62)

$$\mathfrak{T}=\mathfrak{S}+\tau_0\,\mathfrak{S}\,.$$

Thus by Theorem 2 *G/G^0 is of order* 2. The typical outer automorphism is the one which is induced in a natural way from A_* defined in the case (a).

(d) $n=2f$; even integer, $\rho=\rho_0$, $h_0=0$. *$\mathfrak{G}$ is isomorphic to the Lie algebra of the real unimodular group $SL(n+1, R)$ of degree $n+1$.*

Again by (61) and (64) we see in this case that

$$\nu_{ij}=(-1)^{i+j+1} \qquad (1\leqq i,\ j\leqq n+1).$$

Therefore

$\sum_1$ is empty,

$\sum_2=\{\alpha_{ii*}\,;\ 1\leqq i\leqq n+1\}$, and

$$\textstyle\sum_3=\{\alpha_{ij}\,;\ 1\leqq i,\ j\leqq n+1,\ i\neq j^*\}\,.$$

By the same argument as in the case (b), we see that (62) holds and $\mathfrak{T}_1$ consists of inner rotations σ_π where π runs over the group X considered there. To determine the group $\mathfrak{S}$, take a root $\xi=\alpha_{ij}$ in $\sum_3$. Then

$$(\xi,\xi^*)=(\alpha_{ij},\ \alpha_{j*i*})$$

$$\begin{cases} =0, & \text{if } i \neq f \text{ and } j \neq f, \\ \neq 0, & \text{if } i=f \text{ or } j=f. \end{cases}$$

For a root $\xi=\alpha_{ij}$ with $(\xi, \xi^*)=0$, we have $\sigma_{\xi^*}\sigma_{\xi}=\sigma_{\pi_\xi}$ where $\pi_\xi=(i,j)(i^*,j^*)$. For a root $\xi=\alpha_{ij}$ with $(\xi,\xi^*)\neq 0$, i or j, say j, is equal to f, $\xi+\xi^*=\alpha_{ii^*}$ and $\sigma_{\xi+\xi^*}=\sigma_{\pi_\xi}$ where $\pi_\xi=(i,i^*)$. In virtue of Theorem 4 the group $\mathfrak{S}$ is generated by these rotations $\sigma_{\pi_\xi}(\xi \in \sum_3)$, so that $\mathfrak{S}$ consists of rotations $\sigma_\pi(\pi \in X)$. Therefore, $\mathfrak{S}=\mathfrak{T}_1$ and by (62)

$$\mathfrak{T}=\mathfrak{S}+\tau_0\mathfrak{S}.$$

By Theorem 2 the group G/G^0 *is of order* 2. The typical outer automorphism is given by A_* defined in the case (b).

Thus we have completely determined the structure of the groups G/G^0 for the real forms of the complex simple Lie algebras of type A_n. For the real forms of the complex simple Lie algebras of other types, we may determine the groups G/G^0 in a similar way.

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