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Isoperimetric Inequalities.

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1. The most simple isoperimetric problem formulated by Steiner[1] may be stated as follows: *Among all rectifiable closed Jordan curves on a plane with assigned length, determine the one which maximizes the enclosing area.*

The solution of the problem is, as is well known, given by *circle*; in other words, *if F denotes the area of any (finite) domain bounded by a rectifiable closed Jordan curve with length L, then the isoperimetric inequality*

$$(1) \qquad 4\pi F \leqq L^2$$

always holds, and the equality sign appears when and only when the surrounding curve is a circle.

Beside a purely geometrical proof due to Edler[2], an elegant analytical proof of this fact has been given by Hurwitz[3]. But Hurwitz, making use of Fourier series, assumed in his proof the piecewise smoothness of boundary curves in order to confirm the termwise differentiability of the series. And then various generalizations and brief proofs of the proposition have been published by Brunn[4], Minkowski[5], Carathéodory-Study[6] and others.

On the other hand, Bieberbach[7] has shown an analogous isoperimetric

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1) J. Steiner, Einfache Beweise der isoperimetrischen Hauptsätze. Journ. reine u. angew. Math. **18**, (1838), 289-296. Cf. throughout this Note as a reference the excellent report by T. Bonnesen u. W. Fenchel, Theorie der konvexen Körper. (Ergebn. d. Math. III 1.) Berlin (1934).

2) F. Edler, Vervollständigung der Steinerschen elementargeometrischen Beweise für den Satz, dass der Kreis grösseren Flächeninhalt besitzt als jede andere Figur gleich grossen Umfangs. Nachr. Ges. Wiss. Göttingen (1882), 73-80.

3) A. Hurwitz, Sur le problème des isopérimètres. C. R. Acad. Sci. Paris **132** (1901), 401-403; Sur quelques applications géométriques des séries de Fourier. Ann. École Norm. Sup. (3) **19** (1902), 357-408.

4) H. Brunn, Über Ovale und Eiflächen. Inaug. Diss. München (1887), 42 S.

5) H. Minkowski, Allgemeine Lehrsätze über konvexe Polyeder. Nachr. Ges. Wiss. Göttingen (1897), 198-219; Volumen und Oberfläche. Math. Annalen **57** (1903), 447-495.

6) C. Carathéodory u. E. Study, Zwei Beweise des Satzes, dass der Kreis unter allen Figuren gleichen Umfangs den grössten Inhalt hat. Math. Annalen **68** (1909), 133-140.

7) L. Bieberbach, Über eine Extremaleigenschaft des Kreises. Jahresb. Deutsch. Math-Verein. **24** (1915), 247-250.

inequality:

Amang all two-dimensional point sets with given diameter, the circular disc possesses the largest area; in other words, *if F denotes the area (outer measure) of any point set with assigned diameter D, then it holds the inequality*

$$(2) \qquad 4F \leqq \pi D^2,$$

where the equality sign appears only for the circular disc.

This theorem has been generalized by Kubota[8] to the case of point sets of arbitrary dimensions, and a precision has been added in the two-dimensional case by the method of central symmetrization customary in the theory of ovals.

This Note presents function-theoretic proofs of the above two theorems, based on the distortion formulae in the theory of conformal mapping. The method is closely related to the one used previously by the author[9].

2. We begin with the proper isoperimetric inequality (1). Let the boundary of the given point set E be a closed Jordan curve with length L. We can now suppose, without loss of generality, the set to be a continuum, i.e. a connected closed set, and denote its *outer mapping radius* (capacity or transfinite diameter) by C. Then the complementary domain of E, laid on complex w-plane, can be mapped conformally and univocally onto the exterior of the circle with radius C around the origin of the complex z-plane by a function of the form

$$(3) \qquad w=g(z)\equiv z+\sum_{n=0}^{\infty}a_n z^{-n}.$$

Let $F(r)$ denote the area of the finite domain bounded by the image-curve Γ_r of the circumference $|z|=r(>C)$. Then by the method used in the proof of Bieberbach's *area theorem* on schlicht functions, we get

$$F(r)=\pi(r^2-\sum_{n=1}^{\infty}n|a_n|r^{-2n}),$$

8) T. Kubota, Über konvexgeschlossene Mannigfaltigkeiten im n-dimensionalen Raume. Sci. Rep. Tôhoku Univ. **14** (1925), 85–99.

9) Y. Komatu, Einige Anwendungen der Verzerrungssätze auf Hydrodynamik. Proc. Imp. Acad. Tokyo **19** (1943), 454–461.

and consequently

$$(4) \qquad F=F(C+0)=\pi(C^2-\sum_{n=1}^{\infty}n|a_n|C^{-2n})\leqq\pi C^2.$$

The equality sign in the last inequality appears if, and only if, $a_n=0$ for all $n\geqq 1$, and so $g(z)=z+a_0$ and the set E coïncides with the circular disc around the point $w=a_0$.

Next, the length $L(r)$ of the curve Γ_r has the expression

$$L(r)=\int_{|z|=r}|g'(z)||dz|=\int_0^{2\pi}|g'(re^{i\theta})|rd\theta.$$

Since the derivative $g'(z)$ can vanish nowhere for $|z|>C$, because of the univalency of $g(z)$, we can put

$$g'(z)^{\frac{1}{2}}=(1-\sum_{n=1}^{\infty}na_nz^{-n-1})^{\frac{1}{2}}=1+\sum^{\infty}c_nz^{-n},$$

and hence write, for $r>C$,

$$\int_0^{2\pi}|g'(re^{i\theta})|d\theta=\int_0^{2\pi}|g'(re^{i\theta})^{\frac{1}{2}}|^2d\theta$$

$$=2\pi(1+\sum_{n=2}^{\infty}|c_n|^2r^{-2n}).$$

Since $g(z)$, as a mapping function onto Jordan domain, is continuous for $C\leqq|z|<\infty$, the real valued function of z:

$$G_n(z)=\sum_{\nu=1}^{n}\left|\frac{1}{z}(g(\omega_n^{\nu}z)-g(\omega_n^{\nu-1}z))\right|$$

with $\omega_n=e^{2\pi i/n}$ is also continuous and subharmonic for $|z|>C$, the point at infinity being included. The function $G_n(z)$ attains therefore its maximum value at some point z_n^* on the boundary $|z|=C$, i.e. there exists a point z_n^* with $|z_n^*|=C$ such that the inequality

$$G_n(z)\leqq G_n(z_n^*)\leqq\frac{L}{C}$$

is valid for any z with $|z|\geqq C$, the last relation being immediate.

Now, for $r>C$, we have

$$\lim_{n\to\infty} G_n(r) = \int_0^{2\pi} |g'(re^{i\theta})| d\theta,$$

and consequently

$$L \geqq C \int_0^{2\pi} |g'(re^{i\theta})| d\theta = 2\pi C(1+\sum_{n=2}^{\infty} |c_n|^2 r^{-2n}),$$

which, for $r \to C+0$, implies the evaluation

$$L \geqq 2\pi C(1+\sum_{n=2}^{\infty} |c_n|^2 C^{-2n}). \tag{5}$$

Here we may add in passing that, for any set of curves, the length functional of the curve being lower semi-continuous for the strong convergence (in the sense of Fréchet's *écart*[10]), we obtain

$$L \leqq \underline{\lim}_{r\to C+0} L(r) = \lim_{r\to C+0} L(r)$$

$$= 2\pi C(1+\sum_{r=2}^{\infty} |c_n|^2 C^{-2n});$$

and hence, comparing with (5),

$$L = 2\pi C(1+\sum_{n=2}^{\infty} |c_n|^2 C^{-2n}). \tag{6}$$

Two inequalities (4) and (6) (or merely (5)) lead us to the desired result (1). The equality sign in (1) can be, as is obvious from the above procedure, attained only by the circle.

3. We shall proceed to the proof of the inequality (2), continuing to use the above notations. We can here also suppose E to be a continuum; moreover, we may restrict ourselves with a *convex* continuum, since otherwise we have only to replace E by the smallest convex closed set containing it.

Let $D(r)$ denote the diameter of the curve Γ_r, then

$$D(r) = \overline{\text{fin}}_{|z_1|=|z_2|=r} |g(z_1) - g(z_2)| \leqq \overline{\text{fin}}_{|z|=r} |g(z) - g(-z)|.$$

Since the function

$$g(z) - g(-z) = 2z + 2\sum_{n=0}^{\infty} a_{2n+1} z^{-2n-1}$$

10) M. Fréchet, Sur l'écart de deux courbes et sur les courbes limites. Trans. Amer. Math. Soc. 6 (1905), 435–449.

is regular analytic for $C<|z|<\infty$, it follows

$$2=\left[\frac{d}{dz}(g(z)-g(-z))\right]^{z=\infty}=\frac{1}{2\pi i}\int_{|z|=r>C}(g(z)-g(-z))\frac{dz}{z^2},$$

integration in the right side being taken in the negative sense around $z=\infty$. It follows that

$$2\leqq\frac{1}{2\pi}\int_0^{2\pi}|g(re^{i\theta})-g(-re^{i\theta})|\frac{d\theta}{r}\leqq\frac{D(r)}{r},$$

and hence, for $r\to C+0$,

(7) $$2C\leqq D.$$

The last inequality is nothing but a result due to Landau and Toeplitz[11]; it can also be shown that the equality in (7) occurs only for $g(z)=z+a_0$. From both relations (4) and (7) we obtain the desired result (2). That the equality sign there can appear only when E (in general, the closure of E) is a circular disc, is immediately recognized from the procedure of above mentioned proof.

4. The relation (4), combined with (6) and (7), yields

(8) $$L^2-4\pi F=4\pi^2C^2(2+\sum_{n=2}^{\infty}|c_n|^2C^{-2n})\sum_{n=2}^{\infty}|c_n|^2C^{-2n}+4\pi^2\sum_{n=1}^{\infty}n|a_n|^2C^{-2n},$$

(9) $$\pi D^2-4F\geqq 4\pi\sum_{n=1}^{\infty}n\,|a_n|^2C^{-2n},$$

respectively. Hence we shall obtain a sort of precision for corresponding inequalities, whenever an evaluation from below for absolute value of expansion coefficients $\{a_n\}_{n=1}^{\infty}$, $\{c_n\}_{n=2}^{\infty}$ is found with aid of geometrical quantities associated with the set E. To give an example, we shall now turn our attension to the *breadth* $\varDelta$ of E, defined as the minimum of widths of parallel strips containing E in its interior.

We suppose, without loss of generality, E to be put in such a position that the direction of a narrowest parallel strip containing E is parallel to the real axis of the w-plane. Now, the domain $|z|>C$ is mapped univocally by the function

(10) $$\zeta=z+C^2z^{-1}$$

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11) E. Landau u. O. Toeplitz, Über die grösste Schwankung einer analytischen Funktion in einem Kreise. Arch. f. Math. u. Phys (3) 11 (1906), 302–307.

onto the whole ζ-plane cut along the segment $\mathfrak{J}\zeta=0$, $-2C\leqq\mathfrak{R}\zeta\leqq 2C$. Let the function

$$(11) \qquad w=f(\zeta)\equiv\zeta+\sum_{n=0}^{\infty}b_n\zeta^{-n}$$

map this slit domain onto the complementary domain of E conformally and univocally. The relations (3), (10) and (11) imply

$$g(z)=f(z+C^2z^{-1})=z+b_0+(C^2+b_1)z^{-1}+\dots$$

by virtue of the uniqueness of mapping function, and hence, in particular, $a_1=C^2+b_1$. We may combine this relation with the right side of (9), but we will now combine it rather with the relation (4) from which (9) is derived. With abbreviation $\beta=\mathfrak{R}b_1$, it follows from (4) the inequality

$$(12) \qquad \frac{F}{\pi}\leqq C^2-\frac{|a_1|^2}{C^2}=C^2-\frac{|C^2+b_1|^2}{C^2}$$
$$=-2\mathfrak{R}b_1-\frac{|b_1|^2}{C^2}\leqq-2\beta-\frac{\beta^2}{C^2}.$$

We may add here in passing that it also follows particularly the inequality

$$(13) \qquad \beta\leqq-\frac{F}{\pi}\Big/\left(1+\sqrt{1-\frac{F}{\pi C^2}}\right)\leqq-\frac{F}{2\pi},$$

which is a precision of a de Possel's inequality[12] $\beta\leqq 0$.

Let now $\varDelta(r)$ denote the breadth in the direction of imaginary axis of the image curve of the ellipse

$$(14) \qquad \zeta=re^{i\theta}+C^2(re^{i\theta})^{-1}, \qquad 0\leqq\theta<2\pi \quad (r<C)$$

by the mapping (3), then

$$\varDelta(r)=\overline{\text{fin}}_{|z_1|=|z_2|=r}\ \mathfrak{J}\left(f(z_1+C^2z_1^{-1})-f(z_2+C^2z_2^{-1})\right)$$
$$\geqq\overline{\text{fin}}_{|z|=r}\ \mathfrak{J}\left(f(z+C^2z^{-1})-f(\bar{z}+C^2\bar{z}^{-1})\right).$$

Hence the coefficient

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12) R. de Possel, Zum Parallelschlitztheorem unendlich vielfach zusammenhängender Gebiete. Nachr. Ges. Wiss. Göttingen (1931), 192–202.

$$h_1=\frac{1}{2\pi i}\int_{|z|=r}f(\zeta)d\zeta=\frac{1}{2\pi i}\int_{|z|=r,\ \Im z>0}f(\zeta)f-(\bar{\zeta})d\zeta$$

in the expansion (11) has the real part β satisfying

$$\beta=\frac{1}{2\pi}\Im\int_{|z|=r}(f(\zeta)-f(\bar{\zeta}))d\zeta$$

$$\geqq-\frac{1}{2\pi}\left(\varDelta(r)2\left(r+\frac{C^2}{r}\right)+D(r)2\left(r-\frac{C^2}{r}\right)\right).$$

Remembering (7), we obtain, for $r\to C+0$,

(15) $$\beta\geqq-\frac{2C}{\pi}\varDelta\geqq-\frac{1}{\pi}D\varDelta.$$

Again using (7), it follows from (12)

$$\frac{F}{\pi}\leqq-2\beta-\frac{\beta^2}{C^2}\leqq-2\beta-\frac{4\beta^2}{D^2}=\frac{1}{4}D^2-\left(\frac{D}{2}+\frac{2\beta}{D}\right)^2$$

Finally we reach, by (15), the result: *If* $\pi D\geqq 4\varDelta$, *then the inequality*

(16) $$\pi D^2-4F>\frac{1}{\pi}(\pi D-4\varDelta)^2$$

is valid, the equality sign, as is easily seen, being excluded.

But the last result (16) is not precise enough. In fact, Kubota[13] has already proved the sharp inequality

(17) $$F\leqq\frac{1}{2}\varDelta\sqrt{D^2-\varDelta^2}+\frac{1}{2}D^2\arcsin\frac{\varDelta}{D}$$

and the equality sign here occurs when (the closure of) E is the part common to a circular disc with diameter D and a parallel strip with breadth $\varDelta$ laid symmetrically with respect to the center of that circle.

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13) T. Kubota, Einige Ungleichheitsbeziehungen über Eilinien und Eiflächen. Sci. Rep. Tôhoku Univ. **12** (1923), 45–65.