Journal of the Mathematical Society of Japan Vol. 1, No. 4, April, 1950.

On affine collineations in projectively related spaces.

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(Received Sept. 1, 1949)

§1. **Introduction.** M. S. Knebelman[1] proved a theorem on motions in conformally related Riemannian spaces: If an n-dimensional Riemannian space V_n admits an r-parameter group G_r of motions $(r<n)$, then there exist $n-r$ independent Riemannian spaces which are conformal to the Riemannian space V_n and admit the group G_r as group of motions.

One of the present authors has given in his forthcoming book a simple proof of this theorem. Let X_a be the r infinitesimal operators of the group G_r of motions in V_n whose fundamental tensor is $g_{\mu\nu}$, then we have

(1.1) $$X_a g_{\mu\nu}=0, \qquad (a,b,c,\ldots=1,2,\ldots r\,;\ \lambda,\mu,\nu,\ldots=1,2,\ldots,n).$$

In order that a Riemannian space which is conformal to V_n and consequently whose fundamental tensor is of the form $\rho^2 g_{\mu\nu}$ admit the G_r as a group of motions, it is necessary and sufficient that we have

$$X_a(\rho^2 g_{\mu\nu}) \doteq 0,$$

from which,

(1.2) $$X_a \rho^2=0,$$

because of (1.1). But, X_a being the operators of a group, that is, X_a satisfying the relations

(1.3) $$(X_b X_c)f=c_{bc}^{\cdot\cdot a}X_a f,$$

the equations (1.2) are completely integrable. Thus the theorem of Knebelman is proved.

The purpose of the present Note is to give a simple proof of an analogous theorem for group of affine collineations in projectively related spaces which is also due to Knebelman.[3]

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1) M. S. Knebelman: On groups of motions in related spaces. Amer. Jour. of Math., 52(1930), 280–282.

2) K. Yano: Groups of transformations in generalized spaces, in press.

3) M. S. Knebelman: Collineations of projectively related affine connections. Annals of Math., 29 (1928), 389—394.

§2. *Group of affine collineations in projectively related spaces.*

Let A_n be an n-dimensional affinely connected space whose components of the connection are $\Gamma^{\lambda}_{\mu\nu}$, and suppose that A_n admit an r-parameter group $G_r (r<n)$ of affine collineations. Then, deonting by X_a the infinitesimal operators of the group G_r, we have (1.3) and

$$X_a \Gamma^{\lambda}_{\mu\nu}=0. \tag{2.1}$$

Now, in order that an n-dimensional affinely connected space $\bar{A}_n$ which is projectively related to A_n admit the group G_r as a group of affine collineations, it is necessary and sufficient that there exist a covariant vector φ_ν such that

$$\bar{\Gamma}^{\lambda}_{\mu\nu}=\Gamma^{\lambda}_{\mu\nu}+\delta^{\lambda}_{\mu}\varphi_\nu+\delta^{\lambda}_{\nu}\varphi_\mu \tag{2.2}$$

and

$$X_a \bar{\Gamma}^{\lambda}_{\mu\nu}=0. \tag{2.3}$$

Subsituting (2.2) into (2.3), we find

$$X_a \Gamma^{\lambda}_{\mu\nu}+\delta^{\lambda}_{\mu} X_a \varphi_\nu+\delta^{\lambda}_{\nu} X_a \varphi_\mu=0,$$

from which

$$X_a \varphi_\nu=0, \tag{2.4}$$

because of (2.1).

Now, according to (1.3), the differential equations

$$X_a \varphi=0 \tag{2.5}$$

are completely integrable and admit $n-r$ independent solutions.

But, for one of these solutions, we have

$$X_a(\varphi_{;\nu})-(X_a\varphi)_{;\nu}=0, \quad \left(\varphi_{;\nu}=\frac{\partial\varphi}{\partial x^\nu}\right)$$

from which

$$X_a \varphi_{;\nu}=0. \tag{2.6}$$

Thus the gradient of the solution φ of (2.5) satisfies the (2.4). Thus we have the theorem of Knebelman:

Theorem: If an n-dimensional affinely connected space A_n admits an r-parameter group G_r of affine collineations $(r<n)$, then there exist $n-r$ independent affinely connected spaces, which are projectively related to A_n and admit the group G_r as group of affine collineations.

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