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On Continuous Geometries, II.

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In Part I of this paper[1] we have introduced a dimension function with values in a conditionally complete lattice-group, into an arbitrarily given continuous geometry, and imbedded the geometry into the direct sum of irreducible ones. We have proved, thereby, that the dimension is restrictedly additive[2], whence follows immediately the unrestricted additivity of perspectivity[3] This latter additivity had been already proved, however, as we were informed of after the publication of part I by I. Halperin[4]. In the following lines we shall show that the former additivity can be deduced easily from the latter (as was remarked in Part I). Also we shall give a new proof to Halperin's theorem of superposition of decompositions as an application of our theory.

All this will be done in generalizing the method of Part I in a certain sense. We shall namely show to what extent our previous method can be applied to obtain a generalized dimension function and the imbedding theorem, when we replace the perspective relation by an equivalence relation with some natural restrictions. In particular, it should be an extension of perspectivity, that is, any two elements should be in this relation if they are perspective. An example of such extension is that induced by a group of automorphisms of the geometry, considered by Halperin[5] and F. Maeda[6]. In this specified case, our restrictions are stronger than Maeda's, and weaker than Halperin's, and our dimension function can be obtained from Maeda's by means of the representation of a conditionally complete lattice-group by real-valued continuous functions. But, this being concerned only with the dimension function, the subject of this note may, as we hope, appeal to wider interest.

§ **I.** This section is devoted to some preparatory considerations about a conditionally complete, and so abelian, lattice-group, which may be of some interest in themselves, The letter $\mathfrak{G}$ will denote throughout this paper, unless otherwise qualified, always such a group and $f, g, h, \ldots\ldots$ its elements. These letters will be used with or without indices. If we write f', we mean an element of such a lattice-group $\mathfrak{G}'$ with the above mentioned property.

Given a non-empty system of elements $f_\gamma \leqq 0$ such that the set of sums $f_{\gamma_1}+\dots\dots+f_{\gamma_n}$ $(\gamma_i \neq \gamma_j$ for $i \neq j)$ of all its finite subsystems has an upper bound in $\mathfrak{G}$, we denote by $\sum f_\gamma$ the supremum of this set of sums. When the system is finite, this is nothing but the ordinary sum. If γ ranges over all ordinal numbers $<$a limit number λ, then (cf. Part I, §7)

$$\sum_{\alpha<\lambda} f_\gamma = \sup_{\mu<\lambda} \sum_{\gamma<\mu} f_\gamma = \sup_{\mu<\lambda} \sum_{\gamma<\mu} f_\gamma.$$

The *generalized commutative-associative law* holds in the following sense: Let $f_{\alpha\beta} \geq 0$ be elelments with double suffixes. If either $\sum_{\alpha\beta} f_{\alpha\beta}$ or $\sum_\alpha \sum_\beta f_{\alpha\beta}$ exists, then the other also exists, and they coincide. We shall omit here the easy proof.

Lemma 1.1. If $0 \leq g_\gamma$ and $0 \leq f \leq \sum g_\gamma$, there exists a system of elements $h_j \geq 0$ such that $f=\sum h_j$, $h_j \leq g_j$.

Proof. We shall suppose, as it is obviously permitted, that γ ranges over all ordinal numbers $<\alpha$, where α is an ordinal number>0. Define by transfinite induction on $\beta<\alpha$:

$$h_0 = f \cap g_0, \quad h_\beta = (f - \sum_{\gamma<\beta} h_\gamma) \cap g_\beta,$$

where the summands h_γ should be ≥ 0. Let β be an ordinal number such that $0<\beta<\alpha$. Suppose that, for all $\gamma<\beta$, $h_\gamma \geq 0$ are defined and

$$(1) \qquad f \cap \sum_{\xi<\gamma} g_\xi = \sum_{\xi<\gamma} h_\xi.$$

This holds for $\beta=1$. When we have shown that this yields (1) for $\gamma=\beta$ again, we can conclued immediately that the assumption holds for $\beta+1$ in place of β, and consequently for $\beta=\alpha$, which proves the lemma. Now. if β is a limit number, we nave only to take the supremum with respect to all $\gamma<\beta$ on both sides of (1). Otherwise, let η be the immediate predecessor of β, and put

$$\bar{h} = \sum_{\xi<\eta} h_\xi, \quad \tilde{g} = \sum_{\xi<\eta} g_\xi.$$

Then we have $f \cap \tilde{g} = \bar{h}$, $(f-\bar{h}) \cap g_\eta = h_\eta$ and

$$f \cap (\tilde{g}+g_\eta) = f \cap (f+g_\eta) \cap (\tilde{g}+g_\mu) = f \cap ((f \cap \tilde{g})+g_\eta)$$
$$= (\bar{h}+f-\bar{h}) \cap (\bar{h}+g_\eta) = \bar{h} + ((f-\bar{h})+g_\eta) = \bar{h}+h_\eta.$$

Hence $f \cap \sum g_\xi = \sum h_\xi$ with $\xi \leq \eta$, i.e. $\xi < \beta$. q.e.d.

Theorem 1. *If* $0 \leq f_\alpha$, $0 \leq g_\beta$ and $\sum f_\alpha \leq \sum g_\beta$, *where* α *and* β *are independent, then there exists a system of elements* $h_{\alpha\beta} \leq 0$ *such that*

$$f_\alpha = \sum_\beta h_{\alpha\beta}, \quad \sum_\alpha h_{\alpha\beta} \leq g_\beta. \tag{2}$$

If $\sum f_\alpha = \sum g_\beta$, *then from* (2) *follows*

$$g_\beta = \sum_\alpha h_{\alpha\beta}. \tag{3}$$

Proof. We shall suppose, without loss of generality, that α ranges over all ordinal numbers $< \alpha_0$, where α_0 is an ordinal number > 0. Let us define $h_{\alpha\beta}$ by transfinite induction as follows. By $f_0 \leq \sum g_\beta$ we obtain a system of elements $h_{0\beta} \leq 0$ such that $f_0 = \sum h_{0\beta}$, $0 \leq h_{0\beta} \leq g_\beta$. Let α be an ordinal number, $0 < \alpha < \alpha_0$. Suppose that $h_{\xi\beta} \leq 0$ are defined for all β and for all $\xi < \alpha$. Suppose further

$$\sum_{\xi<\alpha} h_{\xi\beta} \geq g, \quad f_\xi = \sum_\beta h_{\xi\beta}.$$

Then, by the commutative-associative law,

$$\sum_\beta (g_\beta - \sum_\xi h_{\xi\beta}) + \sum_\beta \sum_\xi h_{\xi\beta} = \sum_\beta g_\beta$$
$$\geq \sum_\xi f_\xi + f_\alpha = \sum_\xi \sum_\beta h_{\xi\beta} + f_\alpha = \sum_\beta \sum_\xi h_{\xi\beta} + f_\alpha,$$

and consequently $f_\alpha \leq \sum_\beta (g_\beta - \sum_\xi h_{\xi\beta})$. Hence we obtain by the lemma a system of elements $h_{\alpha\beta} \geq 0$ such that

$$h_{\alpha\beta} \leq g - \sum_{\xi<\alpha} h_{\xi\beta}, \quad f_\alpha = \sum_\beta h_{\alpha\beta}.$$

Thus we have a system of elements $h_{\alpha\beta}$ defined for all α and β for which the property (2) is obvious from construction. As for the second part of the theorem, we make use of the commutative-associative law as above, and obtain $\sum_\beta (g - \sum_\alpha h_{\alpha\beta}) = 0$, which implies (3). q.e.d.

Remark. The assumption that $\mathfrak{G}$ be conditionally complete is needed only for the commutativity of $\mathfrak{G}$ and for the existence of the sum $\sum$ when there is an infinite number of summands. This remark is useful in the proof of the following lemma.

Lemma 1.2. *Let* $\mathfrak{G}$ *be an abelian* (*not necessarily conditionally complete*) *lattice-group with an archimedian unit e, and let* $\mathfrak{G}'$ *be any (partially) ordered abelian group. If* $f \rightarrow f'$ *is a mapping of the subset* $(f;\ 0 \leq f \leq e)$ *of* $\mathfrak{G}$ *into* $\mathfrak{G}'$ *such that* $f' \geq 0$ *in* $\mathfrak{G}'$ *and* $(f+g)' = f'+g'$ *when* $0 \leq f, g, f+g \leq e$, *then the mapping can be uniquely extended to an order preserving homomorphism of* $\mathfrak{G}$ *into* $\mathfrak{G}'$. *If, moreover,* $\mathfrak{G}'$ *also has an archimedian unit* e' *and if every element* $f' \in \mathfrak{G}'$ *such that* $0 \leq f' \leq e'$ *is an image of the given mapping, then the extension maps* $\mathfrak{G}$ *onto* $\mathfrak{G}'$.

Proof. The theorem 1, together with the above remark, implies

$$\sum f'_i \leq \sum g'_j \text{ when } \sum f_i \leq \sum g_j,\ 0 \leq f_i \leq e,\ 0 \leq g_j \leq e,$$

where $\sum$ denotes now the ordinary sum of a finite number of summands; in particular, we have $\sum f'_i = \sum g'_j$ when $\sum f_i = \sum g_j$. Hence, if $f = f_1 + \dots$ $\dots + f_m - g - \dots\dots - g_n$, $0 \leq f_i \leq e$, $0 \leq g_j \leq e$, then $f' = f_1' + \dots\dots + f' - g_1 - \dots$ $\dots - g_n'$ is determined by f and does not depend on particular choice of its expression $f = f_1 + \dots\dots + f_m - g_1 - \dots\dots - g_n$, and we have $f' \leq 0$ if $f \leq 0$. Such an expression exists for every $f \in \mathfrak{G}$, since e is an archimedian unit of the lattice-group $\mathfrak{G}$. Thus the existence and uniqueness of the extension to an order preserving homomorphism is proved. The second part of the lemma is obvious from the fact that, in this case, every element $f' \in \mathfrak{G}'$ admits an analogous expression $f' = f_1' + \dots\dots + f'_m - g_1' - \dots\dots - g_n'$ with $0 \leq f' \leq e'$, $0 \leq g \leq e'$. q.e.d.

§ 2. Let L be a continuous geometry. We denote its elements by a, b, c, x, y, with or without indices. We denote by $\sum^{\perp} x_\gamma$ the sum $\sum x_\gamma$ of an independent system of elements x_γ; when, moreover, the system is finite, we write also e.g. $x_1 \dot{+} \dots\dots \dot{+} x_n$. For each pair of elements $a \leq b$ we fix once for all, an element c such that $a = b \dot{+} c$, i.e. $a = b + c$ and $bc = 0$, and denote it by $a - b$.

Remark. The fact that the element $c = a - b$ is subject to no other restriction than $a = b \dot{+} c$ will be made use of in the proof of Corollary 4 to Lemma 4.8.

A function $\delta(x)$ defined on L and with values in a lattice-group $\mathfrak{G}$ will be called a *δ-function* when it satisfies the following the conditions.

(4) $$0 \leq \delta(x) \quad \text{in } \mathfrak{G};$$

(5) $$\delta(x \dot{+} y) = \delta(x) + \delta(y);$$

(6) $$\delta(\sum_\gamma{}^{\perp} x_\gamma) = \delta(\sum_\gamma{}^{\perp} y_\gamma) \text{ if } \delta(x_\gamma) = \delta(y_\gamma) \text{ for all } \gamma;$$

(7) if $\delta(a) \leq \delta(b)$ there exists an element $x \leq b$ such that $\delta(x) = \delta(a)$;

(8) if $0<f\in\mathfrak{G}$ there exists an element x such that $0<\delta(x)\leq f$.
From (5) follows that $\delta(0)=0$ and that

(9) $\delta(x)=\delta(y)$ if x and y are perspective.

When $\delta(1)$ is an archimedian unit of $\mathfrak{G}$. we call $\mathfrak{G}$ the *domain-group* of the δ-function. Given a δ-function, its domain-group is uniquely determined as the set of all $f\in\mathfrak{G}$ such that $-n\cdot\delta(1)\leq f\leq n\cdot\delta(1)$ for some integer n. It is obvious that this set is a conditionally complete sublattice and a subgroup of $\mathfrak{G}$.

When

(10) $$0<x \text{ implies } 0<\delta(x)$$

or equivalently,

(11) $$x<y \text{ implies } \delta(x)<\delta(y),$$

we call the δ-function a *dimension function*. This is a generalization of the concept of dimension function introduced in part I, for which the converse of (9) holds as well. By (9) and its converse, the propeıty (6) of this special dimension funtion is equivalent to the unrerstricted additivity of perspectivity :

(12) $\sum^{\perp}x_\gamma$ and $\sum^{\perp}y_\gamma$ are perspective, if, for each γ, x_γ and y_γ are perspective.

The additivity (12) was established by Halperin without the aid of dimension function. The following theorem 2, therefore, affords a new proof to the unrestricted additivity of the special dimension function :

(13) $$\delta(\sum^{\perp}x_\gamma)=\sum\delta(x_\gamma),$$

which we proved in Part I and from which we deduced (12). (12) will not be used in the following proof of the theorem 2.

Theorem 2. *Every δ-function satisfies* (13).

Proof, Let Γ be the range of γ. When Γ is a finite set, (13) follows immediately from (5). Hence we have only to consider the case where Γ is of potency $\aleph\geq\aleph_0$ such that (13) holds whenever Γ is of potency $<\aleph$. Fuıther, we can and shall suppose that Γ is the set of all ordinal numbers $<\lambda$, where λ is the first ordinal number such that the set of all ordinal numbers $<\lambda$ is of potency $\aleph$.

Put

$$f=\delta(\sum_{\gamma<\lambda}^{\perp}x_\gamma)-\sum_{\gamma<\lambda}\delta(x_\gamma)$$

and suppose $f\neq 0$. Then, since (13) holds for any finite system and since $\sum f_\gamma$ in $\mathfrak{G}(f_\gamma\geq 0)$ is defined as the supremum of the sums of finite number of elements f_γ, we have $0<f\leq\delta\ (\sum^{\perp}x_\gamma)$. By (7) and (8) there exists an element $x\leq\sum^{\perp}x_\gamma$ such that $0<\delta(x)\leq f$. Put $y=\sum^{\perp}x_\gamma-x$. By transfinite induction we can define y_γ for *all* $\gamma<\lambda$ as an element $\leq y-\sum_{\alpha<\gamma}y_\alpha$ with $\delta(y_\gamma)=\delta(x_\gamma)$, because, as will be shown presently, if $\beta<\lambda$ and if y_γ are defined for all $\gamma<\beta$ then $\delta(x_\beta)\leq\delta(y-\sum_{\gamma<\beta}y)$ and consequently, by (7) and (8) again, y_β can be defined.

In fact, the system $(y_\gamma\ ;\ \gamma<\beta)$ is indepedent and its suffix γ ranges over a set of potency $<\aleph$; hence (13) can be applied to it and yields

$$\begin{aligned}\delta(x_\beta)&\leq\sum_{\gamma<\lambda}\delta(x_\gamma)-\sum_{\gamma<\beta}\delta(x_\gamma)=\delta(\sum_{\gamma<\lambda}^{\perp}x_\gamma)-f-\sum_{\gamma<\beta}\delta(x_\gamma)\\&\leq\delta(\sum_{\gamma<\lambda}^{\perp}x_\gamma)-\delta(x)-\sum_{\gamma<\beta}\delta(x_\gamma)=\delta(y)-\delta(\sum_{\gamma<\beta}^{\perp}y_\gamma)\\&=\delta(y-\sum_{\gamma<\beta}y_\gamma)\end{aligned}$$

Thus we obtain an independent system of elements y_γ defined for all $\gamma<\lambda$ and, by (4) and (5),

$$\delta(\sum^{\perp}x_\gamma)=\delta(\sum^{\perp}y_\gamma)\leq\delta(y)=\delta(\sum^{\perp}x_\gamma)-\delta(x),$$

which is in contradiction to $0<\delta(x)$. Hence we should have $f=0$, q.e.d.

Lemma 2.1. *An element $f\in\mathfrak{G}$ is a value of the δ-function $\delta(x)$ if and only if $0\leq f\leq\delta(1)$.*

Proof. If $f=\delta(x)$, then by (4) and (5)

$$0\leq f\leq\delta(x)+\delta(1-x)=\delta(1).$$

Conversely, suppose $0\leq f\leq\delta(1)$ and consider any independent set S of elements of L such that $\sum\delta(y_i)\leq f$ for any finite number of different elements $y_i\in S$. We speake of a *set* S and not a *system* S to imply that no element is to be considered twice or more times as belonging to S. Put $x=\sum^{\perp}y : y\in S$.

Then $\delta(x)\leq f$ by (13). If $\delta(x)<f$, then there exists an element y such that $0<\delta(y)\leq f-\delta(x)$, and we can choose $y\leq 1-x$, since $f<\delta(x)<\delta(1-x)$. From $0<\delta(y)$ and follows $y\notin S$, for, otherwise we should have $y\leq x$ and consequently $y=0$. Hence we can add y to S, to obtain a larger set with the same property as above. But, by Zorn's lemma, there exists a maximal one among such sets S, and we have $\mathring{\delta}(x)=f$ for this maximal S, q. e. d.

An expression $a=\sum^{\perp}a_{\alpha}$ will be called a *decomposition* of a. It will be called a *refinement* of an expression $a=\sum b_{\beta}$ when every $a_{\alpha}\leq$ some b_{β} (depending on a_{α}).

Lemma 2.2. *Let $\delta(x)$ be a dimension function. If $\delta(a)=\sum f_{\gamma}$ and $f_{\gamma}\geq 0$ in its domain group, then there exists a decomposition $a=\sum^{\perp}a_{\gamma}$ with $\delta(a_{\gamma})=f_{\gamma}$.*

Proof. Consider an independent system of elements $a_{\gamma'}\geq a$ with $\delta(a_{\gamma'})=f_{\gamma'}$, γ' ranging over a subset Γ' of the range Γ of γ in $\sum f_{\gamma}$. If $\sum^{\perp}a_{\gamma'}\neq a$ then $\sum^{\perp}a_{\gamma'}<a$, $\sum\delta(a_{\gamma'})<\delta(a)$ and consequently $\Gamma'\neq\Gamma$. When $\Gamma'\neq\Gamma$, let β be an element of Γ which is not contained in Γ'. Then

$$f_{\beta}+\delta(\textstyle\sum^{\perp}a_{\gamma'})=f_{\beta}+\sum f_{\gamma'}\leq\delta(a),$$
$$f_{\beta}\leq\delta(a)-\delta(\textstyle\sum^{\perp}a_{\gamma'})=\delta(a-\sum^{\perp}a_{\gamma'})$$

and, by Lemma 2.1. and (7), $f_{\beta}=\delta(x)$ for some $x\leq a-\sum^{\perp}a_{\gamma'}$, which implies that the system of elements $a_{\gamma'}$ can be augmented. Taking a maximal system, therefore, we have $\Gamma'=\Gamma$ and $a=\sum^{\perp}a_{\gamma}$, q.e.d.

The maximal method in these two lemmas was already used in the proof of Lemma 9.1, Part I. The proof consisted essentially in the following fact, which we formulate here for later use.

Lemma 2.3. *Any expression $a=\sum b_{\beta}$ admits a refinement, that is, for any suah expression there exists a decomposition $a=\sum^{\perp}a_{\alpha}$ such that every $a\leq$ some b_{β}.*

§ 3. Closely related to the concept of dimension function is that of *p-relation*, which is a generalization of perspectivity. By this we mean a binary relation $x\sim y$ defined in a continuous geometry L, satisfying the following conditions:

(14) $x\sim y$ is an equivalence relation;

(15) it is an extension of perspectivity, that is, if x, y are perspective, then holds $x\sim y$;

(16) it is unrestrictedly additive, i,e. $\sum^{\perp}x_{\gamma}\sim\sum^{\perp}y_{\gamma}$ if $x_{\gamma}\sim y_{\gamma}$ for each γ;

(17) if $x\sim\sum^{\perp}y_{\gamma}$, there exists a decomposition $x=\sum^{\perp}x_{\gamma}$ with $x_{\gamma}\sim y_{\gamma}$;

(18) every element x is incompressible in the sense that $x\sim$no element $<x$.

It is easily seen that (18) can can be replaced by a seemingly weaker condition :

$$(18)' \qquad y \sim 1 \text{ implies } y=1.$$

Suppose, in fact, $x \sim x' \leq x$. Then we have, by (14) and (16), $1=(1-x) \dot{+} x \sim (1-x)+x'$, and by (18)' $(1-x)\dot{+}x'=1$, which implies $x'=x$.

If a dimension function $\delta(x)$ is given and $x \sim y$ is defined to mean $\delta(x)=\delta(y)$, then $x \sim y$ is obviously a p-relation. In particular, (17) follows from Lemma 2.2. This will be called the p-relation *induced* by $\delta(x)$. It will be shown that every p-relation can be induced by some dimension function (cf. Theorem 5). The dimension function that induces a given p-relation is uniquely determined up to order preserving isomorphisms of its domain group.
This fact follows from

Lemma 3.1. *If $\delta(x)$ and $\delta'(x)$ are δ-functions defined on a continuous geometry, with domain groups $\mathfrak{G}$ and $\mathfrak{G}'$ and if $\delta(x)=\delta(y)$ implies $\delta'(x)=\delta'(y)$, then there exists an order preserving homomorphism of $\mathfrak{G}$ onto $\mathfrak{G}'$ which carries $\delta(x)$ into $\delta'(x)$ for every x. Such a homomorphism is uniquely determined. On the functions $\delta(x)$ $\delta'(x)$ and we have only to assume that*

(*i*) *The values of $\delta(x)$ are contained in an abelian lattice-group $\mathfrak{G}$ with $\delta(1)$ as an archimedian unit,*

(*ii*) *An element $f \in \mathfrak{G}$ is a value of $\delta(x)$ if and only if $0 \leq f \leq \delta(1)$.*

(*iii*) $\delta(x+y)=\delta(x)+\delta(y)$,

and the corresponding properties of $\delta'(x)$ and $\mathfrak{G}'$, (These conditions are clearly verified, if $\delta(x)$ and $\delta'(x)$ are δ-functions.)

This is an obvious consequence of Lemma 1.1 and Lemma 2.1.

We shall consider, from now on, a continuous geometry with a p-relation $x \sim y$. We write $a \lesssim b$ when $a \sim$ some $x \leq b$. This relation will be called *p-inclusion.* It is the generalization of the perspective inclusion, corresponding to the generalization of perspectivity to p-relation. It is also an extension of perspective inclusion since p-relation is an extension of perspectivity ; in particular, $a \leq b$ implies $a \lesssim b$.

Lemma 3.2. (*i*) $a \leq b \sim b'$ implies $a \lesssim b'$. (*ii*) *$a \gtrsim b \gtrsim c$ implies $a \gtrsim c$.* (*iii*) *$a \leq b \sim a$ implies $a=b$.* (*iv*) *$a \lesssim b \lesssim a$ implies $a \sim b$.*

Proof. (*i*) : Corresponding to the decomposition $b=a\dot{+}(b-a)$ we have a decomposition $b'=a'+c$ such that $a \sim a'$, $b-a \sim c$, hence $a \lesssim b'$.

(*ii*) : We have $a \sim$ some $x \leq b$ and $b \sim$ some $b' \leq c$ and, by (*i*), $x \lesssim b'$, i.e. $x \sim$ some $y \leq b$, which implies $a \sim y \leq c$. (*iii*) is an immediate consequence of the incompressibility of b. (*iv*) : We have $a \sim$ some $a' \leq b$ and, for such a', we have $a' \leq b \sim a'$, which implies $a' = b$ by (*iii*) : hence $a \sim b$.

Remark. (*iv*) can be proved without the assumption of incompressibility. The proof will then be analogous to the usual proof of Bernstein's theorem in the theory of sets.

Theorem 3. *If* $a = \sum a_\alpha \lesssim b = \sum b_\beta$, *there exists a pair of refinements* $a = \sum^{\perp} a'_\gamma$, $b = \sum^{\perp} b_\gamma$, of $a = \sum a_\alpha$, $b = \sum b_\beta$ *such that* $a'_\gamma \lesssim b'_\gamma$ *for every* γ. *If, moreover,* $a = \sum^{\perp} a_\alpha$ *and* $b = \sum^{\perp} b_\beta$, *then we can choose* γ *as pairs* (α, β) *in such a way that* $a_\alpha = \sum^{\perp} a'_{\alpha\beta}$ *and* $b_\beta = \sum^{\perp} b'_{\alpha\beta}$. *If* $a \sim b$, *we can replace p-inclusion by p-relation in both statements above.*

Remark. This is a slight generalization of Halperin's theorem of superposition of decompositions loc. cit. But the following proof is based on a new idea.

Proof. Replacing $a = \sum a_\alpha$ and $b = \sum b_\beta$ by their refinements, we can reduce the first part of the theorem to the second part. We shall prove only the second part, because the last part can be obtained by replacing p-relations with p-inclusions in the following considerations.

Let us first consider the case when the given p-inclusion is the perspectivity. Let $\delta(x)$ be the dimension function defined in part I. Then we have

$\sum \delta(a_\alpha) = \delta(a) \leq \delta(b) = \sum \delta(b_\beta)$, $0 \leq \delta(a_\alpha)$, $0 \leq \delta(b_\beta)$ and consequently, by Theorem 1., there exists a system of elements $h_{\alpha\beta} \geq 0$ in the domain group of $\delta(x)$ such that $\delta(a_\alpha) = \sum_\beta h_{\alpha\beta}$, $\sum_\alpha h_{\alpha\beta} \leq \delta(b_\beta)$. Let b_β' be elements $\leq b_\beta$ with $\delta(b'_\beta) = \sum_\beta h_{\alpha\beta}$ (cf. Lemma 2.1.), and let $b'_\beta = \sum_\alpha b''_{\alpha\beta}$ be their decompositions with $\delta(b''_{\alpha\beta}) = h_{\alpha\beta}$ (cf. Lemma 2.2). From these we can easily construct the decompositions $b_\beta = \sum_\alpha^{\perp} b'_{\alpha\beta}$ with $\delta(b'_{\alpha\beta}) \leq h_{\alpha\beta}$. Further, let $a_\alpha = \sum^{\perp} a_{\alpha\beta}'$ be decompositions with $\delta(a'_{\alpha\beta}) = h_{\alpha\beta}$, then we have a desired pair of decompositions, since $\delta(a'_{\alpha\beta}) \leq \delta(b'_{\alpha\beta})$ implies $a'_{\alpha\beta} \lesssim b'_{\alpha\beta}$.

Let us now consider the general case. Let a' be an element such that $a \sim a' \leq b$. and $a' = \sum^{\perp} a'_\alpha$ be a decomposition such that $a_\alpha \sim a'_\alpha$. Of course a' is perspectively included in b, and by the part of theorem proved above for the case of perspective inclusion, we obtain a pair of refinements $a' = \sum^{\perp} a''_{\alpha\beta}$, $b = \sum^{\perp} b'_{\alpha\beta}$ such that, for any α and β, $a''_{\alpha\beta}$ is perspective to some element $\leq b''_{\alpha\beta}$, and $a'_\alpha = \sum^{\perp} a''_{\alpha\beta}$, $b_\beta = \sum^{\perp} b'_{\alpha\beta}$. In particular, we have

$a''_{\alpha\beta} \lesssim b'_{\alpha\beta}$. Let $a_\alpha = \sum^{\perp} a'_{\alpha\beta}$ be decompositions such $a'_{\alpha\beta} \sim a''_{\alpha\beta}$. Then we have $a'_{\alpha\beta} \lesssim b'_{\alpha\beta}$. q.e.d.

§4. We now generalize the concept of *centre* to that of *relative centre* with respect to the given p-relation. The center of L will be, as before, denoted by Z. We define the relative center Z^* as the set of all $z \in L$ such that

(19) $$x \sim z \text{ implies } x = z.$$

Lemma 4.1. *An element z of L belongs to Z^* if and only if*

(19)′ $$x \lesssim z \text{ implies } x \leq z$$

Proof. By Lemma 3.2 (iii) we have only to show that every element $z \in Z^*$ has the property (19)′. Suppose $x \lesssim z \in Z^*$. Then, from $x - xz \leq x$ follow successively $x - xz \lesssim x$, $x - xz \lesssim z$, $x - xz \sim$ some $y \leq z$, $(x - xz) \dot{+} (x - y) \sim y \dot{+} (z - y) = z$ and, by (19), $(x - xz) \dot{+} (z - y) = z$. Hence $x - xz \leq z$, that is $x \leq z$. q.e.d.

Elements of Z^* will be denoted by z, z_1, z', etc.

Lemma 4.2. *Z^* is a subset of Z and closed in Z and closed in Z with respect to the operations $1 - z$, $\Pi\, z_\gamma$ and $\sum z_\gamma$. It contains* 0 *and* 1 05 *L, and constitutes a complete Boolean algebra.*

Proof. $0 \in Z^*$ and $1 \in Z^*$ follow from the incompressibility. The definition of Z^* implies that any element of Z^* is perspective to no other element than itself. Hence $Z^* \subseteq Z$. From Lemma 4.1 follows that Z^* is closed under the operation Πz_γ, that is, $\Pi z_\gamma \in Z^*$ for any set of elements $z_\gamma \in Z^*$. Now we have only to show that it is closed under the operation $1 - z$, since $\sum z_\gamma = 1 - \Pi (1 - z_\gamma)$. Suppose $x \leq 1 - z$; then $xz \lesssim 1 - z$, i.e. $xz \sim$ some $y \leq 1 - z$, and for such y we have $y \gtrsim z$, since $y \sim xz \leq z$. Hence $y \leq z$ by Lemma 4.1, and consequently $y = 0$. It follows that $xz \sim 0$, which implies $xz = 0$, i.e. $x \leq 1 - z$ (note that $z \in Z$). By Lemma 4.1, therefore, $1 - z \in Z^*$. q.e.d.

Lemma 4.3. *$a \lesssim b$ implies $za \lesssim zb$ for any $z \in Z^*$. In particular, $a \sim b$ implies $za \sim zb$.*

Proof. We have only to prove the first part. From $za \leq a \lesssim b$ follows $za \lesssim b$, i.e. $za \sim$ some $x \geq b$; for such x we have $x \lesssim z$, $x \geq z$ and so $x \leq zb$. Hence $za \leq zb$. q.e.d.

Corresponding to the "central cover" in von Neumann's theory[8] we define, for every $a \in L$,

$$a^*=\Pi z: \quad z \leq a,$$

or, what is the same,

$$a^*=\Pi z: \quad a \lesssim z.$$

Lemma 4.4. *(i)* $a \leq a^* \in Z^*$. *(ii)*. $za^*=(za)^*$. *(iii)* $a \lesssim b$ *implies* $a^* \leq b^*$. *(iv)* $(a+b)^*=a^*+b^*$.

Proof is obvious from Lemma 4.3.. and Lemma 4.3.

Lemma 4.5. $a^*=\sum x$, *where* $\sum$ *extends over all* $x \lesssim a$.

Proof. We have only to show $\sum x \in Z^*$, because $a \leq \sum x \leq a^*$ is obvious. Suppose $y \lesssim \sum x$, then, by Theorem 3, there exists decomposition $y=\sum^{\perp} y_\alpha$ such that every $y_\alpha \lesssim$ some x, and consequently $y_\alpha \lesssim a$, from which follows $y \leq \sum x$. By Lemma 4.1, we have therefore, $\sum x \in Z^*$.

Lemma 4.6. $a^*b^*=\sum w$, *where* $\sum$ *extends over all* w *such that both* $w \lesssim a$ *and* $w \lesssim b$ *hold.*

Proof. We have only to show $a^*b^* \leq \sum w$, since $\sum w \leq a^*$ and $\sum w \leq b^*$ by Lemma 4.5. Let x and y denote arbitrary elements $\lesssim a$ and $\lesssim b$ respectively. Then $a^*b^* \leq \sum x$, $a^*b^* \leq \sum y$ and, by Theorem 3, there exist two decompositions $a^*b^*=\sum^{\perp} u$, $a^*b^*=\sum^{\perp} v$ such that every $u \lesssim$ some x, every $v \lesssim$ some y. Further, there exists a pair of their refinements $a^*b^*=\sum^{\perp} u'_\gamma$, $a^*b^*=\sum^{\perp} v'_\gamma$ respectively, such that $u'_\gamma \sim v'_\gamma$. From $u'_\gamma \leq u \lesssim x \lesssim a$ follows $u'_\gamma \lesssim a$, and from $u'_\gamma \sim v'_\gamma \leq v \lesssim y \lesssim b$ follows $u' \lesssim b$. Hence every u'_γ is a w. There fore $a^*b^*=\sum u'_\gamma \leq \sum w$. q.e.d.

Lemma 4.7. *If* $z=\sum z_\gamma$ *and* $z_\gamma a \lesssim z_\gamma b$ *for all* z_γ, *then* $za \lesssim zb$. *If, in particular,* $z=\sum z_\gamma$ *and* $z_\gamma a \sim z_\gamma b$, *then* $za \sim zb$.

Proof. We have only to prove the first part. Since Z^* itself is clearly a continuous geometry, we can construct a refinement $z=\sum^{\perp} z'_\alpha$ of $z=\sum^{\perp} z_\gamma$ in Z^*. Then we have $za=\sum^{\perp} z'_\alpha a$, $zb=\sum^{\perp} z'_\alpha b$, since z'_α are in Z. For every z'_α there is a z_γ such that $z'_\alpha \leq z_\gamma$, and $z'_\alpha a=z'_\alpha z_\gamma a \lesssim z'_\alpha z_\gamma b=z'_\alpha b$ by Lemma 4.3. Hence $za \lesssim zb$. Q.e.d.

We write $a<b$ when $a \sim$ some $x<b$. By Lemma 3.2 (iii), this is equivalent to the condition that $a \lesssim b$ holds while $a \sim b$ does not.

We write $a \ll b$ when, for every $z \in Z^*$, either $za<zb$ or $za=zb=0$ holds. Of couse, $a \ll b$ implies $a \lesssim b$ and $z_0 a \ll z_0 b$ for all $z_0 \in Z^*$.

Theorem 4. *For any pair of elements* a, b *there exists a decompesition* $1=z_1 \dot{+} z_2 \dot{+} z_3$ *such that*

$$z_1 a \ll z_1 b, \quad z_2 b \ll z_2 a, \quad z_3 a \sim z_3 b.$$

Prooof. There exists, by Zorn's lemma, a maximal set of pairs (a_γ, b_γ) with the following propeities:

(20) the elements a_γ are independent,

(20)′ the elements b_γ are independent,

(21) $a \geq a_\gamma \sim b_\gamma \leq b$.

Take such a maximal set, and put $a_0 = \sum^{\perp} a_\gamma$, $b_0 = \sum^{\perp} b_\gamma$. Then $a \geq a_0 \sim b_0 \leq b$. Put $z_1' = (b-b_0)^*$, $z'_2 = (a-a_0)^*$. Then $z_1' z_2' = 0$ by Lemma 4.6 and by our choice of a maximal set. Put $z'_3 = 1-(z_1'+z_2')$. From $z_3'(a-a_0) \leq z_2' z_3' = 0$ follows $z_3' a = z_3' a_0$; similarly we obtain $z_3' b = z_3' b_0$, $z_2' b = z_2' b_0$, $z_1' a = z_1' a_0$, we have therefore

$$z'_1 a \lesssim z_1' b,\ z_2' b \lesssim z_2' a,\ z_3' a \sim z_3' b.$$

Now we define $z_3 = \sum z'$, $z_1 = z_1'\ (1-z_3)$, $z_2 = z_2'(1-z_3)$, where $\sum$ extends over all z' such that $z' a \sim z' b$. Then, by Lemma 4.7, we get $z_3 a \sim z_3 b$, and $1 = z_1 \dot{+} z_2 \dot{+} z_3$, since $z'_3 \leq z_3$. Of course $zz_1 a \leq zz_1 b$ holds for any z; but if $zz_1 a \sim zz_1 b$ then $zz_1 \leq z_3$, $zz_1 = 0$, $zz_1 a = zz_4 b = 0$. Therefore $z_1 a \lessdot z_1 b$; similarly $z_2 b \lessdot z_2 a$. q.e.d.

Lemma 4.8. *If* $b_1 \dot{+} b_2 \lesssim a_1 \dot{+} a_2$ *and* $a_1 \lesssim b_2$ *then* $b_2 \lesssim a_2$.

Proof. Obviously, we have $y_1 \dot{+} y_2$ non $\lesssim x_1 \dot{+} x_2$ if $x_1 \lesssim x_2$, $y_1 < y_2$. Now let $1 = z_1 \dot{+} z_1 \dot{+} z_3$ be a decomposition satisfying $z_1 a_2 \lessdot z_1 b_2$, $z_2 b_2 \lessdot z_2 a_2$, $z_3 a_2 \sim z_3 b_2$. Then $z_1 a_2 < z_1 b_2$ would imply $z_1 b_1 \dot{+} z_1 b_2$ non $\lesssim z_1 a_1 \dot{+} z_1 a_2$ in contradiction to $z_1(b_1 \dot{+} b_2) \lesssim z_1(a_1 \dot{+} a_2)$, which folllows from $b_1 \dot{+} b_2 \lesssim a_1 \dot{+} a_2$. Hence $z_1 a_2 = z_1 b_2 = 0$, and $b_2 = z_2 b_2 + z_3 b_2 \lesssim z_2 + z_3 a_2 = a_2$. q.e.d.

Corollary 1. *If* $a_1 \dot{+} a_2 \sim b_1 \dot{+} b_1$ *and* $a_1 \sim b_1$ *then* $a_2 \sim b_2$.

Corollary 2. $x \sim y$ *and* $1-x \sim 1-y$ *are equivalent.* $x \lesssim y$ *and* $1-x \lesssim 1-y$ *are equivalent.*

Thus the lemma affords a sort of dualty principle for p-relation and p-inclusion. Another example is

Corollary 3. *If* $a + a' = b + b' = 1$, $a \sim b$, $a' \sim b'$, *then* $aa' \sim bb'$.

We defined $a \lesssim b$ by $a \sim$ some $a' \leq b$ and not by $a \leq$ some $b' \sim b$. which is dual to the former condition. But by Lemma 3.2. the latter implies the former, and we can now prove the converse. Thus these two conditions are equivalent:

Corollary 4. *If* $a \lesssim b$, *there exists an element* b' *such that* $a \leq b' \sim b$. In fact, we have $1-b \lesssim 1-a$, i.e. $1-b \sim$ some $x \leq 1-a$. We can suppose $1-x \geq a$ and $1-(1-b) = b$, according to the Remark in §2.

We have then $b \sim 1-x \geq a$.

Remark. Consideratisns in this § could be much more visualized if we had regarded $x \sim y$ as an equivalence relation between the values $\delta(x)$ and $\delta(y)$, where $\delta(x)$ is the dimension function defined in part I. Such a version is admitted since $x \sim y$ is an extension of perspectivity and $\delta(x) = \delta(y)$ is equivalenty to perspectivity of x and y. However it would have slightly complicated our statements.

§ 5. If we call a *p-class* a set of the from $K(a) = (x\,;\ x \sim a)$, the geometry L is decomposed into mutually exclusive p-classes $A, B, C, \ldots\ldots$, since $x \sim y$ is an equivalence ralation. We write $K(a) \leq K(b)$ when $a \lesssim b$, $K(a) \ll K(b)$ when $a \ll b$, and we denote $K(a+b)$ by $K(a)+K(b)$ when there exists a pair of representative elements a, b of $K(a)$, $K(b)$ such that $ab=0$. For these definitions the particular choice of representative elements a, b of p-classes is obviously irrelevamt. As in part I, § 1, we can define the multiplication of p-classes by elements of Z^* and by rational numbers. We then define *p-types* of a geometry with respect to the given p-relation, in the same way as we have defined the 'type' of a geometry in part I, § 2.

Then we can prove, following the analogy to von Neumann loc. cit, that any geometry L is isomorphic to a direct sum $\sum \oplus L_k$ of p-type k, where the isomorphism is considered together with p-relations, and the p-relation in the direct sum is defined component by component, that is, $\sum \oplus x_k \sim \sum \oplus y_k$ if and only if $x_k \sim y_k$ in each L_k.

Therefore, we can and shall consider only a geometry of some p-type k. We denote by Δ^* the set of all real numbers or of all rational numbers $\frac{n}{k}$ $(n=0, \pm 1, \pm 2, \pm 3, \ldots\ldots)$, according as $k=\infty$ or $k<\infty$. Let Ω^* be the Boolean space corresponding to the Boolean algebra Z^*, and let $\mathfrak{G}^*$ be the lattice-group of all continuous functions on Ω^* with values in Δ^*. Finally, let us call L *p-irreducible* when Z^* contains only the elements O and 1.

Then we obtain the following theorem just as we have obtained the theorem 6 in part I or by the more elegant method of Y. Kawada, Y, Matsushima and K. Higuchi[9].

Theorem 5. *We can attach to each point* $M \in \Omega^*$ *a continuous geometry* L_M *and a mapping* $x \rightarrow x_M$ *of* L *onto* L_M *in such a manner that*

(22) *L is lattice-isomorphically imbedded into the direct sum $\sum_M \oplus L_M$ by the mapping $x \to \sum \oplus x_M$,*

(23) *Each L_M is p-irreducible with respect to a p-relation induced by a dimension function δ^*,*

(24) *For each fixed $x \in L$ the function $\delta^*(x_M)$ of $M \in \Omega^*$ belongs to $\mathfrak{G}^*$*

(25) *if we denote this element of $\mathfrak{G}^*$ by $\delta^*(x)$ then we obtain a dimension function of $x \in L$ with $\mathfrak{G}^*$ as its domain group,*

(26) *The given p-relation in L coincides with that induced by $\delta^*(x)$.*

(27) *For any fixed $z \in Z^*$, the function $\delta^*(z_M)$ of M is the characteristic function of the open-and-closed subset $\Omega(z) \subseteq \Omega$ corresponding to z by the complete representation of Z^* in Ω^*,*

(28) $$\delta^*((za)_M) = \delta^*(z_M) \cdot \delta^*(a_M).$$

From these properties follows furthermore:

(29) Each L_M is of the same p-type as L.

(30) *If* $a = \sum x$ and $b = \Pi x$, x ranging over any given subset of L, then the set of all points $M \in \Omega^*$, for which $a_M \neq \sum x_M$ or $b_M \neq \Pi x_M$, is of first category in Ω^*.

Remark. This theorem is concerned with a geometry of some p-type. But it affords a criterion of p-irreducibility of any geometry: *A geometry is p-irreducible if and only if its p-relation is induced by a real valued dimension function.* For the proof we have only to remark that if the geometry is p-irreducible or its p-relation is induced by a real valued dimension function, it must be of some p-type.

Let us write Ω, $\mathfrak{G}$, δ and p instead of Ω^*, $\mathfrak{G}^*$, δ^* and M respectively, when we take perspectivity for p-relation. This coincides the notation used in Part I, except that Ω was identified with 1 previously; this is the Boolean space corresponding to the Boolean algebra Z.

Now let us suppose that L is of some type and some p-type at the same time, and let us observe the relation between the 'components' x_M and x_p.

If M is fixed,

$$\mu_M(\Omega(e)) = \delta^*(e_M) \quad (e \in Z)$$

is a finitely additive measure defined for the open-and-closed sets $\Omega(e)$ in

Ω. As $\Omega(e)$ is bicompact, we have

$$\mu_M(\Omega(e))=\sum_{n=1}^{\infty} \mu_M(\Omega(e_n))$$

if $\Omega(e_n)$ $(e_n \in Z)$ are disjoint and

$$\Omega(e)=\bigcup_{n=1}^{\infty} \Omega(e_n).$$

So μ_M can be extended to a completely additive measure in Ω, which also will be denoted by μ_M.

By Lemma 3.1 and by the fact that the p-relation is an extension of perspectivity, there exists an order-preserving homomorphis $f \rightarrow f^*$ of $\mathfrak{G}$ onto $\mathfrak{G}^*$, which carries $\delta(x)$ into $\delta^*(x)$.

This homomoephism yields an additive functional $f \rightarrow f^*(M)$ and the functional can be represented, as is easily seen, by the integration

$$f^*(M)=\int_{\Omega} f(p) \ \mu_M(dp). \tag{31}$$

Hence

Theorem 6. *Any dimension function δ^* is determined by its values $\delta^*(e)$ for $e \in Z$.*

Further we denote by $\bar{M}$ the intersection of all $\Omega(e)$ with $e \in Z$ and

$$\delta^*(e_M)=\mu_M(\Omega(e))=1.$$

As a closed subspace of Ω, $\bar{M}$ is a Boolean space, and relatively open-and-closed subsets U of $\bar{M}$ are of the from $\Omega(e) \cap \bar{M}$.

Suppose

$$U=\Omega(e_1) \cap M=\Omega(e_2) \cap \bar{M}.$$

Then the symmetric difference $X=(\Omega(e_1)) \cup \Omega(e_2))-(\Omega(e_1) \cap \Omega(e_2))$

is bicompact and contained in the complement of $\bar{M}$ in Ω; hence X is covered by the open sets $\Omega(e)$ with $\mu_M(\Omega(e))=0$ and so by a finite number of these. Therefore $\mu_M(\Omega(e_1))=\mu(\Omega(e_2))$.

So we can define a finitely additive measure m_M in $\bar{M}$ by

$$m_M(U)=\mu_M(\Omega(e)), \quad U=\Omega(e) \cap M,$$

and we can extend it, as before, to a completely additive one, which again will be denoted by m_M. It is positive for non-empty open sets in $\bar{M}$, since such a set contains a non-empty U.

Now (31) becomes

$$f^*(M)=\int_{\bar{M}} f(p)\ m(dp). \tag{32}$$

When $f \geq 0$, therefore, we have $f^*(M)=0$ if and only if $f(p)=0$ for all $p \in \bar{M}$. Put $f=\delta(a+b)-\varrho(ab)$. Then $f \geq 0$, and we have a series of equivalent conditions: $a_M=b_M$; $(a+b)_M=(ab)_M$; $f^*(M)=0$; $f(p)=0$ for all $P \in \bar{M}$; $(a+b)_p=(ab)_p$ for all $p \in \bar{M}$; $a_p=b_p$ for all $p \in \bar{M}$.

Therefore the correspondence

$$a_M \longleftrightarrow \sum_{p \in M} \oplus a_p$$

is one-to-one, and so lattice-isomorphic. Thus we have obtained the following theorem except the last part.

Theorem 7. *L_M is lattice-isomorphically imbedded in $\sum \oplus L_p$ by $a \rightarrow \sum \oplus a_p$, where $\sum$ ranges over all $p \in \bar{M}$. The dimension $\delta^*(a_M)$ is obtained by integrating the function $\delta(a_p)$ of p over $\bar{M}$ by a completely additive measure m_M which is positive for non-empty open sets. This imbeding coincides with the one obtained by Theorem 5 (or Theorem 6, part I) for L_M considered as a geometry with perspectivity as p-relation.*

As for the last part we have only to remark that $\bar{M}$ can be considered as the representation space of the center of L_M, as the central elements of L_M are of the from e_M $(e \in Z)$ and vice versa.

It may be of some interest that the sets $\bar{M}$, $\bar{M}'$ corresponding to different points M, M' of Ω^* can be separated by the sets $\Omega(z)$. In fact, if $M \neq M'$ there is an element $z \in Z^*$ such that $M \in \Omega^*(z)$ and $M' \in \Omega - \Omega^*(z)=\Omega^*(1-z)$; but $M \in \Omega^*(z)$ implies $\delta^*(z)=1$ and so $\bar{M} \subseteq \Omega(z)$; $M' \in \Omega(1-z)$ implies $\bar{M} \subseteq \Omega(1-z)$ i.e. $\bar{M}' \cap \Omega(z)=0$.

References and Bibliography

1) T. Iwamura: On Continuous Geometries, I; Jap. Jour. Math. vol. XIX, 1944.
2) Theorem 3, § 8 in part I.
3) Corollary to the theorem above.
4) I. Halperin: Additivity and Continuity of Perspectivity; Duke Math. Jour. vol. 5, no. 3, 1939.

5) I. Halperin: Dimensionality in Reducible Geometries: Ann. Math. vol. 40 no. 3, 1939.

6) F. Maeda: Dimension-lattice of Reducible Geometries (in Japanese), Science Report of Hiroshima Bunrika-Daigaku vol. 13, 1939.

7) Cf. K. Iwasawa: On the Structure of Conditionally Complete Lattice-Group; Jap. Jour. Math, Vol. XVIII, 1943.

8) J. von Neumann: Lectures on Continuous Geometry, III.

9) Y. Kawada, K. Higuchi, Y. Matsushima: Bemerkungen über die voranghenden Arbeit von Herrn T. Iwamura:, Jap. Jour. Math, vol. XIX, 1944.