Journal of the Mathematical Society of Japan Vol. 4, No. 1, July, 1952.

On the Schur relations for the representations of a Frobenius algebra.

By Masaru OSIMA

(Received April, 2, 1952)

The Schur relations for the representations of a Frobenius algebra was studied in [1], [6][1]. In the present note we shall prove the Schur relations by a new method. Some supplementary results are also obtained. In § 1 we shall study the properties of corresponding bases[2] of a Frobenius algebra. § 2 deals with the Cartan basis[3] of an algebra. Using the results obtained in §§ 1 and 2, we shall derive in § 3 the Schur relations for the representations of a Frobenius algebra.

1. Corresponding bases of a Frobenius algebra. We consider an algebra A with unit element over a given field K. Let $u_1, u_2, \cdots, u_n$ be a basis of A. Let us denote by $S(a)$ and $R(a)$ the left and the right regular representations of A defined by the basis (u_i):

(1) $$a(u_i)\,b=(u_i)\,S(a)\,R'(b) \qquad (a,\ b\ in\ A)$$

where $R'(b)$ is the transpose of $R(b)$. A is called a Frobenius algebra if $S(a)$ is similar to $R(a)$:

(2) $$S(a)=P^{-1}\,R(a)\,P\,.$$

We then have

(3) $$(P')^{-1}\,R(a)\,P'=S(a^{\varphi}) \qquad (a^{\varphi}\ in\ A)\,.$$

The mapping $a\rightarrow a^{\varphi}$ forms an automorphism φ of A. This automorphism is completely determined by A, apart from an inner automorphism. We see that

(4) $$(u_i^{\varphi})=(u_i)\,(P')^{-1}\,P$$

where (u_i^{φ}) is obtained from (u_i) by application of the automorphism $\varphi: a\rightarrow a^{\varphi}$. If we set

---

1) The numbers in the brackets refer to the references at the end of the paper.

2) Brauer [1].

3) Nesbitt [4], Nesbitt and Scott [5].

(5) $$(\tilde{u}_i)=(u_i)(P')^{-1},$$

then we have from (2) and (3)

(6) $$a^{\varphi}(\tilde{u}_i)b=(\tilde{u}_i)R(a)S'(b).$$

We say that (u_i) and $(\tilde{u}_i)$ are corresponding bases of A belonging to the automorphism φ. Generally (u_i) and (v_i) are called corresponding bases of A belonging to φ, if (v_i) is a basis such that $a^{\varphi}(v_i)b=(v_i)R(a)S'(b)$.

LEMMA 1. *If there exists a second matrix Q such that $S(a)=Q^{-1}R(a)Q$, then $Q=S'(t^{-1})P$ (t in A) and conversely*[4].

We denote by $\varphi(t)$ the automorphism of A: $a\rightarrow ta^{\varphi}t^{-1}$. Then $\varphi=\varphi(1)=\varphi(c)$ if and only if a regular element c lies in the center C of A. From Lemma 1 we have

LEMMA 2. *(u_i) and (v_i) are corresponding bases belonging to $\varphi(t)$ if and only if $v_i=ct\tilde{u}_i$ where c is a regular element in C. In particular, (u_i) and $(c\tilde{u}_i)$ are corresponding bases belonging to φ.*

Let (p_i) be any basis of A: $(p_i)=(u_i)T$. We set $(\tilde{p}_i)=(\tilde{u}_i)(T')^{-1}$. Then we have

LEMMA 3. *(p_i) and $(\tilde{p}_i)$ are corresponding bases of A belonging to φ.*

PROOF. Let $S_1(a)$ and $R_1(a)$ be the left and the right regular representations of A defined by the basis (p_i). Then

$$S_1(a)=T^{-1}S(a)T, \qquad R_1(a)=T'R(a)(T')^{-1}.$$

Hence $S_1(a)=M^{-1}R_1(a)M$ where $M=T'PT$. We see that $(M')^{-1}R_1(a)M'=S_1(a^{\varphi})$ and $(\tilde{p}_i)=(p_i)(M')^{-1}$. This implies that (p_i) and $(\tilde{p}_i)$ are corresponding bases belonging to φ.

LEMMA 4. *Let (u_i) and (v_i) be corresponding bases belonging to φ. Then (v_i) and (u_i^{φ}) are corresponding bases belonging to φ.*

PROOF. From Lemma 2 we have $v_i=c\tilde{u}_i$. Hence $(v_i)=(u_i)S(c)(P')^{-1}$. If we set $T=S(c)(P')^{-1}$, then

$$(\tilde{v}_i)=(\tilde{u}_i)(T')^{-1}=(u_i)(P')^{-1}S'(c^{-1})P=(u_i)(P')^{-1}S'(c^{-1})P'(P')^{-1}P$$
$$=(u_i)R'(c^{-1})(P')^{-1}P=(c^{-1}u_i^{\varphi}).$$

---

4) Osima [7].

By Lemma 3, (v_i) and $(\tilde{v}_i)=(c^{-1}u_i^{\varphi})$ are corresponding bases belonging to φ. Then Lemma 2 shows that (v_i) and (u_i^{φ}) are corresponding bases belonging to φ.

LEMMA 5.[5] *Let σ be an automorphism of A which leaves invariant every element in K and let $(u_i^{\sigma})=W_{\sigma}(u_i)$. Then there exists a regular element b_{σ} in A such that*

$$P' W_{\sigma}'(P')^{-1} W_{\sigma}=R(b_{\sigma}) .$$

THEOREM 1. *Let (u_i) and (v_i) be corresponding bases of A belonging to φ. If σ is an automorphism of A, then (u_i^{σ}) and (v_i^{σ}) are corresponding bases belonging to $\varphi(b_{\sigma}^{\varphi})$ where b_{σ} has the same significance as in Lemma* 5.

Proof. We set $(w_i)=(\tilde{u}_i)W_{\sigma}^{-1}$. By Lemma 3, (u_i^{σ}) and (w_i) are corresponding bases belonging to φ. We have from (6) and Lemma 5

$$\begin{aligned}(b_{\sigma}^{\varphi} w_i)&=(\tilde{u}_i)R(b_{\sigma})W_{\sigma}^{-1}=(u_i)(P')^{-1}R(b_{\sigma})W_{\sigma}^{-1}\\ &=(u_i)W_{\sigma}'(P')^{-1}=(u_i^{\sigma})(P')^{-1}=(\tilde{u}_i^{\sigma}) .\end{aligned}$$

Lemma 2 shows that (u_i^{σ}) and $(\tilde{u}_i^{\sigma})=(b_{\sigma}^{\varphi} w_i)$ are corresponding bases belonging to $\varphi(b_{\sigma}^{\varphi})$. Since $v_i=c\tilde{u}_i$ (c in C), we have $v_i^{\sigma}=c^{\sigma}\tilde{u}_i^{\sigma}$. It follows that (u_i^{σ}) and (v_i^{σ}) are corresponding bases belonging to $\varphi(b_{\sigma}^{\varphi})$.

LEMMA 6. *If (u_i) and (v_i) are corresponding bases belonging to φ, then*

(i) $$\sum v_i u_i=\sum u_i^{\varphi} v_i=(\sum v_i u_i)^{\varphi} .$$

(ii) $$\sum u_i v_i=\sum v_i u_i^{\varphi}=(\sum u_i v_i)^{\varphi} .$$

PROOF. From (4) and (5) we have $(u_i^{\varphi})=(\tilde{u}_i)P$. Further (5) yields $(\tilde{u}_i^{\varphi})=(u_i^{\varphi})(P')^{-1}$. We then see that

$$\begin{aligned}&\sum u_i^{\varphi}\tilde{u}_i=(u_i^{\varphi})(\tilde{u}_i)'=(\tilde{u}_i)PP^{-1}(u_i)'=(\tilde{u}_i)(u_i)'=\sum \tilde{u}_i u_i ,\\ &(\sum \tilde{u}_i u_i)^{\varphi}=(\tilde{u}_i^{\varphi})(u_i^{\varphi})'=(u_i^{\varphi})(P')^{-1}P'(\tilde{u}_i)'=(u_i^{\varphi})(\tilde{u}_i)'=\sum u_i^{\varphi}\tilde{u}_i ,\\ &\sum \tilde{u}_i u_i^{\varphi}=(\tilde{u}_i)(u_i^{\varphi})'=(u_i)(\tilde{u}_i)'=\sum u_i\tilde{u}_i ,\\ &(\sum u_i\tilde{u}_i)^{\varphi}=(u_i^{\varphi})(\tilde{u}_i^{\varphi})'=(\tilde{u}_i)(u_i^{\varphi})'=\sum \tilde{u}_i u_i^{\varphi} .\end{aligned}$$

By Lemma 2, we have $v_i=c\tilde{u}_i$ $(i=1, 2, \cdots, n)$ where c is a regular element in C. This proves our assertions (observe that $c^{\varphi}=c$).

---

5) Osima [7].

Generally we have for any element a in A

(7) $$\sum u_i a\tilde{u}_i = \sum \tilde{u}_i a u_i^{\varphi} = \sum u_i^{\varphi} a \tilde{u}_i^{\varphi} = (\sum u_i a\tilde{u}_i)^{\varphi}.$$

THEOREM 2. *Let (p_i) and (q_i) be corresponding bases belonging to φ, then*

$$\sum p_i q_i = c \sum u_i \tilde{u}_i, \qquad \sum q_i p_i = c \sum \tilde{u}_i u_i$$

where c is a regular element in the center of A.

PROOF. If $(p_i)=(u_i)T$, then Lemma 3 yields $q_i = c\tilde{p}_i$ where $(\tilde{p}_i) = (\tilde{u}_i)(T')^{-1}$. Now we have

(8) $$\sum p_i \tilde{p}_i = (p_i)(\tilde{p}_i)' = (u_i) T T^{-1} (\tilde{u}_i)' = (u_i)(\tilde{u}_i)' = \sum u_i \tilde{u}_i,$$

(9) $$\sum \tilde{p}_i p_i = (\tilde{u}_i)(T')^{-1} T' (u_i)' = (\tilde{u}_i)(u_i)' = \sum \tilde{u}_i u_i.$$

We obtain generally for any element a in A

(10) $$\sum p_i a \tilde{p}_i = \sum u_i a \tilde{u}_i.$$

COROLLARY. *Let (u_i), (v_i) and (p_i), (q_i) be a pair of corresponding bases belonging to $\varphi(t_1)$ and $\varphi(t_2)$ respectively. Then*

$$\sum v_i u_i = ct \sum q_i p_i, \qquad \sum u_i v_i = c \sum p_i t q_i$$

where $t = t_1 t_2^{-1}$.

PROOF. First formula follows readily from $v_i = c_1 t_1 \tilde{u}_i$ and $q_i = c_2 t_2 \tilde{p}_i$. Since $tq_i = c_2 t_1 \tilde{p}_i$, we have from (10)

$$\sum p_i t q_i = c_2 \sum p_i t_1 \tilde{p}_i = c_2 \sum u_i t_1 \tilde{u}_i = c_2 c_1^{-1} \sum u_i v_i.$$

If S is a set of elements in A, then we denote by $r(S)$ [$l(S)$] the set of all right [left] annihilators of S in A. We have $r(N)=l(N)$ for the radical N of a Frobenius algebra A.

LEMMA 7. *Let (u_i) and (v_i) be corresponding bases belonging to φ. Then, for any element b in A*

(i) $$\sum u_i b v_i \in C,$$

(ii) $$a^{\varphi}(\sum v_i b u_i) = (\sum v_i b u_i) a,$$

(iii) $$\sum v_i u_i \in r(N) = l(N).$$

PROOF. (i) We have for any element a in A

$$a(\sum u_i b v_i) = a(u_i b)(v_i)' = (u_i b) S(a) (v_i)' = (u_i b)(v_i)' a = (\sum u_i b v_i) a.$$

(ii) $$a^{\varphi}(\sum v_i\, bu_i)=a^{\varphi}(v_i\, b)\,(u_i)'=(v_i\, b)\,R(a)\,(u_i)'=$$
$$=(v_i\, b)\,(u_i)'a=(\sum v_i\, bu_i)\,a\,.$$

(iii) Let

(11) $$A=\mathfrak{l}_1\supset\mathfrak{l}_2\supset\cdots\cdots\supset\mathfrak{l}_t\supset 0$$

be a composition series of the left-module A. Let us denote by (w_i) a basis of A defined by (11). If $S_1(a)$ and $R_1(a)$ are the left and the right regular representations defined by the basis (w_i), then there exists a matrix Q such that $S_1(a)=Q^{-1}R_1(a)\,Q$. We set $(\tilde{w}_i)=(w_i)\,(Q')^{-1}$. With a suitable choice of Q, corresponding bases (w_i) and $(\tilde{w}_i)$ belong to φ. Further $(\tilde{w}_i)$ is a basis defined by a composition series[6]

(12) $$0\subset r(\mathfrak{l}_2^{\varphi})\subset\cdots\cdots\subset r(\mathfrak{l}_t^{\varphi})\subset A\,.$$

Let $w_i\in\mathfrak{l}_u$ and $w_i\notin\mathfrak{l}_{u+1}$. Then $\tilde{w}_i\in r(\mathfrak{l}_{u+1}^{\varphi})$. As $nw_i\in\mathfrak{l}_{u+1}$ for any n in N, we find

$$(nw_i)^{\varphi}\,\tilde{w}_i=0\,,\qquad\qquad (i=1,2,\cdots,n)\,.$$

Hence $n^{\varphi}(\sum w_i^{\varphi}\,\tilde{w}_i)=0$. This implies

$$\sum w_i^{\varphi}\,\tilde{w}_i=\sum\tilde{w}_i\,w_i\in r(N)\,.$$

It follows from $\sum v_i\,u_i=c\sum\tilde{w}_i\,w_i$ that our assertion is valid.

The algebra A is called a symmetric algebra, when the matrix P in (2) can be chosen as a symmetric, non-singular matrix. Then $a^{\varphi}=a$ and φ becomes the identical automorphism. The equation (6) now reads

(13) $$a(\tilde{u}_i)\,b=(\tilde{u}_i)\,R(a)\,S'(b)\,.$$

We then say that (u_i) and $(\tilde{u}_i)$ are quasi-complementary bases[7]. If (u_i) and (v_i) are quasi-complementary bases, then (v_i) and (u_i) are also quasi-complementary. From Lemmas 6 and 7 we have

(14) $$\sum u_i\,v_i\in C\cap r(N)\,.$$

If (p_i) and (q_i) are any quasi-complementary bases, then from Theorem 2

(15) $$\sum p_i\,q_i=c\sum u_i\,v_i$$

---

6) Osima [7].
7) Brauer [1].

where c is a regular element in C.

Let G be a group of finite order g and let A be the group ring of G over an arbitrary field K:

$$A=G_1K+G_2K+\cdots\cdots+G_gK, \qquad G_1=1. \tag{16}$$

As is well known, A is a symmetric algebra and (G_s) and (G_s^{-1}) are quasi-complementary bases. Hence, by (14)

$$\sum G_s^{-1}G_s=g\in r(N).$$

If the characteristic of the underlying field K is zero or a prime p which does not divide g, then $\sum G_s^{-1}G_s=g$ is a regular element in A. This implies $N=0$, that is, A is semisimple.

In what follows we assume that the group ring A is a semisimple algebra over an algebraically closed field K. Let

$$A=A_1+A_2+\cdots\cdots+A_k$$

be a decomposition of A into a direct sum of simple two-sided ideals. We denote by $e_{i,\alpha\beta}\left(\alpha,\beta=1,2,\cdots,f(i)\right)$ a set of matrix units for the simple algebra A_i. $E_i=\sum e_{i,\alpha\alpha}$ is a unit element of A_i. Let F_1, $F_2,\cdots\cdots,F_k$ be the distinct irreducible representations of A. We set

$$F_i(G_s)=\left(f_{\alpha\beta}^i(G_s)\right). \tag{17}$$

Then $G_s=\sum_i\sum_{\alpha,\beta}f_{\alpha\beta}^i(G_s)\,e_{i,\alpha\beta}$ or in matrix form

$$(G_s)=(e_{i,\alpha\beta})\left(f_{\alpha\beta}^i(G_s)\right) \tag{18}$$

(i,α,β row indices, s column index). We then have

$$(e_{i,\alpha\beta})=(G_s)\left(f_{\alpha\beta}^i(G_s)\right)^{-1}. \tag{19}$$

If we set

$$(v_{i,\beta\alpha})=(G_s^{-1})\left(f_{\alpha\beta}^i(G_s)\right)', \tag{20}$$

then, by Lemma 3, $(e_{i,\alpha\beta})$ and $(v_{i,\beta\alpha})$ are quasi-complementary. Since $(e_{i,\alpha\beta})$ and $(e_{i,\beta\alpha})$ are also quasi-complementary, Lemma 2 yields

$$v_{i,\beta\alpha}=ce_{i,\beta\alpha} \tag{21}$$

where c is a regular element in the center of A. Now (20) becomes

$$(e_{i,\beta\alpha})\,S(c)=(G_s^{-1})\left(f_{\alpha\beta}^i(G_s)\right)'. \tag{22}$$

On the other hand we have

(23) $$(G_s^{-1})=(e_{i,\,\alpha\beta})\left(f^i_{\alpha\beta}(G_s^{-1})\right)=(e_{i,\,\beta\alpha})\left(f^i_{\beta\alpha}(G_s^{-1})\right).$$

(22) and (23) yield

(24) $$\left(f^i_{\beta\alpha}(G_s^{-1})\right)\left(f^i_{\alpha\beta}(G_s)\right)'=S(c).$$

By (8), we find

$$\sum G_s^{-1}\,G_s=g=c\sum_i\sum_{\alpha,\beta}e_{i,\,\beta\alpha}\,e_{i,\,\alpha\beta}=c\sum_i f(i)\,E_i\,.$$

Hence $c=\sum g/f(i)\,E_i$. This shows that $f^i_{\beta\alpha}(c)=g/f(i)\,\delta_{\alpha\beta}$ and hence we have from (24)

(25) $$\sum_s f^i_{\alpha\beta}(G_s)\,f^j_{\mu\nu}(G_s^{-1})=g/f(i)\,\delta_{ij}\,\delta_{\alpha\nu}\,\delta_{\beta\mu}\,.$$

The same arguments may be also applied to the representations of a Frobenius algebra.

2. Cartan basis. Let A be an algebra with unit element over an algebraically closed field, and let

(26) $$A=\bar{A}+N$$

be a splitting of A into a direct sum of a semisimple subalgebra $\bar{A}$ and the radical N of A. We shall denote by

(27) $$\bar{A}=\bar{A}_1+\bar{A}_2+\cdots\cdots+\bar{A}_k$$

the unique splitting of $\bar{A}$ into a direct sum of simple invariant subalgebras $\bar{A}_\kappa$. Let

(28) $$A=A_1\supset A_2\supset\cdots\cdots\supset A_r\supset 0$$

be a composition series for A considered as the A-A-module. Let $e_{\kappa;\,\alpha\beta}\left(\alpha,\beta=1,2,\cdots\cdots,f(\kappa)\right)$ denote a set of matrix units for the simple algebra $\bar{A}_\kappa$. We set $e=\sum_\kappa e_{\kappa,\,11}$. Then $A^0=eAe$ becomes an algebra with unit element e. The algebra A^0 is called the basic algebra[8] of A. Corresponding to (28) we obtain a composition series for A^0 considered as the A^0-A^0-module:

(29) $$A^0=A_1^0\supset A_2^0\supset\cdots\cdots\supset A_r^0\supset 0\,,\qquad A_u^0=eA_ue\,.$$

Let composition factor group A_u/A_{u+1} be of type (ρ_u,σ_u), $(u=1,$

8) See Nesbitt and Scott [5].

2, ……, r). Then we can choose a basis $b_1, b_2, \cdots, b_r$ of A^0 corresponding to (29) such that $b_u \in A^0_u$, $b_u \notin A^0_{u+1}$ and $e_{\rho_u, 11} b_u e_{\sigma_u, 11} = b_u$. Further we may choose $b_1, b_2, \cdots, b_k$ such that $b_\kappa = e_{\kappa, 11}$ ($\kappa = 1, 2, \cdots, k$). The elements

$$e_{\rho_u, \alpha 1} b_u e_{\sigma_u, 1\beta} \qquad \begin{array}{l} u=1,2,\cdots,r \\ \alpha=1,2,\cdots,f(\rho_u) \\ \beta=1,2,\cdots,f(\sigma_u) \end{array} \tag{30}$$

form a basis of A. This basis is called the Cartan basis of A. In regard to this basis an element a in A may be expressed as

$$a = \sum_{u, \alpha\beta} h^u_{\alpha\beta}(a) e_{\rho_u, \alpha 1} b_u e_{\sigma_u, 1\beta}. \tag{31}$$

The additive group formed by the matrices $H_u(a) = \left(h^u_{\alpha\beta}(a)\right)$ is called an elementary module of A belonging to b_u. Let $F_1, F_2, \cdots, F_k$ be the distinct irreducible representations of A. We set

$$F_\kappa(a) = \left(f^\kappa_{\alpha\beta}(a)\right). \tag{32}$$

$F_\kappa(a)$ is the elementary module belonging to $e_{\kappa, 11}$, that is,

$$h^\kappa_{\alpha\beta}(a) = f^\kappa_{\alpha\beta}(a) \qquad (\kappa = 1, 2, \cdots, k).$$

The number of b_u which are of type (κ, λ) is denoted by $c_{\kappa\lambda}$. The $c_{\kappa\lambda}$ are called the Cartan invariants of A. We have

$$\sum_{\kappa, \lambda} c_{\kappa\lambda} = r. \tag{33}$$

Let us assume that $\sigma_{\lambda(i)} = \lambda$ ($i = 1, 2, \cdots, t = \sum_\kappa c_{\kappa\lambda}$) where $\lambda(1) < \lambda(2) < \cdots < \lambda(t)$. Then

$$A_{\lambda(1)} e_{\lambda, 11} \supset A_{\lambda(2)} e_{\lambda, 11} \supset \cdots\cdots \supset A_{\lambda(t)} e_{\lambda, 11} \supset 0 \tag{34}$$

is a composition series of $Ae_{\lambda, 11}$ considered as the left-module. The elements

$$e_{\kappa_i, \alpha 1} b_{\lambda(i)} e_{\lambda, 11}, \qquad (\kappa_i = \rho_{\lambda(i)}) \tag{35}$$

$\left((i = 1, 2, \cdots, t;\ \alpha = 1, 2, \cdots, f(\kappa_i)\right)$ form the basis of $Ae_{\lambda, 11}$. Let U_λ be the indecomposable representation of A defined by (35). Then we see that

$$U_\lambda = \begin{pmatrix} F_\lambda & & \\ H_{\lambda(2)} & F_\nu & \\ \cdots\cdots & \cdots\cdots & \\ H_{\lambda(t)} & \cdots\cdots & F_\mu \end{pmatrix} \tag{36}$$

where $H_{\lambda(i)}$ is the elementary module of A belonging to $b_{\lambda(i)}$.

In what follows we assume that A is a Frobenius algebra over an algebraically closed field. Let $S(a)$ and $R(a)$ be the left and the right regular representations of A defined by the Cartan basis $(e_{\rho_u, \alpha 1}\, b_u\, e_{\sigma_u, 1\beta})$. Then we have $S(a)=P^{-1}R(a)P$ and $(P')^{-1}R(a)P'=S(a^{\varphi})$. Let $U_1, U_2, \cdots, U_k$ and $V_1, V_2, \cdots, V_k$ be the indecomposable parts of $S(a)$ and $R(a)$ respectively. Then

(37) $$U_\lambda \cong V_{\pi(\lambda)}$$

where $(\pi(1), \pi(2), \cdots, \pi(k))$ is a permutation of $(1, 2, \cdots, k)$.

In [7] we have proved that the automorphism φ may be chosen such that

(38) $$e_{\pi(\lambda), \alpha\beta} \equiv e^{\varphi}_{\lambda, \alpha\beta} \quad (\text{mod } N).$$

From now on we consider only the automorphism φ: $a \rightarrow a^{\varphi}$ which satisfies (38). The element a^{φ} will be denoted simply by a^*. We obtain the irreducible representation $a^* \rightarrow F_\lambda(a)$ which will be denoted by $F_{\lambda*}(a)$. If we set $F_{\lambda*}(a)=\left(f^{\lambda*}_{\alpha\beta}(a)\right)$, then, by (38), we see that $f^{\lambda*}_{\alpha\beta}(a)=f^{\pi(\lambda)}_{\alpha\beta}(a)$. Hence we may set

(39) $$U_\lambda=\begin{pmatrix} F_\lambda & & \\ H_{\lambda(2)} & F_\nu & \\ \cdots\cdots & \cdots\cdots & \\ H_{\lambda(t)} & \cdots\cdots & F_{\lambda*} \end{pmatrix}.$$

If we set

(40) $$(v_{u, \beta\alpha})=(e_{\rho_u, \alpha 1}\, b_u\, e_{\sigma_u, 1\beta})\,(P')^{-1},$$

then $(v_{u, \beta\alpha})$ is a basis of A corresponding to a composition series

(41) $$0 \subset r(A^*_2) \subset \cdots\cdots \subset r(A^*_r) \subset A\,.$$

Further $(e_{\rho_u, \alpha 1}\, b_u\, e_{\sigma_u, 1\beta})$ and $(v_{u, \beta\alpha})$ are corresponding basis of A belonging to φ. (1) and (6) yield

(42) $$v_{u. \beta\alpha}=e^*_{\sigma_u, \beta 1}\, d_u\, e_{\rho_u, 1\alpha}$$

where $d_1, d_2, \cdots, d_r$ form a basis of the $(A^0)^*$-A^0-module e^*Ae. Further from (34) we see that

(43) $$0 \subset e^*_{\lambda, 11}\, r(A^*_{\lambda(2)}) \subset \cdots\cdots \subset e^*_{\lambda, 11}\, r(A^*_{\lambda(t)}) \subset e^*_{\lambda, 11}\, A$$

is a composition series of the right-module $e^{*}_{\lambda,11}A$ and

(44) $$e^{*}_{\lambda,11}\, d_{\lambda(i)}\, e_{\kappa_{i},1\alpha}\,, \qquad (\kappa_i=\rho_{\lambda(i)})$$

form a basis of $e^{*}_{\lambda,11}A$. We can see that $e_{\kappa_{i},1\alpha}=e_{\pi(\lambda),1\alpha}$. Thus we obtain

(45) $$\begin{cases} a(e_{\kappa_{i},\alpha 1}\, b_{\lambda(i)}\, e_{\lambda,11})=(e_{\kappa_{i},\alpha 1}\, b_{\lambda(i)}\, e_{\lambda,11})\, U_\lambda(a)\,, \\ (e^{*}_{\lambda,11}\, d_{\lambda(i)}\, e_{\kappa_{i},1\alpha})\, a=(e^{*}_{\lambda,11}\, d_{\lambda(i)}\, e_{\kappa_{i},1\alpha})\, U_\lambda'(a)\,, \end{cases}$$

where U_λ is written in the form (39). We set in (39)

(46) $$G_\lambda(a)=\left(g^{\lambda}_{\alpha\beta}(a)\right)=\left(h^{\lambda(i)}_{\alpha\beta}(a)\right).$$

Let (p_s) and (q_s) be any corresponding bases of A belonging to φ. Then, by (31)

(47) $$(p_s)=(e_{\rho_{u},\alpha 1}\, b_u\, e_{\sigma_{u},1\beta})\left(h^{u}_{\alpha\beta}(p_s)\right)$$

(u, α, β row indices, s column index). Hence

(48) $$(e_{\rho_{u},\alpha 1}\, b_u\, e_{\sigma_{u},1\beta})=(p_s)\left(h^{u}_{\alpha\beta}(p_s)\right)^{-1}.$$

If we set

(49) $$(w_{u,\beta\alpha})=(q_s)\left(h^{u}_{\alpha\beta}(p_s)\right)',$$

then $(e_{\rho_{u},\alpha 1}\, b_u\, e_{\sigma_{u},1\beta})$ and $(w_{u,\beta\alpha})$ are corresponding bases belonging to φ. Hence we may assume without restriction that

(50) $$w_{u,\beta\alpha}=e^{*}_{\sigma_{u},\beta 1}\, d_u\, e_{\rho_{u},1\alpha}\,.$$

In regard to this new basis an element a in A may be expressed as

(51) $$a=\sum_{u,\alpha\beta} k^{u}_{\beta\alpha}(a)\, e^{*}_{\sigma_{u},\beta 1}\, d_u\, e_{\rho_{u},1\alpha}\,.$$

Then

(52) $$(q_s)=(e^{*}_{\sigma_{u},\beta 1}\, d_u\, e_{\rho_{u},1\alpha})\left(k^{u}_{\beta\alpha}(q_s)\right).$$

Now (49), (50) and (52) yield

$$\left(k^{u}_{\beta\alpha}(q_s)\right)\left(h^{u}_{\alpha\beta}(p_s)\right)'=I$$

that is,

(53) $$\sum_{s} k^{u}_{\mu\nu}(q_s)\, h^{v}_{\alpha\beta}(p_s)=\delta_{uv}\,\delta_{\alpha\nu}\,\delta_{\beta\mu}\,.$$

3. The Schur relations. From (38) and $f_{\alpha\beta}^{\lambda *}(a)=f_{\alpha\beta}^{\pi(\lambda)}(a)$ we have

$$e_{\lambda,\beta 1}^{*}\, d_{\lambda(t)}\, e_{\pi(\lambda),1\alpha} \equiv c_\lambda\, e_{\lambda,\beta\alpha}^{*} \qquad (\mathrm{mod}\ N) \tag{54}$$

where $c_\lambda \neq 0$ is an element of K. It follows from (51) that

$$c_\lambda\, k_{\beta\alpha}^{\lambda(t)}(a)=f_{\beta\alpha}^{\lambda *}(a)\,. \tag{55}$$

Using (53) and (55) we have the following

THEOREM 3. *Let A be a Frobenius algebra which has (p_s) and (q_s) as corresponding bases belonging to $\varphi: a\rightarrow a^*$. If U_λ is written in the form* (39), *then*

$$\sum_s f_{\mu\nu}^{\lambda *}(q_s)\, g_{\alpha\beta}^{\lambda}(p_s)=c_\lambda\, \delta_{\alpha\nu}\, \delta_{\beta\mu}\,, \tag{56}$$

$$\sum_s f_{\mu\nu}^{\lambda *}(q_s)\, h_{\alpha\beta}^{u}(p_s)=0 \qquad (H_u \neq G_\lambda) \tag{57}$$

where the element $c_\lambda \neq 0$ of K is independent of α, β, μ, ν.

Further we obtain

THEOREM 4. *If $U_\kappa(a)=\big(u_{mn}^{\kappa}(a)\big)$ (m row index, n column index), then, under the assumptions of Theorem* 3, *we have*

$$\sum_s f_{\mu\nu}^{\lambda *}(q_s)\, u_{mn}^{\kappa}(p_s)=0 \qquad (\kappa=1, 2, \cdots, k) \tag{58}$$

provided that u_{mn}^{κ} does not belong to the elementary module G_λ in the lower left corner in U_λ, (39).

PROOF. If $\big(u_{mn}^{\kappa}(a)\big)$ belongs to the elementary module $H_u(a) \neq G_\lambda(a)$, then (58) becomes (57). Hence we consider $u_{mn}^{\kappa}(a)$ which does not belong to the elementary module $H_u(a)$. From (31) we see that $u_{mn}^{\kappa}(p_s)$ is a linear combination of $h_{\alpha\beta}^{u}(p_s)$ where the coefficients of $g_{\alpha\beta}^{\lambda}(p_s)=h_{\alpha\beta}^{\lambda(t)}(p_s)$ vanish. Observe that the elementary module $G_\lambda(a)$ belongs to $b_{\lambda(t)} \in l(N)$. Thus (57) shows that our assertions are valid.

Now we set

$$U_\lambda=\begin{pmatrix} H_\lambda^{(1,1)} & & & \\ H_\lambda^{(2,1)} & H_\lambda^{(2,2)} & & \\ \cdots & \cdots & \cdots & \\ H_\lambda^{(t,1)} & H_\lambda^{(t,2)} & \cdots\cdots & H_\lambda^{(t,t)} \end{pmatrix} \tag{59}$$

where $H_\lambda^{(1,1)}=F_\lambda$, $H_\lambda^{(t,t)}=F_{\lambda*}$, $H_\lambda^{(t,1)}=G_\lambda$ and $H_\lambda^{(i,1)}=H_{\lambda(i)}$. We write

(60) $$H_\lambda^{(i,j)}(a)=\left(h_{\mu\nu}^{\lambda(i,j)}(a)\right).$$

We see from (45) and (51) that $h_{\mu\nu}^{\lambda(i,j)}(a)$ is a linear combination of $k_{\beta\alpha}^u(a)$.

LEMMA 8. *$h_{\mu\nu}^{\lambda(t,j)}(a)$ is a linear combination of $k_{\beta\alpha}^u(a)$ where the coefficient of $k_{\beta\alpha}^{\lambda(j)}(a)$ is $c_\lambda\,\delta_{\alpha\nu}\,\delta_{\beta\mu}$ and the coefficients of $k_{\beta x}^u(a)$ $\left(u<\lambda(j)\right)$ vanish.*

PROOF. Since $r(A_u^*)$ in (41) are the two-sided ideals of A, $Nr(A_u^*)\subset r(A_{u-1}^*)$. From (54) we then have

$$e_{\lambda,11}^*\,d_{\lambda(t)}\,e_{\pi(\lambda),1\mu}\cdot e_{\lambda,\beta1}^*\,d_{\lambda(j)}\,e_{\kappa_j,1\alpha}$$

$$\equiv\begin{cases}c_\lambda\,e_{\lambda,11}^*\,d_{\lambda(j)}\,e_{\kappa_j,1\alpha} & \left(\operatorname{mod} r(A_{\lambda(j-1)}^*)\right) \quad (\mu=\beta)\\ 0 & \left(\operatorname{mod} r(A_{\lambda(j-1)}^*)\right) \quad (\mu\neq\beta).\end{cases}$$

This proves the first part of the lemma. The second part follows from

$$e_{\lambda,11}^*\,d_{\lambda(t)}\,e_{\pi(\lambda),1\mu}\cdot e_{\sigma_u,\beta1}^*\,d_u\,e_{\rho_u,1\alpha}\in e_{\lambda,11}^*\,r(A_u^*)\subseteqq e_{\lambda,11}^*\,r(A_{\lambda(j-1)}^*).$$

According to (53) and Lemma 8, we have

(61) $$\sum_s h_{\mu\nu}^{\lambda(t,i)}(q_s)\,h_{\alpha\beta}^{\lambda(i,1)}(p_s)=c_\lambda\,\delta_{\alpha\nu}\,\delta_{\beta\mu},$$

(62) $$\sum_s h_{\mu\nu}^{\lambda(t,j)}(q_s)\,h_{\alpha\beta}^{\lambda(i,1)}(p_s)=0 \qquad (i<j).$$

LEMMA 9. *$h_{\mu\nu}^{\lambda(i,j)}(a)$ $(1<j)$ is a linear combination of $h_{\alpha\beta}^u(a)$ where the coefficients of $h_{\alpha\beta}^u(a)$ $\left(\lambda(i)\leqq u\right)$ vanish.*

PROOF. (31) and (45) show that $h_{\mu\nu}^{\lambda(i,j)}(a)$ $(1<j)$ is a linear combination of $h_{\alpha\beta}^u(a)$. We have from (28), $A_uN\subset A_{u+1}$. Since $e_{\kappa_j,\mu1}\,b_{\lambda(j)}\,e_{\lambda,11}\in A_{\lambda(j)}\subset N$, it follows from $\lambda(i)\leqq u$ that

$$e_{\rho_u,\alpha1}\,b_u\,e_{\sigma_u,1\beta}\cdot e_{\kappa_j,\mu1}\,b_{\lambda(j)}\,e_{\lambda,11}\in A_{u+1}\,e_{\lambda,11}\subseteqq A_{\lambda(i+1)}\,e_{\lambda,11}.$$

This implies that the coefficients of $h_{\alpha\beta}^u(a)$ $\left(\lambda(i)\leqq u\right)$ vanish.

LEMMA 10. *$h_{\mu\nu}^{\lambda(i,j)}(a)$ $(i<t)$ is a linear combination of $k_{\beta\alpha}^u(a)$ where the coefficients of $k_{\beta x}^u(a)$ $\left(u\leqq\lambda(j)\right)$ vanish.*

PROOF. Since $e_{\lambda,11}^*\,d_{\lambda(i)}\,e_{\kappa_i,1\mu}\in N$,

$$e_{\lambda,11}^*\,d_{\lambda(i)}\,e_{\kappa_i,1\mu}\cdot e_{\sigma_u,\beta1}^*\,d_u\,e_{\rho_u,1\alpha}\in e_{\lambda,11}^*\,r(A_{u-1}^*)\subseteqq e_{\lambda,11}^*\,r(A_{\lambda(j-1)}^*).$$

This proves the lemma.

Lemmas 8, 9, 10, combined with (53), yield

$$\sum_s h^{\lambda(m,\,l)}_{\mu\nu}(q_s)\,h^{\lambda(i,\,j)}_{\alpha\beta}(p_s)=0 \qquad \text{for} \begin{cases} \text{(i)} & i<l, \\ \text{(ii)} & i=l,\ \ j>1, \\ \text{(iii)} & i=l,\ j=1,\ m<t. \end{cases} \tag{63}$$

We denote by $u(\lambda)$ the degree of U_λ. If we set $U_\lambda(a)=\left(u^\lambda_{mn}(a)\right)$ (m row index: n column index), then above arguments show that

$$\sum_s u^\lambda_{mn}(q_s)\,u^\lambda_{nl}(p_s)= \begin{cases} c_\lambda & 1\leqq l\leqq f(\lambda),\quad m=u(\lambda)-f(\lambda)+l, \\ 0 & \text{otherwise.} \end{cases} \tag{64}$$

This implies

$$U_\lambda(\sum_s q_s p_s)=\sum_s U_\lambda(q_s)\,U_\lambda(p_s)= \begin{pmatrix} 0 & 0 \\ a_\lambda I & 0 \end{pmatrix} \tag{65}$$

where I is the unit matrix of degree $f(\lambda)$ and $a_\lambda=c_\lambda u(\lambda)$.

Department of Mathematics
Okayama University

References

[1] R. Brauer, On hypercomplex arithmetic and a theorem of Speiser, Festschrift zum 60 Geburtstag von Prof. Dr. Andreas Speiser (1945).

[2] T. Nakayama, On Frobeniusean algebras I, Ann. of Math., 40 (1939).

[3] ———, On Frobeniusean algebras II, Ann. of Math, 42 (1941).

[4] C. Nesbitt, On the regular representations of algebras, Ann. of Math., 39 (1938).

[5] ——— and W. M. Scott, Some remarks on algebras over an algebraically closed field, Ann. of Math., 44 (1943).

[6] ——— and R. M. Thrall, Some ring theorems with applications to modular representations, Ann. of Math., 47 (1946).

[7] M. Osima, Some studies on Frobenius algebras, forthcoming.