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# Some Remarks concerning p-adic Number Field

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The content of this paper is already published in 1942 in Japanese ([1]) but as after that time several remarks and developments in this direction were made by other authors, most of them being also Japanese, it would not be meaningless to translate the results in European language. Especially Nakayama, who obtained a generalizaton of my result, hoped me to quote his result together, if the translation is attempted in future.

**1.** The results obtained by myself are the following two theorems. In this paper all the concerning fields are p-adic number fields and $G(k/K)$ denote the group of numbers $A$ in $K$ such that $N_{Kk}(A)=1$.

**Theorem 1.** (*Ordering theorem*). *If the fields $K/k$ and $K/k'$ are both relative abelian, then the necessary an a sufficient condition that $k'\subset k$ holds is $G(k/K)\subset G(k'/K)$.*

**Theorem 2.** *If $K/k$ is abelian of degree $n$ and a factor set of exponent $n$, then the group $G(k/K)$ is generated by the numbers of the form $a_{\sigma,\tau}/a_{\tau,\sigma}$ and $b^{1-\sigma}$:*

$$G(k/K)=\{a_{\sigma,\tau}/a_{\tau,\sigma},\ K^{1-\sigma}\}.$$

I called the latter theorem "Hauptgeschlechtssatz im Minimalen". Clearly this is a generalization of the well-known Hilbert's norm theorem concerning cyclic fields.

**2.** Next I will give a brief sketch of the further developments obtained by other authors.

T. Nakayama [2] generalized the theorem 2 as follows.

**Theorem 2'.** *If $K/k$ is a normal field with the Galois group $\mathfrak{G}=\{\rho,\sigma,\tau,\ldots\}$, then the group $G(k/K)$ consists of the products of the elements*

$$b^{1-\rho}(b\in K,\ \rho\in\mathfrak{G})$$

*and*

$$b(\sigma,\tau)/b(\tau,\sigma)\ \ (\textit{where}\ (b)\sim(a)\ \textit{and}\ b(\sigma\tau,\mathfrak{G})=b(\tau\sigma,\mathfrak{G})).$$

*Thereby $(a)$ is a fixed factor set corresponding to a division algebra.*

The proof of this theorem will be given in the below.

In this paper we use the terminology "factor set" in the following sense: i.e. a system of elements $a_{\sigma,\tau}$ (or $a(\sigma, \tau)$ if it is convenient in writing) of $K$ with

$$a_{\rho,\sigma\tau}\ a_{\sigma,\tau}=a_{\rho\sigma,\tau}\ a^{\tau}_{\rho,\sigma}.$$

We use also the abbreviation $a(\sigma, \mathfrak{G})=\Pi a(\sigma, \tau)(\tau \in \mathfrak{G})$, and so on.

Y. Matsushima [3] proved the following theorem, which constitutes a converse of Hilbert's norm theorem for the special fields.

**Theorem 3.** *If $K/k$ is abelian and $G(k/K)$ consists of the elements of the form $\Pi b^{1-\sigma}$, then the Galois group of $K/k$ is cyclic.*

He presented also the question, if in this theorem the restriction "$K/k$ is abelian" could be omitted or not.

G. Toyoda [4] treated this problem, but his proof contained a small mistake.

H. Kuniyoshi [5] proved the following theorem, which solves Matsushima's problem negatively.

**Theorem 4.** If $\mathfrak{G}$ is the Galois group of $K/k$, and its successive commutator factors

$$\mathfrak{G}/\mathfrak{G}',\ \mathfrak{G}'/\mathfrak{G}'',\ldots$$

are all cyclic, then the group $G(k/K)$ consists of the elements of the form $\Pi b^{1-\sigma}$.

Kuniyoshi [6] also decided the factor group $G(k/K)/K^{1-\sigma}$ completely in terms of the Galois group ($K/k$ being abelian), and his result contains theorem 3 as an obvious consequence.

My theorem 2 would become easier to understand if we recall the fact

$$a_{\sigma,\tau}/a_{\tau,\sigma}=u_{\sigma}^{-1}\ u_{\sigma}^{-1}\ u_{\sigma}\ u_{\tau},$$

where $u_{\sigma}, u_{\tau}\ldots$ are the usual basis for crossed product with

$$u_{\sigma}^{-1}\ a\ u_{\sigma}=a^{\sigma},\quad u_{\sigma}\ u_{\tau}=u_{\sigma\tau}\ a_{\sigma,\tau}.$$

From this standpoint I presented a conjecture, which was proved afterward to be true by Nakayama and Matsushima [7], and the result is

**Theorem 5.** *If $D$ is a division algebra over $k$, then all the elements of $D$ with the reduced norm 1 belong to the commutator group of multiplicative group $D$.*

Recently S. Wang [8] showed that this theorem is also true for the case of simple algebras over algebraic number fields.

**3.** We shall now prove the theorem 1.

**Lemma 1.** *From the assumptions* $K \supset k$, $K \neq k$ *follows* $(K^* : k^*) = \infty$, *where* $*$ *means the multiplicative group of non zero elements.*

*Proof.* We assume $(K^* : k^*) = n < \infty$ contrary to our assertion, and select a system of representatives $\alpha_0 = 1, \alpha_1, \ldots, \alpha_{n-1}$ of the factor group $K^*/k^*$. As $\alpha = \alpha_1 \notin k$, we have also $1+\alpha, 2+\alpha, \ldots \notin k$. If we put $m+\alpha = \beta_m a_m$ ($a_m \in k$, $\beta_m$ being one of the representatives $\alpha_1, \alpha_2, \ldots$), all the numbers $\beta_1, \beta_2, \ldots$ are different, for if $\beta_l = \beta_m$ ($l \neq m$) we have

$$\beta_m = \frac{l-m}{a_l - a_m} \in k$$

contrary to the assumption, and our lemma is also proved.

**Lemma 2.** *If the fields* $K/k$ *and* $K/k'$ *are both relative abelian and* $k' \subset k$, $k' \neq k$, *then the (proper) implication relation* $G(k/K) \subset G(k'/K)$ *holds.*

*Proof.* Obviously it is sufficient to prove that $G(k/K) \neq G(k'/K)$. We assume that the equality holds. We denote by $N(A) = a$ and $N'(A) = a'$ the norms with respect to $K/k$ and $K/k'$ respectively, and put $(K : k) = n$, $(k : k') = n'$. Then we have

$$N(a^{n'}/a') = 1,$$

so that it follows from the translation theorem of the local class field theory

$$a^{n'}/a' \in N(K^*), \quad a' = N(B) \in N(K^*)$$

and from which

$$N'(A^{n'}/B) = 1.$$

From the assumption $G(k/K) = G(k'/K)$ we have then $N(A^{n'}/B) = 1$, that is $a^{n'} = a' \in k'$, $N(K^*)^{n'} \subset k'^*$.

We have especially

$$k^{*nn'} \subset N(K^*)^{n'} \subset k'^* \subset k^*,$$

and as $(k^* : k^{*nn'}) < \infty$ holds,

$$(k^* : k'^*) < \infty.$$

From the lemma 1 then follows $k = k'$ as was required.

**Proof of the theorem 1.** It remains only to prove following fact: If $K=kk'$ and $K/k$, $K/k'$ are relatively abelian, then from the assumption $G(k/K) \subset G(k'/K)$ we can infer $k' \subset k$.

We put $k''=k \cap k'$, $(K:k)=n$, $(K:k')=n'$ and $K=k''(\theta)$. If $\theta=\theta^{(1)}, \theta^{(2)}, \ldots, \theta^{(n)}$ are the relative conjugates with respect to $K/k$,

$$f(x)=(x-\theta^{(1)})(x-\theta^{(2)})\ldots(x-\theta^{(n)})$$

is the irreducible polynomial in $k[x]$, with $\theta$ as a root, and

$$f(a)=N(a-\theta),$$

if $a \in k''$, where $N$ denotes the norm with respect to $K/k$, and likewise $N'$ for the field $K/k'$. The equality $N\{H^n/f(a)\}=1$ and the assumption $G(k/K) =G(k/K)$ yield $N'\{H^n/f(a)\}=1$, hence

$$N'\{f(a)\}=N'(H^n)$$

or

$$f_1(a)f_2(a)\ldots f_{n'}(a)=(a-\theta_1)^n(a-\theta_2)^n\ldots(a-\theta_{n'})^n,$$

where $f_i(x)$ and $\theta_i$ means the conjugate polynomials and numbers of $f(x)$ and $\theta$ respectively with respect to $K/k'$. As the above equality holds for any $a \in k''$, it holds also for arbitrary $x$ instead of $a$. Especially each root $\theta^{(i)}$ of $f(x)=f_1(x)$ coincide with one of $\theta_j$, and from which we conclude easily the required relation $k' \subset k$, by Galois-theoretical consideration, if we recall that $K/k''$ is a normal field.

**4.** We now proceed to the proof of the theorem 2', and assume that the theorem is already proved to be true for the fields of lower degrees.

As the Galois group $\mathfrak{G}$ of $K/k$ is soluble, we can choose a cyclic intermediate field $Z/k$, corresponding subgroup of which being denoted by $\mathfrak{N}$. We then have the decomposition of the form

$$\mathfrak{G}=\mathfrak{N}+\mathfrak{N}\zeta+\mathfrak{N}\zeta^2+\ldots+\mathfrak{N}\zeta^{m-1}.$$

Now suppose that the relative norm $N(A)$ of $A$ with respect to $K/k$ is 1. As

$$N(A)=N''(N'(A))=1,$$

where $N'$ and $N''$ denote the norms with respect to $K/Z$ and $Z/k$ respectively, we have by Hilbert's norm theorem

$$N'(A)=z^{1-\zeta} \quad (z \in Z). \tag{1}$$

But since $\{a(x, \lambda)\}$ $(x, \lambda \in \mathfrak{R})$ is a factor set corresponding to a normal division algebra over $Z$, we see by Nakayama and Akizuki's theorem [9], that there exists an element $\mu \in \mathfrak{R}$ with

$$z=a(\mu, \mathfrak{R})\ N'(c) \quad (c \in K), \tag{2}$$

and fiom which we deduce

$$z^{1-\zeta}=a(\mu, \mathfrak{R})^{1-\zeta}\ N'(c^{1-\zeta}). \tag{3}$$

If $\nu$ runs over $\mathfrak{R}$, we have

$$\begin{aligned} a(\mu, \mathfrak{R})^{1-\zeta} &= \Pi(a(\mu, \nu)/a(\mu, \nu)^{\zeta}) \\ &= \Pi(a(\mu, \nu)/a(\mu, \nu\zeta)\ a\ (\nu, \zeta)a(\mu\nu, \zeta)^{-1}) \\ &= a(\mu, \mathfrak{R})/a(\mu, \mathfrak{R}\zeta) \\ &= \Pi(a(\zeta\mu, \nu)a(\zeta, \mu)^{\nu}a(\zeta, \mu\nu)^{-1} \\ &\quad /a(\mu\zeta, \nu)a(\mu, \zeta)^{\nu}a(\zeta, \nu)^{-1}) \\ &= N'(a(\zeta, \mu)/(\mu, \zeta))a(\zeta\mu, \mathfrak{R}) \\ &\quad /a(\mu\zeta, \mathfrak{R}). \end{aligned} \tag{4}$$

We observe a factor set $(b)$, which is associated to $(a)$:

$$b(\sigma, \tau)=a(\sigma, \tau)l_{\sigma}^{\tau}\ l_{\tau}/l_{\sigma\tau}\ ,$$

then

$$\begin{aligned} &b(\zeta\mu, \mathfrak{R})/b(\mu\zeta, \mathfrak{R}) \\ &=a(\zeta\mu, \mathfrak{R})/a(\mu\zeta, \mathfrak{R})\cdot N'(l_{\zeta\mu}/l_{\mu\zeta}). \end{aligned}$$

Now we distinguish two cases ascording as (i) $\zeta\mu=\mu\zeta$, or (ii) $\zeta\mu\neq\mu\zeta$.

First suppose that (i) holds. Then we have for $(a(\sigma, \tau))$ itself as $(b(\sigma, \tau))$,

$$b(\mu, \mathfrak{R})^{1-\zeta}=N'(b(\zeta, \mu)/b(\mu, \zeta)). \tag{5}$$

But if (ii) holds, we have anyhow a relation of the foim

$$a(\zeta\mu, \mathfrak{R})/a(\mu\zeta, \mathfrak{R})=N'(f) \qquad (f \in K)$$

so that for a suitable system $(l_{\sigma})$ with $l_{\zeta\mu}/l_{\mu\zeta}=f^{-1}$ (5) holds likewise.

It is easy to see that the relation of the form

$$z=b(\mu, \mathfrak{R})N'(a) \qquad (a \in K) \tag{6}$$

holds, and from which we have, combining with (5)

$$N'(A)=\varepsilon^{1-\zeta}=b(\mu,\ \mathfrak{N})^{1-\zeta}\ N'(a^{1-\zeta}) \qquad (7)$$
$$=N'(b(\zeta,\ \mu)/b(\mu,\ \zeta)\cdot a^{1-\zeta}).$$

By the assumption of the mathematical induction $A(b(\zeta,\ \mu)/b(\mu,\ \zeta)\cdot a^{1-\zeta})^{-1}$ can be represented as the product of several elements $b^{1-\lambda}(\lambda \in \mathfrak{N})$ and

$$b'(\varkappa,\ \lambda)/b'(\lambda,\ \varkappa) \qquad ((b'(\varkappa,\ \lambda))\sim(a(\varkappa,\lambda)),$$
$$b'(\varkappa\lambda,\ \mathfrak{N})=b'(\lambda\varkappa,\ \mathfrak{N})=1, \qquad \varkappa,\lambda \in \mathfrak{N}).$$

But it is clear that $(b'(\varkappa,\ \lambda))$ can be extended to a factor set $(b')\sim(a)$, and from $N'(b'(\varkappa,\ \lambda)/b'(\lambda,\ \varkappa))=1$ it follows that

$$N(b'(\varkappa,\ \lambda)/b'(\lambda,\ \varkappa))$$
$$=b'(\varkappa\lambda,\ \mathfrak{S})/b'(\lambda\varkappa,\ \mathfrak{S})=1,$$

and the theorem is completely proved.

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