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On the remainder term of Nevanlinna's second fundamental theorem.

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1. Let $w(z)$ be meromorphic in $|z|<R$ ($\leqq\infty$) and $n(r, a)$ be the number of zero points of $w(z)-a$ in $|z|<r$ and

$$[a, b]=\frac{|a-b|}{\sqrt{(1+|a|^{2})(1+|b|^{2})}}\,.$$

We put

$$m(r, a)=\frac{1}{2\pi}\int_{0}^{2\pi}\log\frac{1}{[w(re^{i\theta}), a]}\,d\theta\,,$$

$$N(r, a)=\int_{0}^{r}\frac{n(r, a)-n(0, a)}{r}\,dr+n(0, a)\log r+C\,,$$

where C is a constant, which is determined by $\lim_{r\to 0}\left(m(r, a)+N(r, a)\right)=0$.

Then $T(r)=m(r, a)+N(r, a)$ is independent of a, which is the Nevanlinna's first fundamental theorem.

Let $n_1(r, 0)$, $n_1(r, \infty)$ be the number of zero points and poles of $w'(z)$ in $|z|<r$ and $n_1(r)=n_1(r, 0)-n_1(r, \infty)+2n(r, \infty)\geqq 0$,

$$N_1(r)=\int_{0}^{r}\frac{n_1(r)-n_1(0)}{r}\,dr+n_1(0)\log r\,,$$

Then

NEVANLINNA'S SECOND FUNDAMENTAL THEOREM:

$$(q-2)\,T(r)\leqq\sum_{i=1}^{q}N(r, a_i)-N_1(r)+\Omega(r)\qquad(q\geqq 3)\,,$$

where the remainder term $\Omega(r)$ *satisfies the following condition:*

(i) *If* $R=\infty$,

$$\Omega(r)=O\left(\log r+\log T(r)\right)\,,\tag{I}$$

outside intervals I_ν, *such that* $\sum_\nu \int_{I_\nu} r^{k-1}\, dr < \infty \qquad (k \geqq 0)$.

$$\int_1^r \frac{\Omega(r)}{r^{k+1}}\, dr \leqq O\left(\int_1^r \frac{\log T(r)}{r^{k+1}}\, dr\right) \qquad (k > 0). \qquad (\mathrm{I}')^{1)}$$

(ii) *If* $R=1$,

$$\Omega(r) \leqq (k+\varepsilon) \log \frac{1}{1-r} + O\big(\log T(r)\big), \qquad (\mathrm{II})$$

outside intervals I_ν, *such that* $\sum_\nu \int_{I_\nu} \frac{dr}{(1-r)^k} < \infty \qquad (k \geqq 1)$,

ε *being any positive number.*

$$\int_{r_0}^r \Omega(r)\,(1-r)^{k-1}\, dr \leqq O\left(\int_{r_0}^r \log T(r)\,(1-r)^{k-1}\, dr\right) \quad (k > 0). \quad (\mathrm{II}')^{1)}$$

(I) and (II) was proved by Ahlfors[2] very elegantly. In this note, we will give a simple proof of (I′) and (II′) by Ahlfors' method.

Proof of (I′) and (II′).

Ahlfors proved in the paper cited,

$$\Omega(r) = \frac{1}{2} \log \frac{\lambda(r)}{2\pi} + O\big(\log T(r)\big), \qquad (1)$$

where

$$\lambda(r) = \int_0^{2\pi} \left(\frac{|\,w'(re^{i\theta})\,|}{1+|\,w(re^{i\theta})\,|^2}\right)^2 \rho\big(w(re^{i\theta})\big)\, d\theta,$$

$$\log \rho(w) = 2\sum_{i=1}^q \log \frac{1}{[w, a_i]} - \alpha \log\left(\sum_{i=1}^q \log \frac{1}{[w, a_i]}\right) - 2C \qquad (\alpha > 1),$$

where C is a constant, which is determined by

$$\iint_K \rho(a)\, d\omega(a) = 1,$$

$d\omega(a)$ being the surface element of the Riemann sphere K at a. Then Ahlfors proved,

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1) R. Nevanlinna: Théorèmes de Picard-Borel et la thèorie des fonctions méromorphes. (1929).

2) L. Ahlfors: Über eine Methode in der Theorie der meromorphen Funktionen. Soc. Sci. Fenn. Comment. Phys.-Math. 8. Nr. 10 (1932).

$$\int_1^r \frac{dr}{r}\int_0^r \lambda(r)\, rdr < T(r)+A\,, \quad (r>1)\,, \tag{2}$$

where A is a constant.

(i) First we will prove (I′).

By (1), to prove (I′), it suffices to prove that

$$\int_1^r \frac{\log \lambda(r)}{r^{k+1}}\, dr \leqq O\left(\int_1^r \frac{\log T(r)}{r^{k+1}}\, dr\right) \quad (k>0)\,. \tag{3}$$

We will deduce (3) from (2). Let $r>1$,

$$r_n = r\left(1-\frac{1}{2^n}\right) \qquad (n=1, 2, \cdots)\,,$$

$$\varDelta r_n = r_n - r_{n-1} = \frac{r}{2^n} \qquad (n=2, 3, \cdots)\,.$$

Then since $\dfrac{r}{2} \leqq r_n < r$, we have from (2),

$$T(r_{n+1})+A > \int_{r_n}^{r_{n+1}} \frac{dr}{r}\int_{r_{n-1}}^{r_n} \lambda(r) rdr > \frac{\varDelta r_{n+1}}{r}\int_{r_{n-1}}^{r_n} \lambda(r) rdr$$

$$> \frac{\varDelta r_{n+1}}{r}\cdot\frac{r}{2}\int_{r_{n-1}}^{r_n} \lambda(r) dr = \frac{\varDelta r_n}{4}\int_{r_{n-1}}^{r_n} \lambda(r) dr \qquad (n \geqq 2)\,,$$

$$\frac{4(T(r_{n+1})+A)}{(\varDelta r_n)^2} > \frac{1}{\varDelta r_n}\int_{r_{n-1}}^{r_n} \lambda(r) dr\,.$$

Since

$$\frac{4(T(r_{n+1})+A)}{(\varDelta r_n)^2} = \frac{4.2^{2n}(T(r_{n+1})+A)}{r^2} < e^{4n}\,(T(r_{n+1})+A)\,,$$

we have

$$e^{4n}\big(T(r_{n+1})+A\big) > \frac{1}{\varDelta r_n}\int_{r_{n-1}}^{r_n} \lambda(r) dr\,,$$

so that

$$4n + \log\big(T(r_{n+1})+A\big) > \log\left(\frac{1}{\varDelta r_n}\int_{r_{n-1}}^{r_n} \lambda(r) dr\right)$$

$$\geqq \frac{1}{\varDelta r_n}\int_{r_{n-1}}^{r_n} \log \lambda(r) dr = \frac{2^n}{r}\int_{r_{n-1}}^{r_n} \log \lambda(r) dr\,,$$

$$\int_{r_{n-1}}^{r_n} \log \lambda(r)dr \leqq \frac{4nr}{2^n} + \frac{r}{2^n} \log \big(T(r_{n+1})+A\big) . \tag{4}$$

$$\int_{r_{n+1}}^{r_{n+2}} \log \big(T(r)+A\big)dr \geqq \Delta r_{n+2} \log \big(T(r_{n+1})+A\big)$$

$$= \frac{r}{2^{n+2}} \log \big(T(r_{n+1})+A\big) ,$$

hence from (4),

$$\int_{r_{n-1}}^{r_n} \log \lambda(r)dr \leqq \frac{4nr}{2^n} + 4\int_{r_{n+1}}^{r_{n+2}} \log \big(T(r)+A\big)dr ,$$

so that

$$\int_{\frac{r}{2}}^{r} \log \lambda(r)dr \leqq 4r\sum_{n=1}^{\infty} \frac{n}{2^n} + 4\int_{\frac{r}{2}}^{r} \log \big(T(r)+A\big)dr$$

$$= 8r + 4\int_{\frac{r}{2}}^{r} \log \big(T(r)+A\big)dr .$$

$$\int_{\frac{r}{2}}^{r} \frac{\log \lambda(r)}{r^{k+1}}\, dr \leqq \frac{1}{(r/2)^{k+1}}\int_{\frac{r}{2}}^{r} \log \lambda(r)\, dr \leqq \frac{2^{k+4}}{r^k} +$$

$$+ \frac{2^{k+3}}{r^{k+1}}\int_{\frac{r}{2}}^{r} \log \big(T(r)+A\big)dr \leqq \frac{2^{k+4}}{r^k} + 2^{k+3}\int_{\frac{r}{2}}^{r} \frac{\log \big(T(r)+A\big)}{r^{k+1}}\, dr . \tag{5}$$

We determine n_0, such that $r_0 = \frac{r}{2^{n_0}} \leqq 1 < \frac{r}{2^{n_0-1}}$, then

$$\frac{1}{2} < r_0 \leqq 1 ,\quad 2^{n_0} < 2r .$$

If we apply (5) to $\frac{r}{2^\nu}$, $\frac{r}{2^{\nu-1}}$, we have

$$\int_{r_0}^{r} \frac{\log \lambda(r)}{r^{k+1}}\, dr = \sum_{\nu=1}^{n_0} \int_{\frac{r}{2^\nu}}^{\frac{r}{2^{\nu-1}}} \frac{\log \lambda(r)}{r^{k+1}}\, dr \leqq \frac{2^{k+4}}{r^k} \sum_{\nu=1}^{n_0} 2^{(\nu-1)k} +$$

$$2^{k+3}\int_{r_0}^{r} \frac{\log \big(T(r)+A\big)}{r^{k+1}}\, dr \leqq \frac{2^{k+4}\cdot 2^{n_0 k}}{r^k(2^k-1)} + 2^{k+3}\int_{r_0}^{r} \frac{\log \big(T(r)+A\big)}{r^{k+1}}\, dr$$

$$\leqq \frac{2^{k+4}(2r)^k}{r^k(2^k-1)} + 2^{k+3}\int_{r_0}^{r} \frac{\log (T(r)+A)}{r^{k+1}}\,dr$$

$$= \frac{2^{2k+4}}{2^k-1} + 2^{k+3}\int_{r_0}^{r} \frac{\log (T(r)+A)}{r^{k+1}}\,dr\,. \tag{6}$$

Hence we have (3).

(ii) Next we will prove (II′).

First suppose that $k \geqq 1$, then by (4), we have for $\frac{1}{2} \leqq r < 1$,

$$\int_{r_{n-1}}^{r_n} \log \lambda(r)\,dr \leqq \frac{4n}{2^n} + \frac{1}{2^n}\log\left(T(r_{n+1})+A\right),$$

$$\int_{r_{n-1}}^{r_n} \log \lambda(r)\,(1-r)^{k-1}\,dr \leqq (1-r_{n-1})^{k-1}\int_{r_{n-1}}^{r_n} \log \lambda(r)\,dr$$

$$\leqq \frac{4n}{2^n} + \frac{(1-r_{n-1})^{k-1}}{2^n}\log\left(T(r_{n+1})+A\right). \tag{7}$$

$$\int_{r_{n+1}}^{r_{n+2}} \log\left(T(r)+A\right)(1-r)^{k-1}\,dr \geqq \varDelta r_{n+2}(1-r_{n+2})^{k-1}\log\left(T(r_{n+1})+A\right)$$

$$= \frac{r}{2^{n+2}}(1-r_{n+2})^{k-1}\log\left(T(r_{n+1})+A\right) \geqq \frac{(1-r_{n+2})^{k-1}}{2^{n+3}}\log\left(T(r_{n+1})+A\right), \tag{8}$$

so that

$$\int_{r_{n-1}}^{r_n} \log \lambda(r)\,(1-r)^{k-1}\,dr$$

$$\leqq \frac{4n}{2^n} + 8\left(\frac{1-r_{n-1}}{1-r_{n+2}}\right)^{k-1}\int_{r_{n+1}}^{r_{n+2}} \log\left(T(r)+A\right)(1-r)^{k-1}\,dr\,. \tag{9}$$

Since

$$\frac{1-r_{n-1}}{1-r_{n+2}} = \frac{1-r+r/2^{n-1}}{1-r+r/2^{n+2}} \leqq 8\,,$$

$$\int_{r_{n-1}}^{r_n} \log \lambda(r)\,(1-r)^{k-1}\,dr \leqq \frac{4n}{2^n} + 8^k\int_{r_{n+1}}^{r_{n+2}} \log\left(T(r)+A\right)(1-r)^{k-1}\,dr\,.$$

From this we have

$$\int_{r_0}^{r} \log \lambda(r)\,(1-r)^{k-1}\,dr \leqq O\left(\int_{r_0}^{r} \log\left(T(r)+A\right)(1-r)^{k-1}\,dr\right). \tag{10}$$

Similarly we have the same relation, if $0 < k < 1$.
Hence from (10), we have (II').

REMARK. In case $R=\infty$, we have from (5),

$$\int_{\frac{r}{2}}^{r} \frac{\Omega(r)}{r^{k+1}}\, dr \leqq O\left(\int_{\frac{r}{2}}^{r} \frac{\log T(r)}{r^{k+1}}\, dr\right) \quad (k>0).$$

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