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Conformal Mapping of Polygonal Domains.*

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§ 1. Introduction.

It is well known that a function, which maps a circular disc or a half-plane onto the interior of a polygon, is given by the formula of Schwarz-Christoffel. Let $w=f(z)$ be such a function and let the image-polygon, laid on w-plane, have m vertices corresponding to the points a_{μ} ($\mu=1,\ldots, m$) on z-plane. Denoting by $\alpha_{\mu}\pi\,(0<\alpha_{\mu}<2)$ the interior angle at vertex $f(a_{\mu})$, the Schwarz-Christoffel formula may be written as follows:

$$f(z)=C\int^{z}\prod_{\mu=1}^{m}(a_{\mu}-z)^{\alpha_{\mu}-1}dz+C', \tag{1·1}$$

where C and C' are both constants depending on position and magnitude of image-polygon.

The present author has previously shown that this formula can be generalized to the case of analogous mapping of doubly-connected domains.[1] We may take, as a standard doubly-connected basic domain, an annular domain $q<|z|<1$, $-\lg q$ being a uniquely determined conformal invariant, i. e. the so-called *modulus* (*Modul*) of given polygonal domain. Let the boundary components corresponding to circumferences $|z|=1$ and $|z|=q$ be polygons with m and n vertices respectively. Let further $\alpha_{\mu}\pi$ and $\beta_{\nu}\pi$ denote the interior angles (with respect to each boundary polygon itself) at vertices $f(e^{i\varphi_{\mu}})$ and $f(qe^{i\psi_{\nu}})$ respectively. The mapping function $w=f(z)$ is then given by the formula

*) A preliminary report under the same title has been published in Kōdai Math. Sem. Rep. Nos. 3–4 (1949), 47250.

1) Y. Komatu, Darstellungen der in einem Kreisringe analytischen Funktionen nebst den Anwendungen auf konforme Abbildung über Polygonalringgebiete. Jap. Journ. Math. **19** (1945), 203–215.

$$f(z)=C\int^{z} z^{ic^*-1}\prod_{\mu=1}^{m}\sigma(i\lg z+\varphi_\mu)^{\alpha_\mu-1}\Big/\prod_{\nu=1}^{n}\sigma_3(i\lg z+\psi_\nu)^{\beta_\nu-1}\cdot dz+C', \quad (1\cdot2)$$

where the sigma-functions are those of Weierstrass with primitive periods $2\omega_1=2\pi$ and $2\omega_3=-2i\lg q$, and the constant c^* is given by

$$c^*=\frac{\eta_1}{\pi}\left(\sum_{\mu=1}^{m}(1-\alpha_\mu)\varphi_\mu-\sum_{\nu=1}^{n}(1-\beta_\nu)\psi_\nu\right); \quad (1\cdot3)$$

the constants C and C' having similar meanings as before. It can be shown moreover that Schwarz-Christoffel formula (1·1) for basic domain $|z|<1$ may be regarded as being a limiting case of (1·2) when $q\to0$.

On the other hand, any function $w=f(z)$ which maps a circular disc or a half-plane on z-plane onto the interior of a circular polygonal domain, i. e. the interior of a polygon having circular arcs as sides, is linear-polymorphic. A differential equation of the third order of the form

$$\{f(z),\ z\}=R(z) \quad (1\cdot4)$$

holds good always for such a function $f(z)$. The left hand member of the equation denotes, as usual, Schwarzian derivative of $f(z)$ with respect to z, i. e.

$$\{f(z),\ z\}\equiv\frac{f'''(z)}{f'(z)}-\frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2=\frac{d^2}{dz}\lg f'(z)-\frac{1}{2}\left(\frac{d}{dz}\lg f'(z)\right)^2,$$

and $R(z)$ is a rational function which possesses, as poles of order two at most, the points $a_\mu\ (\mu=1,\dots,m)$ corresponding to the vertices of image-polygon. More precisely, if we denote by $\alpha_\mu\pi$ the interior angle at $f(a_\mu)$ of the image-polygon, we have, at poles in question,

$$\lim_{z\to a_\mu}(z-a_\mu)^2R(z)=\frac{1-\alpha_\mu^2}{2}.$$

The above mentioned results (1·1) and (1·4) are usually derived by making use of analytic continuability of mapping function, that is, by performing successive inversions with respect to boundary arcs. But the author of this paper previously pointed out that Schwarz-Christoffel formula (1·1) is deduced immediately also from Poisson's integral representation of functions analytic in a circular disc.[2] He then derived the formula (1·2) by means of Villat's integral representation[3] of functions analytic in an annular domain. However it will be shown that the formula (1·2) can also be derived by the classical method without difficulty.

We can, on the other hand, consider the problem of generalization of (1·4) corresponding to that of (1·1) to (1·2). In the present Note, we shall derive, from a more general stand-point, general relations corresponding to (1·1) and (1·4) in the case of multiply-connected domains, and then, by specifying them to doubly-connected case, we shall obtain the expression (1·2) again and the sesult generalizing (1·4) too.

§ 2. Mapping onto circular polygonal domains.

Consider, in w-plane, an n-ply-connected domain Δ whose boundary consists of n circular polygons $\Gamma_j(j=1,\dots, n)$, each Γ_j being formed by m_j circular arcs. We can now take several types of domains as standard n-ply-connected basic domains. But here we shall first take a domain D bounded by n full circles.[4] Such a domain D is uniquely determined for the given domain Δ, except possible linear transformations. A domain of this type is defined in general by $3n$ real parameters denoting the coordinates of centres and the radii of n boundary circles. But, since a linear transformation depends on 6 real parameters, essentially $3n-6$ real conformal invariants belong to an n-ply-connected domain (with non-boundary components) as *moduli*, provided $n>2$. In degenerating an exceptional case $n=2$, there exists just one invariant, and in the case $n=1$ there remains freedom corresponding to 3 real parameters.

Now, let the boundary circle of D corresponding to Γ_j be

$$C_j : |z-c_j|=r_j \qquad (j=1,\dots, n), \tag{2·1}$$

2) Y. Komatu, Einige Darstellungen analytischer Funktionen und ihre Anwendugen auf konforme Abbildung. Proc. Imp. Acad. Tokyo **20** (1944), 536-541.

3) For a brief proof of Villat's representation, see Y. Komatu, Sur la représentation de Villat pour les fonctions analytiques définies dans un anneau circulaire concentrique. Proc. Imp. Acad. Tokyo **21** (1945), 64-96.

4) As to the possibility of taking such a domain, see e. g. R. Courant, Plateau's problem and Dirichlet's principle. Ann. of Math. **38** (1937), 679-724; L. R. Ford, Automorphic functions. New York (1929), p. 279 et seq.

and the mapping function be, as before, $w=f(z)$. Let further the points corresponding to vertices of Γ_j be $a_{j\mu}(\mu=1,\dots,m_j)$ and the interior angle of Γ_j at its vertex $f(a_{j\mu})$ with respect to Δ be $\alpha_{j\mu}\pi(0\leqq\alpha_{j\mu}\leqq 2)$. The function $f(z)$ remains, of course, regular even on each interior part of C_j divided by $a_{j\mu}$. If we denote the inversion $z\mid z_j^*$ with respect to C_j by

$$z_j^*=\lambda_j(z)\equiv c_j+\frac{r_j^2}{\bar{z}-\bar{c}_j},$$

then $\lambda_j(z)$ being all linear in $\bar{z}$, the composed functions

$$l_{jk}(z)\equiv\lambda_j(\lambda_k(z)) \qquad (j,\ k=1,\dots,\ n) \tag{2·2}$$

are also all linear with respect to z. The transformation $z\mid l_{jk}(z)$ is composed of successive inversions with respect first to C_k and next to C_j. Since operation of inversion is involutory, i. e. the identity $\lambda_j^{-1}(z)=\lambda_j(z)$ holds, we have $l_{jj}(z)=z$ and

$$l_{jk}^{-1}(z)=\lambda_k^{-1}(\lambda_j^{-1}(z))=\lambda_k(\lambda_j(z))=l_{kj}(z).$$

The aggregate of all linear transformations corresponding to inversions repeated even times with respect to boundary circles (2·1) forms a group $\mathfrak{G}$ generated thus by $\binom{n}{2}$ linear transformations $z\mid l_{jk}(z)\ (j<k)$.

After these preparatory considerations, we shall now state a result generalizing (1·4) :

Theorem 1. Let $w=f(z)$ denote a mapping function from D onto Δ. Then

$$\{f(z),\ z\}dz^2 \tag{2·3}$$

is an automorphic differential belonging to the group $\mathfrak{G}$, whose fundamental domain may be composed of the basic domain D itself and its inverse image-domain with respect to any one of boundary circles of D (speaking more exactly, the fundamental domain must be the open kernel of closure of the above mentioned one). The function $\{f(z), z\}$, being meromorphic in $\bar{D}\equiv D+\sum_{j=1}^{n}C_j$, is regular everywhere possibly except at $a_{j\mu}(\mu=1,\dots,m_j;\ j=1,\dots,\ n)$, where a pole of order at most two appears as shows the following formula

$$\lim_{z\to a_{j\mu}}(z-a_{j\mu})^2\{f(z),\ z\}=\frac{1-\alpha_{j\mu}^2}{2}. \tag{2·4}$$

Proof. The mapping function $w=f(z)$ can be analytically continued to an in general, infinitely many-valued function. It maps thus a domain which results from the original one, D, by successive inversions with respect to C_j, onto a domain which results from Δ by corresponding inversions with respect to Γ_j. The Riemann surface of this analytic function lying on z-plane possesses, as its branch points, the points $a_{j\mu}$ and their equivalents obtained by the transformations of $\mathfrak{G}$. These continuations are performed on z-plane by applying all transformations of $\mathfrak{G}$ to the join of D and its inversion-image with respect to any one of boundary circles of D. Let now $z \mid L(z)$ be any transformation of $\mathfrak{G}$. The value of $f(L(z))$ is then obtained from $f(z)$ by applying a linear transformation which is caused on w-plane by successive inversions repeated even times corresponding to $L(z)$. We therefore have a functional relation of the form

$$f(L(z))=\frac{gf(z)+h}{g'f(z)+h'}, \tag{2·5}$$

the constants g, h, g', h' being determined by the order of inversions which constitute the transformation $L(z)$. If we transform the variables by $\xi=\xi(x)$ and $\eta=\eta(y)$, the Schwarzian derivative is transformed according to the formula

$$\{\eta, \xi\}=\{\eta, y\}\left(\frac{dy}{d\xi}\right)^2+\{y, x\}\left(\frac{dx}{d\xi}\right)^2-\{\xi, x\}\left(\frac{dx}{d\xi}\right)^2. \tag{2·6}$$

Putting for brevity $z^*=L(z)$, we hence have

$$\{f(z^*), z^*\}=\{f(z^*), f(z)\}\left(\frac{df(z)}{dz^*}\right)^2+\{f(z), z\}\left(\frac{dz}{dz^*}\right)^2-\{z^*, z\}\left(\frac{dz}{dz^*}\right)^2.$$

Now, since z^* is a linear function of z and $f(z^*)$ is, by (2·5), also a linear one of $f(z)$, we have $\{z^*, z\}=\{f(z^*), f(z)\}=0$. Consequently, the relation of automorphism

$$\{f(z^*), z^*\}dz^{*2}=\{f(z), z\}dz^2$$

holds good; namely any transformation $z^*=L(z)$ of $\mathfrak{G}$ leaves the differential expression (2·3) invariant. Clearly, Schwarzian derivative $\{f(z), z\}$ can possess its singularities only at $a_{j\mu}$ and their equivalents with respect to $\mathfrak{G}$. In the first place, if $\alpha_{j\mu}\neq 0$, each branch of $(f(z)-f(a_{j\mu}))^{1/\alpha_{j\mu}}$ is regular in a vicinity of $a_{j\mu}$ and possesses this own point as zero point of the first order. Hence $f(z)$ is expressed there as

$$f(z)=f(a_{j\mu})+b_{j\mu}(z-a_{j\mu})^{\alpha_{j\mu}}+\dots \quad (b_{j\mu}\neq 0).$$

We calculate the successive derivatives of this expression and form the Schwarzian derivative as follows:

$$f'(z)=b_{j\mu}\alpha_{j\mu}(z-a_{j\mu})^{\alpha_{j\mu}-1}+\dots,$$

$$\frac{d}{dz}\lg f'(z)=\frac{\alpha_{j\mu}-1}{z-a_{j\mu}}+\dots, \qquad \frac{d^2}{dz^2}\lg f'(z)=-\frac{\alpha_{j\mu}-1}{(z-a_{j\mu})^2}+\dots, \quad (2\cdot 7)$$

$$\{f(z),z\}=-\frac{\alpha_{j\mu}-1}{(z-a_{j\mu})^2}+\dots-\frac{1}{2}\left(\frac{\alpha_{j\mu}-1}{z-a_{j\mu}}+\dots\right)^2=\frac{1}{2}\,\frac{1-\alpha_{j\mu}^2}{(z-a_{j\mu})^2}+\dots.$$

which yields the desired relation (2·4). In the second place, if $\alpha_{j\mu}=0$, we use intermediately the expression exp $(b'_{j\mu}/(f(z)-f(a_{j\mu})))$ in place of $(f(z)-f(a_{j\mu}))^{1/\alpha_{j\mu}}$ and can arrive again to the result (2·4) with $\alpha_{j\mu}=0$.

In a particular case, $n=1$, that is, when Δ is simply-connected, $\mathfrak{G}$ degenerates to a trivial group composed of a unique transformation, the identity. In virtue of this degeneration, the automorphic property (2·3) vanishes out, and the Schwarzian derivative $\{f(z), z\}$ becomes an analytic function possessing $a_\mu(\equiv a_{1\mu})$ $(\mu=1,\dots, m)$ as its poles of order at most two, and hence becomes a rational function.

If the image-domain Δ is particularly bounded by rectilinear polygons, more concrete properties of mapping function $f(z)$ can be derived. In fact, we have the following theroem:

Theorem 2. If, in the theorem 1. the boundary components of Δ are all rectilinear polygons, then the differential expression

$$d_2 \lg d_1 f(z)\equiv d_2 \lg(f'(z)d_1 z)=\left(\frac{f''(z)}{f'(z)}+\frac{d_2 d_1 z}{d_2 z d_1 z}\right)d_2 z \qquad (2\cdot 8)$$

possesses automorphic property, d_1 and d_2 both denoting differentiation operators. The function $f''(z)/f'(z)$ meromorphic in D is regular except at the point $a_{j\mu}$ which are poles of order one with residue $\alpha_{j\mu}-1$.

Proof. By virtue of the assumption on rectilinearity of the boundary polygons, any successive inversion repeated even times with respect to Γ_j

degenerates merely to a translation. Hence, for any transformation $z|L(z)$ of $\mathfrak{G}$, the relation (2·5) takes a simpler form

$$f(L(z))=gf(z)+h \qquad (|g|=1).$$

Consequently we now have

$$d_1f(z^*)=gd_1f(z), \quad d_2 \lg d_1f(z^*)=d_2 \lg d_1f(z),$$

putting $z^*=L(z)$. Any transformation $z \mid z^*$ belonging to $\mathfrak{G}$ leaves therefore the expression (2·8) invariant. Clearly, $f''(z)/f'(z)$ is regular except at the points $a_{j\mu}$ and at their equivalents. In a vicinity of $a_{j\mu}$ we have

$$\frac{f''(z)}{f'(z)}=\frac{\alpha_{j\mu}-1}{z-a_{j\mu}}+\cdots,$$

as was seen in the proof of the preceding theorem (cf. (2·7)), which proves the present theorem.

In the particular case $n=1$, $\mathfrak{G}$ consists of the identical transformation alone. The automorphic property thus vanishes out. and $f''(z)/f'(z)$ becomes an analytic function in the entire plane, possessing $a_\mu(\equiv a_{1\mu})$ ($\mu=1,\dots, m$) as poles of order one. Furthermore, since $f(z)$ remains evidently regular at ∞ ($\neq a_\mu$), we have

$$\frac{f''(z)}{f'(z)}=\sum_{\mu=1}^{m}\frac{\alpha_\mu-1}{z-a_\mu},$$

which, by integration, implies just the Schwarz-Christoffel formula (1·1).

§ 3. Specialization to doubly-coonnected domains.

In the case of doubly-connected domains, we can take the annular domain D: $q<|z|<1$ as a standard basic domain of modulus $-\lg q$. Two general theorems of the last section then take more clear and concrete forms. In the first place, by specializing theorem 1, we obtain the following result:

Theorem 3. Any function $w=f(z)$, mapping annular domain D onto a ring domain Δ bounded by two circular polygons, satisfies the differential equation of the third order

$$\{f(z), z = \}\frac{E(i \lg z)}{z^2}, \tag{3·1}$$

$E(Z)$ being an elliptic function with primitive periods 2π and $-2i \lg q$ (or being a constant). If we now denote by $e^{i\varphi_\mu}$ ($\mu=1,\dots,m$) and $qe^{i\psi_\nu}$ ($\nu=1,\dots,n$) the points corresponding to vertices of boundary polygons Γ_1 and Γ_2 of Δ respectively and further by $\alpha_{1\mu}\pi$ and $\alpha_{2\nu}\pi$ the interior angles of Γ_1 and Γ_2 at vertices $f(e^{i\varphi_\mu})$ and $f(qe^{i\psi_\nu})$ respectively, then the function $E(Z)$ possesses at $Z=-\varphi_\mu$ and at $Z=-\psi_\nu+i\lg q$ its primitive poles of order at most two, and further satisfies

$$\lim_{Z\to-\varphi_\mu}(Z+\varphi_\mu)^2E(Z)=\frac{1-\alpha_{1\mu}^2}{2}, \quad \lim_{Z\to-\psi_\nu+i\lg q}(Z+\psi_\nu-i\lg q)^2E(Z)=\frac{1-\alpha_{2\nu}^2}{2}. \tag{3·2}$$

Proof. Since the basic domain D is supposed to be $q<|z|<1$, inversions with respect to its bounding circles are given by $\lambda_1(z)=1/\bar{z}$ and $\lambda_2(z)=q^2/\bar{z}$. Hence, $\mathfrak{G}$ is a cyclic group generated by a unique transformation

$$z \mid l_{12}(z)=\frac{z}{q^2};$$

that is to say, all the transformations of $\mathfrak{G}$ are expressed in the form

$$z \,\Big|\, \frac{z}{q^{2\varkappa}} \qquad (\varkappa=0, \pm1, \pm2, \dots).$$

Introducing now an auxiliary variable Z, we put

$$Z=i \lg z, \qquad F(Z)=f(e^{-iZ}) \ (=f(z)). \tag{3·3}$$

In virtue of one-valuedness of $f(z)$ in D, $F(Z)$ remains invariant under the transformation $Z \mid Z+2\pi$. But the mapping function $f(z)$ is here supposed to have been analytically continued over the pricked (*punktiert*) plane $0<|z|<\infty$, and is in general infinitely many-valued function. Taking account of the corresponding many-valuedness of $F(Z)$, it must therefore be understood that $F(Z)$ is also subject to a linear transformation

$$F(Z+2\pi)=\frac{g_0F(Z)+h_0}{g'_0F(Z)+h'_0}, \tag{3·4}$$

the coefficients depending on the manner of the continuation in question.

This polymorphism appears even when is made the substitution $Z \mid Z+2\pi$ corresponding to the identical transformation $z \mid l_{11}(z) \equiv z$ of $\mathfrak{G}$. On the other hand, since, to the generating linear transformation $z \mid l_{12}(z) = z/q^2$, corresponds the substitution $Z \mid Z-2i \lg q$, we have a relation

$$F(Z-2i\lg q) = \frac{gF(Z)+h}{g'F(Z)+h'}, \tag{3·5}$$

coefficients depending on the manner of continuation. By virtue of linear-polymorphic properties (3·4) and (3·5), the expression

$$\Phi(Z) = \{F(Z), Z\} \tag{3·6}$$

is a one-valued function of Z possessing double periodicity

$$\Phi(Z+2\pi) = \Phi(Z-2i\lg q) = \Phi(Z). \tag{3·7}$$

By a substitution of variables, Schwarzian derivative undergoes a transformation given by (2·6). Now, since under the substitution (3·3) it holds

$$\{F,f\} = 0, \quad \{Z,z\} = \frac{1}{2z^2}, \quad \left(\frac{dz}{dZ}\right)^2 = -z^2,$$

we have

$$\Phi(Z) \equiv \{F(Z), Z\} = -z^2\{f(z), z\} + \frac{1}{2}.$$

Putting $E(Z) = 1/2 - \Phi(Z)$, we obtain the desired differential equation (3·1). That the function $E(Z)$ also possesses double periodicity $E(Z+2\pi) = E(Z-2i\lg q) = E(Z)$, is an immediate consequence of (3·7). Clearly, its primitive periods are 2π and $-2i\lg q$ (if $\varDelta$ itself is a circular ring, then $E(Z) \equiv 0$). In order to investigate the singularities of $E(Z)$, we restrict ourselves to the periodicity parallelogram (rectangle!)

$$-2\pi \leqq \Re Z < 0, \qquad 2\lg q \leqq \Im Z < 0. \tag{3·8}$$

This rectangle corresponds to the fundamental domain $q^2 < |z| < 1$ adjoined by its interior circumference. The possible singularities of $F(z)$ belonging to (3·8) appear only at $Z = -\varphi_\mu$ and at $Z = -\psi_\nu + i\lg q$ which correspond to $z = e^{i\varphi_\mu}$ and $z = qe^{i\psi_\nu}$ respectively. The method used in the proof of theorom 1 here also leads us immediately to the relations (3·2).

By general theorem 1, the differential expression (2·3) is to be automorphic. If D is in particular an annular domain $q<|z|<1$, the group $\mathfrak{G}$ is generated, as was already seen, by the linear transformation $z \mid z^*=z/q^2$ alone. This transformation gives

$$\{f(z^*),\ z^*\}=\{f(z),\ z\}q^4 \qquad (z^*=z/q^2),$$

and hence $\{f(z^*),\ z^*\}dz^{*2}=\{f(z),\ z\}dz^2$, as is stated just in theorem 1. On the other hand, the second member of (3.1) changes, on account of the periodicity of $E(Z)$, according as

$$\frac{E(i \lg z^*)}{z^{*2}}=\frac{E(i \lg z-2i \lg q)}{(z/q^2)^2}=\frac{E(i \lg z)}{z^2}q^4 \quad (z^*=z/q^2),$$

which coïncides with the above relation. It may be remarked in passing that the existence of another period 2π of $E(Z)$ corresponds to one-valuedness of $\{f(z),\ z\}$.

As was already stated in § 1, if the boundary of doubly-conneted domain Δ consists of two rectilinear polygons, the explicit representation (1·2) is valid. This result has previously been obtained by the present author by means of Villat's integral representation, but we shall now try to derive it again from general theorem 2.

Theorem 4. Any function, which maps the annular domain $q<|z|<1$ onto a ring domain bounded by rectilinear polygons, is expressed by the formula (1·2), the constant c^* being given by (1·3).

Proof. Adopting the same notations as in the proof of the preceding theorem, since the both boundary polygons are rectilinear, the relations (3·4) and (3·5) become simply

$$F(Z+2\pi)=g_0F(Z)+h_0, \quad F(Z-2i \lg q)=gF(Z)+h;$$
$$|g_0|=1, \qquad |g|=1. \tag{3·9}$$

Hence $F''(Z)/F'(Z)$ is an elliptic function with periodicity parallelogram (3·8). If we put $a_{1\mu}=a_\mu(\mu=1,\dots,m)$. $2-a_{2\nu}=\beta_\nu(\nu=1,\dots,n)$, its residues at primitive poles $-\varphi_\mu$ and $-\psi_\nu+i \lg q$ of order one are $a_\mu-1$ and $(2-\beta_\nu)-1$, respectively. This function can therefore be represented in the form

$$\frac{F''(Z)}{F'(Z)}=\sum_{\mu=1}^{m}(a_\mu-1)\zeta(Z+\varphi_\mu)-\sum_{\nu=1}^{n}(\beta_\nu-1)\zeta(Z+\psi_\nu-i \lg q)+c,$$

c being a constant. By means of a well known formula $\zeta(\mu - i \lg q) = \zeta_3(u) + \eta_3$, it may be written as

$$\frac{F''(Z)}{F'(Z)} = \sum_{\mu=1}^{m}(\alpha_\mu - 1)\zeta(Z+\varphi_\mu) - \sum_{\nu=1}^{n}(\beta_\nu - 1)\zeta_3(Z+\psi_\nu) + c^*,$$

and thus, by integration,

$$F'(Z) = -iCe^{c^*Z} \prod_{\mu=1}^{m} \sigma(Z+\varphi_\mu)^{\alpha_\mu - 1} \Big/ \prod_{\nu=1}^{n} \sigma_3(Z+\psi_\nu)^{\beta_\nu - 1}, \qquad (3\cdot10)$$

where $-iC$ denotes a constant resulting from the integration. A further integration and returning to original variables implies the desired formula. Now it remains only to determine the value of constant c^*. In formulae on elliptic functions

$$\sigma(u+2\omega_1) = -e^{2\eta_1(u+\omega_1)}\,\sigma(u), \qquad \sigma_3(u+2\omega_1) = +e^{2\eta_1(u+\omega_1)}\,\sigma_3(u),$$

$$\sigma(u+2\omega_3) = -e^{2\eta_3(u+\omega_3)}\,\sigma(u), \qquad \sigma_3(u+2\omega_3) = -e^{2\eta_3(u+\omega_3)}\,\sigma_3(u),$$

we are here concerning with primitive periods $2\omega_1 = 2\pi$, $2\omega_3 = -2i \lg q$. Hence, remembering the relations between interior angles of each boundary polygon

$$\sum_{\mu=1}^{m}(\alpha_\nu - 1) = \sum_{\nu=1}^{n}(\beta_\nu - 1) = 2,$$

we have from $(3\cdot10)$

$$\frac{F'(Z+2\pi)}{F'(Z)} = e^{2\pi c^*} \prod_{\mu=1}^{m}\left(-e^{2\eta_1(Z+\varphi_\mu+\pi)(\alpha_\mu - 1)}\right) \Big/ \prod_{\nu=1}^{n} e^{2\eta_1(Z+\psi_\nu+\pi)(\beta_\nu - 1)}$$

$$= (-1)^m \exp\Big(2\pi c^* + 2\eta_1\Big(\sum_{\mu=1}^{m}(\alpha_\mu - 1)\varphi_\mu - \sum_{\nu=1}^{n}(\beta_\nu - 1)\psi_\nu\Big)\Big).$$

and

$$\frac{F'(Z - 2i \lg q)}{F'(Z)} = \exp\Big(-2i \lg q \cdot c^* + 2\eta_3\Big(\sum_{\mu=1}^{m}(\alpha_\mu - 1)\varphi_\mu - \sum_{\nu=1}^{n}(\beta_\nu - 1)\psi_\nu\Big)\Big).$$

The absolute value of left hand member of each of these relations is, by $(3\cdot9)$, equal to unity. Since η_3 is purely imaginary, c^* must be a real constant, because of $|g| = 1$. Further, since η_1 is real, we obtain from $|g_0| = 1$

$$2\pi c^* + 2\eta_1\Big(\sum_{\mu=1}^{m}(\alpha_\mu - 1)\varphi_\mu - \sum_{\nu=1}^{n}(\beta_\nu - 1)\psi_\nu\Big) = 0,$$

which yields (1·4).

Now, by substitution (3·3), we have

$$\frac{F''(Z)}{F'(Z)} = -i\Big(1 + z\frac{f''(z)}{f'(z)}\Big).$$

Hence $zf''(z)/f'(z)$ must remain invariant under any transformation $z \mid z/q^{2\kappa}$ of $\mathfrak{G}$. This fact will also be immediately confirmed from general theorem 2. For this end, we have only to consider the generating transformation $z \mid z/q^2$ of $\mathfrak{G}$ together with the identical transformation $z \mid z$. The invariance of $zf''(z)/f'(z)$ under the latter transformation is nothing but the one-valuedness of the function. As to the invariance under the former, putting $z^* = z/q^2$, we have

$$d_2 \lg(f'(z^*)d_1 z^*) = \Big(\frac{f''(z^*)}{f'(z^*)} + \frac{d_2 d_1 z^*}{d_2 z^* d_1 z^*}\Big)d_2 z^* = \Big(\frac{1}{q^2}\frac{f''(z^*)}{f'(z^*)} + \frac{d_2 d_1 z}{d_2 z d_1 z}\Big)d_2 z,$$

and hence, from theorem 2.

$$\frac{1}{q^2}\frac{f''(z^*)}{f'(z)} = \frac{f''(z)}{f'(z)}, \qquad z^*\frac{f''(z^*)}{f'(z^*)} = z\frac{f''(z)}{f'(z)}.$$

§ 4. Another basic domains.

As a standard multiply-connected domain, we can take any one of various possible types other than that used in § 2. For instance, as are often used, parallel slit domain[5), 6)] obtained from entire plane by cutting along parallel segments, circular slit domain or radial slit domain which is obtained from either entire plane, circular disc or annular ring by cutting along circular arcs or radial segments contained in its interior. For such a basic domain, a group $\mathfrak{G}$ with analogous fundamental domain, eventually extending over two sheets on a Riemann surface, can also be constructed in

5) R. de Possel, Zum Parallelschlitztheorem unendlich-vielfach zusammenhängender Gebiete. Göttinger Nachr. (1931), 192–202.

6) E. Rengel, Existenzbeweise für schlichte Abbildungen mehrfach zusammenhängender Bereiche auf gewisse Normalbereiche. Jaresber. Deutsche Math.-Vereinig. **44** (1934), 51–55.

quite similar manner as in theorems 1 and 2. These theorems themselves remain to hold in almost the same form. We have only to carry out a few modifications by considering that the regularity of boundary curves is lost at end points of the slits.

Theorem 5. In any case of such a basic domain of above mentioned type, the conclusion of theorem 1 remains to hold with following modifications. If an end point of a slit coïncides with point $a_{j\mu}$, the relation (2·4) is replaced by

$$\lim_{z\to a_{j\mu}}(z-a_{j\mu})^2\{f(z),z\}=\frac{4-\alpha_{j\mu}^2}{8}, \tag{4·1}$$

and if an end point, say p, of a slit coïncides with none of $a_{j\mu}$, the Schwarzian derivative has there a pole of the second order and satisfies

$$\lim_{z\to p}\ (z-p)^2\{f(z),\ z\}=\frac{3}{8}. \tag{4·2}$$

Proof. It is only to prove the last part of the theorem concerning with end points of the slits. Suppose in the first place that $a_{j\mu}$ is an end point of a slit, then any branch of the function $(f(z)-f(a_{j\mu}))^{2/\alpha_{j\mu}}$ remains regular around $a_{j\mu}$ and has a zero of the first order there. (In case of $a_{j\mu}=0$ an evident modification must be made.) Its effect is therefore to put $\alpha_{j\mu}/2$ in place of $\alpha_{j\mu}$, which leads (2·4) to (4·1). Next, if an end point p of a slit coïncides with none of $a_{j\mu}$, then $(f(z)-f(p))^{1/2}$ is regular around p and vanishes at this point in the first order. Hence, its effect is as if there exists an image-vertex with interior angle $\pi/2$, for which it becomes $(1-(1/2)^2)/2=3/8$.

Theorem 6. If Δ is bounded by rectilinear polygons, the conclusion of theorem 2 remains to be true, in any case of above mentioned basic domains, with following modifications. If an end point of a slit coïncides with $a_{j\mu}$, the relation (2·8) is replaced by

$$\lim_{z\to a_{j\mu}}(z-a_{j\mu})\frac{f''(z)}{f'(z)}=\frac{\alpha_{j\mu}-2}{2}, \tag{4·3}$$

and if an end point p of a slit coïncides with none of $a_{j\mu}$, then $f''(z)/f'(z)$ has the point p as a pole of the first order with residue $-1/2$, that is,

$$\lim_{z\to p}(z-p)\frac{f''(z)}{f'(z)}=-\frac{1}{2}. \tag{4·4}$$

Proof. We have again only to investigate the last part of the theorem concerning with end points of the slits. And the quite same reasoning will apply as in the proof of the preceding theorem.

In conclusion, we remark that a circular disc with n sheets may also be taken as a standard type of n-ply-connected domains.[7],[8] The group $\mathfrak{G}$ considered in theorem 1 then consists of a unique transformation $z \mid z$, all inversions $z \mid \lambda_j(z)$ referring to a common circumference. Hence, the group degenerates to a trivial one, while the mapping function becomes n-valued one on the projection of the disc in question. In this case a corresponding theorem may be stated as foollows:

Theorem 7. Let $w=f(z)$ be a function which maps a disc with n sheets covering a circle D_0 on z-plane onto an n-ply-connected circular polygonal domain $\varDelta$. Then, each branch $f_j(z)$ $(j=1,\dots, n)$ of $f(z)$ satisfies a differential equation of the third order

$$\{f_j(z),\ z\}=M_j(z), \tag{4·5}$$

where $M_j(z)$ is a one-valued meromorphic function. Denoting by Γ_j a boundary polygon of $\varDelta$ mapped from boundary circle C_j of D by $w=f(z)$ (i. e. by $w=f_j(z)$), and by $a_{j\mu}$ a point lying on C_j and corresponding to a vertex of Γ_j, the function $M_j(z)$ possesses at $a_{j\mu}$ a pole of order at most two and satisfies

$$\lim_{z\to a_{j\mu}}(z-a_{j\mu})^2 M_j(z) = \frac{1-\alpha_{j\mu}^2}{2}, \tag{4·6}$$

where $\alpha_{j\mu}$ denotes, as before, the interior angle at $f_j(a_{j\mu})$ with respect to $\varDelta$. Let further t be a branch point of D of order $\tau-1$, then, for all the branches $f_j(z)$ relating to this branch point, the function $M_j(z)$ possesses there a pole of order at most two and satisfies

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7) L. Bieberbach, Über einen Riemannschen Satz aus der Lehre von der konformen Abbildung. Sitzungsber. preuss. Akad. Wiss. Berlin (1925), 6-9.

8) H. Grunsky, Über die konforme Abbildung mehrfach zusammenhängender Bereiche auf mehrblättrige Kreise. Sitzungsber. preuss. Akad. Wiss. Berlin (1937), 1-9.

$$\lim_{z\to t}(z-t)^2 M_j(z)=\frac{\tau^2-1}{2\tau^2}. \tag{4·7}$$

Excepting those points, $M_j(z)$ is regular everywhere.

Proof. By repeating inversions with respect to boundary circle, $f(z)$ is analytically continued to an infinitely many-valued function. By successive inversions repeated even times, though the affix z return back to its original position, each branch of $f(z)$ undergoes a linear transformation (the same for all). But the function $M_j(z)$ defined by (4·5) remains thereby invariant, and hence is a one-valued function. For the relation (4·6), the reasoning is the same as in theorem 1. As to (4·7), since $(f_j(z)-f_j(t))^\tau$ is regular around t and possesses this point as a pole of the first order, can be there expressed in the form

$$f_j(z)=f_j(t)+b_j(z-t)^{1/\tau}+\dots .$$

Direct calculation will show that the relation (4·7) holds good.

Theorem 8. If, in the preceding theorem, Δ is bounded particularly by rectilinear polygons, then we have, for each branch of the mapping function, an explicit expression of the form

$$f_j(z)=C\int^z dz\ \exp\int^z N_j(z)dz+C', \tag{4·8}$$

where $N_j(z)$ is a one-valued meromorphic function. Corresponding to (4·6) and (4·7) we have, at its poles $a_{j\mu}$ and t, the relations

$$\lim_{z\to a_{j\mu}}(z-a_{j\mu})N_j(z)=\alpha_{j\mu}-1, \tag{4·9}$$

$$\lim_{z\to t}(z-t)N_j(z)=\frac{1-\tau}{\tau}, \tag{4·10}$$

respectively. $N_j(z)$ is, except those points, regular everywhere.

Proof The theorem relates to theorem 7 as if theorem 6 does to theorem 5. We have only to observe that $f_j''(z)/f_j'(z)$, having no branch point, appears here as one-valued and meromorphic function.

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