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Riemann Spaces of Class Two and Their Algebraic Characterization.

Part II.

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In this paper we give a necessary and sufficient condition that a Riemann space $R_n(n \geqq 8)$ be of class two, making use of the type number discussed in a preceding paper[1].

§ 1. A reality condition

Suppose that a Riemann space $R_n(n \geqq 6)$ of class two is of type $\geqq 3$, and put

$$K_{ij}=p_{ij}+eq_{ij} \quad (e^2=-1); \tag{1·1}$$

where the tensor K_{ij} is the solution of (1·9) of the part I, i. e.

$$M_{ijkl}=K_{i(j}K_{kl)}, \tag{1·2}$$

and p's and q's are all real. The tensor M_{ijkl} in (1·2) is defined by (1·10) of the part I, i. e.

$$M_{ijkl}=\frac{1}{2}(R_{c\cdot i(j}^{\ a}R_{|a|\cdot kl)}^{\ \ c}). \tag{1·3}$$

Substituting (1·1) in (1·2) and equating to zero the imaginary parts we have

$$p_{i(j}q_{kl)}+q_{i(j}p_{kl)}=0. \tag{1·4}$$

(A) Suppose that det. $|q|\neq 0$. Contracting (1·4) by q^{kl} we have

$$(n-4)\ p_{ij}+q^{ab}p_{ab}q_{ij}=0,$$

and contracting it by q^{ij} we have $q^{ab}p_{ab}=0$ for $n>2$. Therefore all of p_{ij} are zero for $n\geqq 6$. Hence the K's are pure imaginary except zero.

(B) Suppose that det. $|q|=0$. If the rank of $\|q\|$ is $2\sigma(n>2\sigma\geqq 6)$, we have similarly $p_{ij}=0$ for $i, j=1,\ldots\ldots, 2\sigma$.

Next putting $k, l=1,\ldots\ldots, 2\sigma$ and $i, j>2\sigma$ in (1·4) we have $q_{kl}p_{ij}=0$, and therefore $p_{ij}=0$ for $i, j>2\sigma$ if $2\sigma\geqq 2$.

Finally putting $i, j, k=1,\ldots\ldots 2\sigma$ and $l>2\sigma$ in (1.4) we have

$$p_{il}q_{jk}+p_{kl}q_{ij}+p_{lj}q_{ik}=0,$$

Contracting this by q^{ik} we have $p_{il}=0$ for $i=1,\ldots\ldots 2\sigma$ and $l>2\sigma$ if $2\sigma\geqq 4$.

Hence the following cases may occur :

(I) $2\sigma\geqq 6$; then $p_{ij}=0$ $(i, j=1,\ldots\ldots, n)$,

(II) $2\sigma = 4$; then $p_{ij}=0$ $(i=1,\ldots\ldots, n\,;\, j=5,\ldots\ldots n)$;

(III) $2\sigma = 2$; then $p_{ij}=0$ $(i, j=3,\ldots\ldots n)$,

(IV) $2\sigma = 0$; then $p_{ij}=0$ $(i, j=1,\ldots\ldots n)$.

If the cases (II) or (III) are assumed to hold, we have immediately that a maximum value of rank of $\|K\|$ is four in contradiction to hypothesis on the type number and we have the

Lemma :......*If a Riemann space* $R_n(n\geqq 6)$ *of class two is of type* $\geqq 3$, *the solutions* K*'s of* (1.2) *are all real or pure imaginary except zero.*

Now we put

$$K^2_{(h,i,j,k,l,m)}=\begin{vmatrix} 0 & K_{hi} & K_{hj} & K_{hk} & K_{hl} & K_{hm} \\ -K_{hi} & 0 & K_{ij} & K_{ik} & K_{il} & K_{jm} \\ -K_{hl} & K_{ij} & 0 & K_{jk} & K_{jl} & K_{jm} \\ -K_{hk} & -K_{ik} & -K_{jk} & 0 & K_{kl} & K_{km} \\ -K_{hl} & -K_{il} & -K_{jl} & -K_{kl} & 0 & K_{lm} \\ -K_{hm} & -K_{im} & -K_{jm} & -K_{km} & -K_{lm} & 0 \end{vmatrix} \tag{1·5}$$

and

$$M^2_{(h,i,j,k,l\,m)}=\begin{vmatrix} 0 & M_{jklm} & M_{iklm} & M_{ijlm} & M_{ijkm} & M_{ijkl} \\ -M_{jklm} & 0 & M_{hklm} & M_{hjlm} & M_{hjkm} & M_{hjkl} \\ -M_{iklm} & -M_{hklm} & 0 & M_{hilm} & M_{hikm} & M_{hikl} \\ -M_{ijlm} & -M_{hjlm} & -M_{hilm} & 0 & M_{hijm} & M_{hijl} \\ -M_{ijkm} & -M_{hjkm} & -M_{hikm} & -M_{hijm} & 0 & M_{hijk} \\ -M_{ijkl} & -M_{hjkl} & -M_{hikl} & -M_{hijl} & -M_{hijk} & 0 \end{vmatrix} \tag{1·6}$$

and let $K_{(h,i,j,k,l,m)}$ and $M_{(h,i,j\,k,l,m)}$ be respectively the Pfaff's aggregate of $K^2_{(h,i,j,k,l,m)}$ and $M^2_{(h,i,j,k,l,m)}$.

From the theory[2] of determinant, for example, let $\bar{K}_{hi}$ be the algebraic complement of K_{hi} in det. $K^2_{(h,i,j,k,l,m)}$. The algebraic complement K_{hi} in $K_{(h,i,j,k,l,m)}$ is M_{jklm} as it is seen from (1·2). Then we have

$$\bar{K}_{hi}=M_{jklm}\ K_{(h,i,j,k,l,m)}, \tag{1·7}$$

and so on. Let $\bar{K}^2_{(h,i,j,k,l,m)}$ be the determinant, whose elements are $\bar{K}_{hi},\cdots$, ..., $\bar{K}_{lm}$, then we have

$$\bar{K}^2_{(h,i,j,k,l,m)}=M^2_{(h,i,j,k,l,m)}\{K^2_{(h,i,j,k,l,m)}\}^3. \tag{1·8}$$

As $K^2_{(h,i,j,k,l,m)}$ is of order six, we have

$$\bar{K}^2_{(h,i,j,k,l,m)}=\{K^2_{(h,i,j,k,l,m)}\}^5,$$

Now, from the hypothesis on the type number, we can take h, i, j, k, l, m so that $K^2_{(h,i,j,k,l,m)}$ is not zero. Hence we have from (1·8)

$$\{K^2_{(h,i,j,k,l,m)}\}^2=M^2_{(h,i,j,k,l,m)} \tag{1·9}$$

$K^2_{(h,i,j,k,l,m)}$ is positive, because it is square of real polynomial of $K_{hi},\ldots\ldots, K_{lm}$. On the other hand we have from (1·2)

$$K^2_{(h,i,j,k,l,m)}=M_{(h,i,j,k,l,m)}. \tag{1·10}$$

Then we have

$$M_{(h,i,j,k,l,m)}\geqq 0. \tag{1·11}$$

If K's be pure imaginary except zero and the rank of $\|K\|$ be $\geqq 6$, there is one of $K^2_{(h,i,j,k,l,m)}$ which is negative. Hence from (1·10), (1·11) and lemma we have the

Theorem I. I......*If a real Riemaun space R_n $(n\geqq 6)$ of class two is of type $\geqq 3$, solutions K's shall be real if, and only if, the inequality* (1.11) *is satisfied.*

§ 2. The resultant system

From (1.9) we have the

Lemma: ......*If a real Riemann space $R_n (n\geqq 6)$ of class two is of type $\geqq 3$, it is necessary that*

$$\sum M^2_{(h,i,j,k,l,m)} > 0\ ; \tag{2·1}$$

where the summation is to be extended over all possible values of the indices appearing in the above determinant.

Further necessary conditions in the form of a system of linear homogeneous equations can be derived as follows. Let us write (1·2) in their homogeneous form namely,

$$A^2\ M_{ijkl} = K_{i(j}K_{kl)}. \tag{2·2}$$

We multiply (1·2) by K_{hm} and subtract the expression obtained by interchanging m and l and then we have by means of (1·2)

$$K_{ij}M_{hmkl} + K_{jk}M_{hmil} + K_{ki}M_{hmjl} + K_{hm}M_{ijlk} + K_{ml}M_{ijhk} + K_{lh}M_{ijmk} = 0. \tag{2·3}$$

Consider the equations (2·2) and (2·2) as a system for the determination of the unknowns A and K's. Since R_n is of class two and of type $\geqq 3$, the system must admit such a solution that the matrix $\|K\|$ has rank $\geqq 6$, hence the system composed of (2·2) and (2·3) must admit a non-trivial solution (A, K's). Now we know from the theory[3] of a system of homogeneous algebraic equations that the above equations (2·2) and (2·3) must admit a resultant system, i. e. a set of polynomials in the components M's such that the vanishing of these polynomials is necessary and sufficient for the existence of a non-trivial solution. Representing the resultant system of (2·2) and (2·3) by $R(M)$, it follows that

$$R(M) = 0 \tag{2·4}$$

is a necessary condition for a Riemann space R_n to be of class two.

Suppose (2·4) to be satisfied. Let (A, K's) be a non-trivial solution of (2·2) and (2·3). Suppose $A=0$ in this solution. Then by the similar way as in § 1 of the part I we have the rank of matrix $\|K\|$ to be zero or two. If the rank of $\|K\|$ is two, taking the coordinate system such that all of K_{ij} are zero except K_{12} and putting $i=1$, $j=2$ and h, k, l, $m=3,\ldots\ldots, n$ in (2·3) we have $M_{hklm}=0$ for h, k, l, $m=3\ldots\ldots, n$. Next putting $i=h=1$, $j=2$ and k, l, $m=3,\ldots\ldots, n$ in (2·3) we have $M_{1klm}=0$ for k, l, $m=3, \ldots\ldots, n$ and similarly $M_{2klm}=0$. Hence all of $M_{ijkl}=0$ except M_{12ij} in contradiction to (2·1) and it follows that all of K's are zero in contradiction to the hypothesis. We must therefore have $A \neq 0$ so that the quantities K_{ij}/A can be defined and these constitute a solution of (1·2). We thus have from the theorem 2·1 of the part I the

Theorem 2. 1...... *If a real Riemann space* $R_n(n\geqq 6)$ *is of type* $\geqq 3$ *and the left-hand members of* (1·2) *are defined by* (1·3), *then the equations* (1·2) *will have solutions* K*'s which are unique to within algebraic sign if, and only if, the inequality* (2·1) *and the equations* (2·4) *are satisfied.*

When conditions (1·11) are likewise imposed, it follows from the theorem 1. 1 that the above solutions K's will be real. In this case the polynomial inequality (2·1) can be replaced by the polynomial inequality

$$\sum M_{(h,i,j,k,l,m)}>0 \tag{2·5}$$

of lower degree, the summation in this inequality and in (2·1) having the same significance. Hence we have the

Theorem 2. 2...... *If a real Riemann space* $R_n(n\geqq 6)$ *is of type* $\geqq 3$ *and the left-hand members of* (1·2) *are defined by* (1·3), *then the equation* (1·2) *will have a real solutions* K*'s which are unique to within algebraic sign if, and only if, the inequalities* (1·12) and (2·5) *and the equation* (2·4) *are satisfied.*

Moreover we shall derive the explicit expression for K's. From the theory[4] of determinant, we have

$$(K_{hi})^2\ \{K^2_{(h,i,j,k,l,m)}\}^3=\begin{vmatrix} 0 & \bar{K}_{jk} & \bar{K}_{jl} & \bar{K}_{jm} \\ -\bar{K}_{jk} & 0 & \bar{K}_{kl} & \bar{K}_{km} \\ -\bar{K}_{jl} & -\bar{K}_{kl} & 0 & \bar{K}_{lm} \\ -\bar{K}_{jm} & -\bar{K}_{km} & -\bar{K}_{lm} & 0 \end{vmatrix}$$

and from (1·7)

$$=\{K^2_{(h,i,j,k,l,m)}\}^2\begin{vmatrix} 0 & M_{hilm} & M_{hikm} & M_{hikl} \\ -M_{hilm} & 0 & M_{hijm} & M_{hijl} \\ -M_{hikm} & -M_{hijm} & 0 & M_{hijk} \\ -M_{hikl} & -M_{hijl} & -M_{hijk} & 0 \end{vmatrix}$$

Hence, let $M_{(h,i)}$ be the above determinant and we have

$$(K_{hi})^2M_{(h,i,j,k,l,m)}=M_{(h,i)}. \tag{2·6}$$

If we hope to obtain the full expression of K's, we may discuss in similar way as in § 9 in the Thomas's paper for Riemann spaces of class one.[6]

§ 3. Tensor E_{ijkl}

Let a Riemann space $R_n(n \geqq 6)$ of class two be of type $\geqq 3$. We put

$$E_{ijkl}=H^P_{ij}H^P_{kl} \qquad (P=I,\ II\ ;\ i, j, k, l=1,\ldots\ldots, n), \tag{3·1}$$

and shall find the intrinsic expressions of E's. For this purpose, we shall first find the intrinsic expressions of L's defined by (1·1) in the part I, i. e.

$$L_{aibj}=H^I_{ai}H^{II}_{bj}-H^I_{aj}H^{II}_{bi}-H^{II}_{ai}H^I_{bj}+H^{II}_{aj}H^I_{bi}. \tag{3·2}$$

(A) Suppose that det. $|K| \neq 0$ and contract (1·7) of the part I, i. e.

$$N_{abijkl}=L_{aib(j}K_{kl)}+K_{i(j}L_{|a|k|b|l)} \tag{3·3}$$

by K^{kl}. As L_{aibj} is skew-symmetric in i and j, we have

$$(n-4)L_{aibj}+K^{kl}L_{akbl}K_{ij}=K^{kl}N_{abijkl}. \tag{3·4}$$

Contracting (3·4) by K^{ij} we have

$$K^{ij}L_{aibj}=\frac{1}{2(n-2)}K^{ij}K^{kl}N_{abijkl},$$

hence from (3·4) we have

$$L_{aibj}=\frac{1}{n-4}K^{cd}N_{abijcd}-\frac{1}{2(n-2)(n-4)}K_{ij}K^{cd}K^{fg}N_{abcdfg}$$
$$(a, b, c, d, i, j=1,\ldots,n).$$

(B) Suppose that the rank of $\|K\|=2\tau(n>2\tau \geqq 6)$, and take $i, j, k, l=1,\ldots\ldots, 2\tau$ and $a, b=1,\ldots\ldots, n$ in (3·3), and then we have similarly (3·5′) obtained by taking $i, j, c, d, f, g=1,\ldots\ldots, 2\tau$ and $a, b=1,\ldots\ldots, n$ and $n=2\tau$ in (3·5). Next taking $j>2\tau$; $i, k, l=1,\ldots\ldots, 2\tau$ and $a, b=1, \ldots\ldots, n$ in (3·3) we obtain

$$L_{aibj}=\frac{1}{2(\tau-1)}K^{cd}N_{abijcd} \quad (j>2\tau\ ;\ i, c, d=1,\ldots\ldots, 2\tau\ ;\ a, b=1,\ldots\ldots, n)$$

Finally taking $i, j > 2\tau$; $k, l = 1 \ldots\ldots, 2\tau$ and $a, b = 1, \ldots\ldots, n$ in (3·3) we have similarly

$$L_{aibj} = \frac{1}{2\tau} K^{cd} N_{abijcd} \; (i, j > 2\tau \; ; \; c, d = 1, \ldots\ldots 2\tau \; ; \; a, b = 1, \ldots\ldots n).$$

Hence we have uniquely the intrinsic expressions of L's from (3·3), if the conditions of the theorem 2·1 are satisfied.

Now, if we put

$$S_{aibj} = H^{I}_{ai} H^{II}_{bj} - H^{II}_{ai} H^{I}_{bj} (a, b, i, j = 1, \ldots\ldots, n), \tag{3·8}$$

and then we have from (3·2)

$$S_{aibj} = \frac{1}{2} (L_{aibj} + L_{iajb}). \tag{3·9}$$

Next multiplying (1·6) of the part I, i. e.

$$H^{Q}_{a(i} K_{|Q|\cdot jk)}^{\;P} = H^{P}_{c(i} R_{|a|\cdot jk)}^{\;c}, \tag{3·10}$$

for $P = II$ by H^{I}_{bl} and for $P = I$ by H^{II}_{bl} and subtracting, we have from (3·1) and (3·8)

$$E_{a(i|bl|} K_{jk)} = S_{c(i|bl|} R_{|a|\cdot jk)}^{\;c}. \tag{3·11}$$

By the similar way as that of finding L's, when $|K| \neq 0$, we have

$$E_{aibl} = \frac{1}{n-2} K^{jk} (S_{cibl} R_{a\cdot jk}^{\;c} + 2 S_{cjbl} R_{a\cdot ki}^{\;c}), \tag{3·12}$$

and when the rank of $\|K\|$ is 2τ $(n > 2\tau \geqq 6)$, we have (3·12′) obtained by taking $i, j. k = 1, \ldots\ldots, 2\tau$, and $a, b, c, l = 1, \ldots\ldots, n$ and $n = 2\tau$ in (3·12), and

$$E_{aibl} = \frac{1}{2\tau} K^{jk} (S_{cibl} R_{a\cdot jk}^{\;c} + 2 S_{cjbl} R_{a\cdot ki}^{\;c})$$
$$(a, i, c, l = 1, \ldots\ldots, n \; ; \; j, k = 1, \ldots\ldots, 2\tau \; ; \; i > 2\tau). \tag{3·13}$$

On the other hand from (3·1) and (3·8), we have immediately

$$E_{aibj} = E_{bjai} = E_{iabj} \quad (a, b, i, j = 1, \ldots\ldots n), \tag{3·14}$$

and

$$S_{abci}S_{ijkl} = \begin{vmatrix} E_{abij} & E_{abkl} \\ E_{cdij} & E_{cdkl} \end{vmatrix} (a, b, c, d, i, j, k, l=1,\ldots\ldots, n). \quad (3\cdot15)$$

Finally we have from the Gauss equation

$$R_{ijkl}=E_{ikjl}-E_{iljk} \ (i, j, k, l=1,\ldots\ldots, n). \quad (3\cdot16)$$

§ 4. The Gauss equation

In the first place we shall pay attention to the following facts.

Remark I. If we interchange the positive directions of normals $B_P^{\cdot\alpha}$ to a n-dimensional variety S_n in a $(n+2)$-dimensional euclidean space E_{n+2}, the algebraic sign of H_{ij}^P changes according to $H_{ij}^P=B_{i,j}^{\cdot\alpha}B_P^{\cdot\alpha}$.

Remark II. Let $B_P^{\cdot\alpha}$ and $\bar{B}_P^{\cdot\alpha}$ be two systems of mutually orthogonal unit vectors normal to S_n in E_{n+2}. With reference to $B_P^{\cdot\alpha}$ and $\bar{B}_P^{\cdot\alpha}$ we have respectively H_{ij}^P and $\bar{H}_{ij}^P$. The formulae of transformation of normal systems are

$$\bar{B}_P^{\cdot\alpha}=l_P^Q B_Q^{\cdot\alpha} ;$$

where $\|l\|$ is orthogonal matrix. We can deduce

$$\bar{H}_{ij}^P=l_P^Q H_{ij}^Q.$$

If we put, for example, $\bar{H}_{11}^{II}=0$, i. e.

$$l_{II}^I H_{11}^I+l_{II}^{II}H_{11}^{II}=0,$$

from this equation $l_P^Q(P, Q=I, II)$ are determined to within algebraic sign. Therefore, for example, the algebraic sign of

$$\bar{H}_{11}^P=l_I^I H_{11}^I+l_I^{II}H_{11}^{II}$$

can be selected arbitrarily and then other $\bar{H}_{ij}^P$ are all determined.

Now, if a real Riemann space $R_n(n\geqq6)$ is of class two and of type $\geqq3$, it is necessary that (1·11), (2·4) and (2·5) are satisfied. Then K's is expressed intrinsically. Making use of K's, we have intrinsic expressions of L's, S's and E's. Therefore (3·14), (3·15) and (3·16) are also necessary conditions for R_n to be of class two. These necessary conditions (1·11), (2·4), (3·5), (3·14), (3·15) and (3·16) are constracted only by g_{ij} and their partial derivatives.

Conversely we shall prove the

Theorem 4. I: ...... *If a real Riemann space* $R_n(n\geqq 6)$ *is of type* $\geqq 3$, *then there will be a set of functions* $H^P_{ij}(=H^P_{ji})(P=I,\ II: i, j=1,\dots\dots, n)$ *satisfying the Gauss equation if, and only if, the inequalities* (1·11) *and* (2·5), *and the equations* (2·4), (3·14), (3·15) *and* (3·16) *are satisfied.*

Taking $a=i$, $b=j$, $c=k$, $d=l$ in (3·15) we have

$$E_{ijij}E_{klkl}-(E_{ijkl})^2=(S_{ijkl})^2 \tag{4·1}$$

Suppose that all of $E_{ijij}=0$ and then all of $E_{ijkl}=0$ from (4·1), because E's and S's are real. Therefore from (3·16) R_{ijkl} is zero tensor in contradiction to the hypothesis on the type number. Now suppose, for example, $E_{1111}\gtrless 0$. We take $H^{II}_{11}=0$ and

$$H^I_{11}=\sqrt{E_{1111}}\,; \tag{4·2}$$

where algebraic sign is arbitrary (Cf. *Re. II*). Next we take

$$E_{11ij}=H^I_{11}H^I_{ij}\quad (i, j=1,\dots\dots, n), \tag{4·3}$$

and then we have uniquely such a set of functions $H^I_{ij}(i, j=1,\dots\dots, n)$ that is symmetric in i and j from (3·14). Finally we take

$$S_{11ij}=H^I_{11}H^{II}_{ij}\ (i, j=1,\dots\dots, n), \tag{4·4}$$

and then we have uniquely a set of functions $H^{II}_{ij}(i, j=1,\dots\dots, n)$ and it is symmetric in i and j, because interchanging a and b, or c and d in (3·15) we obtain

$$S_{abcd}=S_{bacd}=S_{abdc}.$$

Now we shall prove that those H^P_{ij} satisfy (3·1). In fact, taking $a=b=i=j=1$ in (3·10) we have from (4·2), (4·3) and (4·4)

$$\begin{vmatrix} (H^I_{11})^2 & H^I_{11}H^I_{jl} \\ H^I_{11}H^I_{cd} & E_{cdkl} \end{vmatrix} =(H^I_{11})^2\cdot H^{II}_{kl}H^{II}_{cd},$$

hence we can deduce (3·1) immediately. Finally from (3·16) those satisfy the Gauss equation and therefore we have the theorem 4.1.

If H^P_{ij} are real, we have from (3·1) $E_{ijij}\geqq 0(i, j=1,\dots\dots n)$ and so, from (3·15), matrix $||E_{ijkl}||$ (i, j: row ; p, l: column) is positive semi-definite. Conversely the matrix $||E_{ijkl}||$ is positive semi-definite, we can have real H^P_{ij} by the above method, hence we have the

Theorem 4. 2 : ...... *If a real Riemann space* $R_n(n\geqq 6)$ *is of type* $\geqq 3$, *then there will be a set of real functions* $H^P_{ij}(=H^P_{ji})$ $(P=I, II;\ i, j=1,\ldots\ldots, n)$ *satisfying the Gauss equation if, and only if, the inequalities* (1·11) *and* (2·5), *and the equations* (2·4), (3·14), (3·15) *and* (3·16) *are satisfied; and* $||E_{ijkl}||$ *is positive semi-definite.*

We have from the theorem 2.4 of the part I and theorem 4.1 the

Theorem 4. 3 : ...... *If a real Riemann space* $R_n(n\geqq 8)$ *is of type* $\geqq 4$, *then there will be two sets of functions* $H^P_{ij}(=H^P_{ji})$ *and* $H^P_{Qi}(=-H^Q_{Pi})$ $(P, Q=I, II;\ i, j=\ldots\ldots, n)$ *satiyfying the Gauss, Codazzi and Ricci equations if, and only if, the inequalities* (1·11) *and* (2·5), *and the equations* (2·4), (3·14), (3·15) *and* (3·16) *are satisfied; and the matrix* $||E_{ijkl}||$ *is positive semi-definite.*

If H^P_{ij} are real, $H^P_{Qi}(P, Q=I, II;\ i=1,\ldots\ldots, n)$ satisfying the Codazzi equaion are also real (Cf. Allendoerfer's paper),[(6)] hence we have the most remarkable theorem:

Theorem 4. 4 : ...... *If a real Riemann space* $R_n(n\geqq 8)$ *is of type* $\geqq 4$, *then* R_n *will be of class two if, and only if, the inequalities* (1·11) *and* (2·5), *and the equations* (2·4), (3·14), (3·15) *and* (3·16) *are satisfied; and the matrix* $||E_{ijkl}||$ *is positive semi-definite.*

For remark, a solution K's of (1·2) is unique to within algebraic sign for type $\geqq 3$ and $g^{ab}(H^{II}_{ai}H^{I}_{bj}-H^{II}_{aj}H^{I}_{bi})$ satisfy (1·2) as the result of the Gauss equation, hence the equation (Cf. §. 1 of the part I)

$$K_{ij}=g^{ab}(H^{II}_{ai}H^{I}_{bj}-H^{II}_{aj}H^{I}_{bi})$$

are satisfied to within algebraic sign.

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