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Conformally Flat Riemann Spaces of Clase One

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When an $n(\geqq 3)$-dimensional Riemann space C_n with a metric defined by a positive-definite quadratic differential form is conformal to a flat space and is of class one, at least $(n-1)$ principal normal curvatures are equal. If all of them are equal, C_n is of constant curvature.

This theorem was proved by J. A. Schouten,[1] only when $(n+1)$-dimensional flat E_{n+1} enclosing C_n is euclidean. But, even if E_{n+1} is not euclidean, we can prove it similarly as follows.

Since C_n is conformally flat, the curvature tensor is written in the from

$$(0\cdot 1)\qquad R_{hijk}=g_{hj}l_{ik}+g_{ik}l_{hj}-g_{hk}l_{ij}-g_{ij}l_{hk}\,;$$

where we put

$$(0\cdot 2)\qquad l_{ij}=\frac{1}{n-2}\left(R_{ij}-\frac{R}{2(n-1)}g_{ij}\right).$$

If C_n is of class one, the Gauss equation

$$(0\cdot 3)\qquad R_{hijk}=e(H_{hj}H_{ik}-H_{hk}H_{ij})\qquad (e=\pm 1)$$

is satisfied; where $e=+1$ if enclosing E_{n+1} is euclidean. Referring to the coordinate system (C) such that coordinate lines are lines of curvature, we have from (0·1) and (0·3),

$$(0\cdot 4)\qquad l_{ii}+l_{jj}=eH_{ii}H_{jj}\qquad (i\neq j)\,.$$

Making use of (0·4), we can prove easily the above theorem.

In this paper we look for the condition that $C_n(n>3)$ not of constant curvature be of class one. It is very difficult to express the resultant system[2] explicitly as in the paper of T. Y. Thomas for general Riemann spaces of class one. C. B. Allendoerfer could avoid this difficulty for the Einstein spaces.[3] We shall obtain an analogous result for conformally flat spaces as follows.

I. We have from (0·3)

$$(1\cdot 1)\qquad H_{lm}R_{hijk}-H_{lk}R_{hijm}-H_{jh}R_{mkli}+H_{ji}R_{mklh}=0\,.^{(4)}$$

Contracting (1·1) by $g^{lm}g^{jh}$ and making use of (0·1) give

$$H_{ai}l_j^a - H_{aj}l_i^a = 0 \quad (l_j^a = g^{ab}l_{jc}) . \tag{1·2}$$

Next, contracting (1·1) by g^{hm} and taking account of (0·1) and (1·2), we have

$$g_{ij}H_{ka}l_i^a - g_{kl}H_{ia}l_j^a + l(H_{ij}g_{kl} - H_{kl}g_{ij}) + (n-3)(H_{ij}l_{kl} - H_{kl}l_{ij}) = 0 \quad (l = l_a^a) . \tag{1·3}$$

Moreover, contracting by g^{kl} gives $nH_{ia}l_j^a = (H_b^a l_a^b - Hl)g_{ij} + (2n-3)lH_{ij} - (n-3)Hl_{ij}$, $(H_b^a = g^{ac}H_{bc}$, $H = H_a^a)$. Substituting from this expression in (1·3) and putting $K_{ij} = nl_{ij} - lg_{ij}$, we have finally

$$n(K_{ij}H_{kl} - K_{kl}H_{ij}) + H(g_{ij}K_{kl} - g_{kl}K_{ij}) = 0 . \tag{1·4}$$

We see immediately that C_n is of constant curvature if, and only if, $K_{ij} = 0$.

II. Referring to the coordinate system (C), we have $g_{ij} = H_{ij} = l_{ij} = K_{ij} = 0$ $(i \neq j)$, $g_{ii} = 1$, $(i = l, \cdots, n)$, $H_{22} = \cdots = H_{nn}$, $l_{22} = \cdots = l_{nn}$, $K_{22} = \cdots = K_{nn}$. Evidently C_n is flat for $H_{22} = 0$. If $H_{11} = 0$, we have $H = (n-1)H_{22}$ and from (1·4) $K_{11} - K_{22} = 0$; and we see easily from (1·4) that $H_{11} = H_{22} = \cdots = H_{nn}$; so that C_n is of constant curvature. Consequently C_n, being not of constant curvature, is of type n,[5] since $H_{11} \neq 0$, $H_{22} = \cdots = H_{nn} \neq 0$, $H_{ij} = 0$. $(i \neq j)$. (C_n of constant curvature, of course, is of type n).

If $K_{22} = 0$, we have $l_{11} = l_{22}$ and therefore C_n is of constant curvature. If $K_{11} = 0$, we have similarly $l_{11} + l_{22} = 0$ and from (0·4) $H_{11}H_{22} = 0$; so that C_n is also of constant curvature.

Consequently we have the

Lemma :—*If conformally flat $C_n (n \geqq 3)$, which is not of constant curvature, is of class one, C_n is of type n and the determinant $|K_{ij}|$ does not vanish.*

From det. $|K_{ij}|$ we make conjugate K^{ij} of K_{ij} and contracting (1·4) by K^{kl} we have

$$H_{ij} = ag_{ij} + bl_{ij} ; \tag{2·1}$$

where a and b are scalars. Substituting (2·1) in (0·3) and making use of (0·1) we have

$$(e-ab)R_{hijk} = a^2(g_{hj}g_{ik} - g_{hk}g_{ij}) + b^2(l_{hi}l_{ik} - l_{hk}l_{ij}) . \tag{2·2}$$

If the matrix $\|l_{ij}\|$ is of rank one, the last term of (2·2) vanishes and hence $(e-ab)$ is equal to zero ; since otherwise C_n would be of constant curvature. Therefore we have $a=0$; so that we meet with contradiction, *i. e.*, $e=0$. Thus we have the necessary condition (A) that $\|l_{ij}\|$ *is of rank* $\geqq 2$.

III. Now, from (2·2), consider a system of linear homogeneous equations

$$(3\cdot 1) \qquad tR_{hijk}=A(g_{hj}g_{ik}-g_{hk}g_{ij})+B(l_{hj}l_{ik}-l_{hk}l_{ij}) \ ;$$

where t, A and B are unknowns. The system has non-trivial solutions t, A and B if, and only if,

$$(3\cdot 2) \qquad \begin{vmatrix} R_{abcd} & g_{ac}g_{bd}-g_{ad}g_{bc} & l_{ac}l_{bd}-l_{ad}l_{bc} \\ R_{hijk} & g_{hj}g_{ik}-g_{hk}g_{ij} & l_{hj}l_{ik}-l_{hk}l_{ij} \\ R_{pqrs} & g_{pr}g_{qs}-g_{ps}g_{qr} & l_{pr}l_{qs}-l_{ps}l_{qr} \end{vmatrix}=0 \, .$$

Consequently it is necessary for C_n of class one that (3·2) are satisfied.

Conversely, we can prove that this condition and (A) are sufficient. In fact, suppose that we have a system of solutions t, A and B; where $t=0$. Making use of the equation $A(g_{hj}g_{ik}-g_{hk}g_{ij})+B(l_{hj}l_{ik}-l_{hk}l_{ij})=0$ and referring to the coordinate system $(\bar{C})$; where $g_{ij}=l_{ij}=0$ $(i\neq j)$, $g_{ii}=1$; we see easily that C_n is of constant curvature, in contradiction to hypothesis. Consequently $t\neq 0$ and hence we have

$$(3\cdot 3) \qquad R_{hijk}=C(g_{hj}g_{ik}-g_{hk}g_{ij})+D(l_{hj}l_{ik}-l_{hk}l_{ij}) \, .$$

Referring to the coordinate system $(\bar{C})$ we know $CD=1$. Now we put

$$(3\cdot 4) \qquad a=+\sqrt{\frac{cC}{2}}\, , \quad b=+\sqrt{\frac{eD}{2}}\ ;$$

where $l=+1$ or -1, according as C (D) is positive or negative. We define H_{ij} by (2·1) and making use of (0·1) we see immediately that those H_{ij} satisfy the Gauss equation (0·3).

From the lemma and the theorems in Thomas's paper,[6] it follows that above H_{ij} satisfying the Gauss equation is unique to within algebraic sign and that satisfies the Codazzi equation

$$H_{ij,k}-H_{ik,j}=0 \, .$$

because of type $n(>3)$. As the result we have the

Theorem:—*If* $n(\geqq 4)$*-dimensional Riemann space* C_n *with a metric defined by a positive definite quadratic differential form is conformal to a flat* *space and not of constant curvature, C_n is of class one if, and only if, the matrix andition (A) and the equation (3·2) are satisfied. Then the type number of C_n is equal to n.*

IV. Now we give another proof of the theorem of Brinkmann that *class number of conformally flat space is at most two.*[7] Let us define H^P_{ij} (P I, II) as follows :

$$H^{\mathrm{I}}_{ij}=\frac{1}{\sqrt{2}}(g_{ij}+l_{ij}), \quad H^{\mathrm{II}}_{ij}=\frac{1}{\sqrt{2}}(g_{ij}-l_{ij}) ; \tag{4·1}$$

then we have from (0·1) the Gauss equation

$$R_{hijk}=l_P(H^P_{hj}H^P_{ik}-H^P_{hk}H^P_{ij}) \quad (l_{\mathrm{I}}=+1,\ l_{\mathrm{II}}=-1), \tag{4·2}$$

and according to (0·2)

$$H^P_{ij,k}-H^P_{ik,j}=\frac{l_P}{\sqrt{2}\,(n-2)}R_{ijk} ; \tag{4·3}$$

where we put

$$R_{ijk}=R_{ij,k}-R_{ik,j}-\frac{1}{2(n-1)}(g_{ij}R_{,k}-g_{ik}R_{,j}). \tag{4·4}$$

We know well that R_{ijk} vanishes for conformally flat space $C_n (n\geqq 3)$. Hence, if we define $H^P_{Qi}=0$, then we have the Codazzi equation $H^P_{ij,k}-H^P_{ik,j}=l_Q(H^Q_{ij}H^Q_{Pk}-H^Q_{ik}H^Q_{Pj})$, and also the Ricci equation

$$H^P_{Qi,j}-H^P_{Qj,i}=g^{ab}(H^Q_{ai}H^P_{bj}-H^Q_{aj}H^P_{bi}) ;$$

so that $C_n (n\geqq 3)$ is imbedded always in flat E_{n+2}.

It will be interesting to compare (2·1) with (4·1).

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References

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(3) Bull. Amer. Math. So., 43 (1937), pp. 265–270.
(4) T. Y. Thomas, l. c., Cf. (8.2).
(5) l. c., Cf. § 5.
(6) l. c., Cf. §§ 5, 6.
(7) Pro. Nat. Acad. Sci. U.S.A., 9 (1923), pp. 1–3.