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Base conditions for hypersurfaces at a point.

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In this paper we shall study systematically the base condition at a given point for hypersurfaces in an n-dimensional affine space over an arbitrary ground field K.

A base condition for a system of hypersurfaces will be expressed by a certain set of linear (homogeneous) relations between the coefficients of the equations of hypersurfaces belonging to the system. Namely, taking the given point O as the origin of the coordinate system $OX_1X_2 \ldots X_n$, and the equation of such hypersurface being $f=\sum a_{ij\cdots k}X_1^iX_2^j\ldots X_n^k=0$ (a' s $\in$ K), the base condition is expressed by a set of equations

$$\sum a_{ij\cdots k}a_{ij\cdots k}^{(\lambda)}=0 \quad (a' \ s \in K) \quad (\lambda=1, 2, \ldots\ldots)$$

for the coefficients of the polynomial f. The totality of polynomials satisfying the base condition forms an ideal in the ring of polynomials.

Since the degree of the polynomial is not assigned by base conditions, it is preferable to deal more generally with formal power series. In §§ 2—3 the base condition will be discussed as a set of linear conditions related with linear mappings of K-vector space into K. In § 4 we shall characterize the base condition by using the Macaulay's inverse system from a new point of view.[1] In § 5 some results concerning with irreducible ideals are obtained.

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1. *The ring of formal power series.*

Let L be the ring $K[[X_1, \ldots\ldots, X_n]]$ of formal power series in $X_1, \ldots\ldots, X_n$ over K. Let us arrange all non-negative power products of $X_1, \ldots\ldots, X_n$ lexicographically and consider them linearly ordered. We denote them, for brevity, by $x_i (i=1, 2, \ldots\ldots)$. Then, any series f of L is expressible in the form $f=\sum_{i=1}^{\infty} a_i x_i$ ($a_i \in K$). If $a_1=\ldots\ldots=a_{r-1}=0$, $a_r\neq 0$, then r will be called the *rank* of f.

Let $f=\sum_{i=1}^{\infty} a_i x_i$ be a series of L. We introduce in L a weak topology, namely we define a neighborhood of f, for each finite set $i_1, \ldots\ldots, i_m$ of positive integers, as being the set of all such series $\sum_{i=1}^{\infty} a'_i x_i$ that a'_i is equal to a_i in case i is equal to one of $i_1, \ldots\ldots, i_m$ and is arbitrary ($\in K$) in other cases. Then, L can be considered as a complete Hausdorff space. All topological notions applied to L will be mentioned with regard to this topology.

LEMMA. *Every ideal of L is a closed linear subspace of L, if L is considered as a topological vector space over K.*

Proof. Let A be an ideal of L, and $f_1, \ldots\ldots, f_m$ a finite set of generators of the ideal A. Any infinite linear combination of the products $x_i f_j$ ($i=1,2,\ldots\ldots; j=1,\ldots\ldots, m$) over K is significant and contained in A, since there can not exist any such infinite subset of the $x_i f_j$ that all of its elements have a same rank. Hence, A is equal to the totality of all (finite or infinite) linear combinations of the $x_i f_j$ over K, and every convergent sequence of elements from A has its limit in A. Thus A is a closed linear subspace of L.

2. *Linear conditions and linear mappings.*

A linear mapping φ of the vector space L in the field K is called *continuous* if φ satisfies the condition:

$$\lim_{k\to\infty} \varphi[f_k]=\varphi[\lim_{k\to\infty} f_k]$$

for every convergent sequence f_k from L, namely, if almost all of $\varphi[f_k]$ ($k=1,2,\ldots\ldots$) are equal to $\varphi[\lim_{k\to\infty} f_k]$. The above condition is satisfied if and only if there exists such a neighborhood of the zero of L that is mapped by φ on the zero of K.

If φ is a continuous linear mapping of L in K, all but a finite number of the images $\varphi[x_i]$ of x_i ($i=1,2,\ldots\ldots$) must be zero. Conversely, a continuous linear mapping φ is uniquely determined by assigning arbitrary values from K as $\varphi[x_i]$ for a finite number of i and zero for the others.

Now, for each $i (i=1,2,\ldots\ldots)$, let ξ_i be such a linear mapping that

$$\xi_i[x_i]=1, \quad \xi_i[x_j]=0 \text{ (for all } j\neq i).$$

If we define as usual the addition and the K-multiplication for linear mappings of L in K, we see at once that *a linear mapping φ is continuous if and only if φ is equal to a finite linear combination of ξ_i ($i=1,2,\ldots\ldots$) over K.*

Let Λ be the totality of all *finite* linear combinations (or all continuous linear mappings) $\varphi=\sum_{i=1}^m a_i \xi_i$ ($a_i \in K$) (the number m depending on φ), and let us consider Λ as a vector space over K with trivial topology.

If $f=\sum_{i=1}^{\infty} a_i x_i$, $\varphi=\sum_{i=1}^{m} \alpha_i \xi_i$ are elements of L, Λ respectively, we get

$$\varphi[f]=\sum_{i=1}^{m} a_i \alpha_i.$$

From now on, we shall express φ and $\varphi[f]$ as $\varphi=\sum_{i=1}^{\infty} \alpha_i \xi_i$ and

$$\varphi[f]=\sum_{i=1}^{\infty} a_i \alpha_i, \tag{1}$$

considering all the α_i $(i>m)$ as zeros. So φ is a K-character of the topological K-module L and Λ the K-character group.

By setting

$$f[\varphi]=\varphi[f],$$

f can be considered as a linear mapping of Λ in K (a K-character of Λ), and we see at once that L *is equal to the totality of all such linear mappings; namely,* L *is the K-character group of* Λ.

3. *Duality between L and Λ.*

If f, φ are elements of L,Λ respectively such that $\varphi[f]=f[\varphi]=0$, then each of f,φ is called an *annihilator* of the other. If Γ is a linear subspace of Λ, the totality of all common annihilators (in L) of all elements of Γ will be denoted by $L(\Gamma)$; and if A is a closed linear subspace of L, we use the similar notation $\Lambda(A)$ for the set of all common annihilators in Λ of elements of A. Clearly $L(\Gamma)$ and $\Lambda(A)$ are always linear subspaces of L and Λ respectively, and the former is always closed on account of the continuity.

Let A be a closed linear subspace of L. Then, it is easily verified that *there exists a set of vectors* f_k $(k=1, 2,\ldots\ldots)$ *of ascending ranks such that* A *is equal to the set of all (finite or infinite) linear combinations of* f_k $(k=1, 2,\ldots\ldots)$ *over* K.[2] Such a set f_k will be called a *normal base* A.

For a vector $\varphi=\sum\alpha_i \xi_i$ of Λ, if we have $\alpha_m\neq 0$, $\alpha_i=0$ $(i>m)$, m will be called the *order* of φ. *If* Γ *is a linear subspace of* Λ, we see easily that *there exists a set of vectors* φ_k $(k=1,2,\ldots\ldots)$ *of ascending orders such that* Γ *consists of all finite linear combinations of the* φ_k *over* K. Such a set φ_k will be called a *normal base* of Γ.

Let A be a closed linear subspace of L with a normal base f_k $(k=1,2,\ldots\ldots)$. and r_k the rank of f_k. Let us now set, for all i $(i=1,2,\ldots\ldots)$,

$$y_i=\sum_{j=1}^{\infty} p_{ij}\, x_j=\begin{cases} f_k(\text{if } i \text{ is equal to } r_k) \\ x_i(\text{if } i \text{ is otherwise}), \end{cases} \tag{2}$$

then we get another normal base y_i of the whole space L. A consists of all linear combinations of y_i $(i=r_1, r_2,\ldots\ldots)$. In the infinite matrix $P=$

$\|p_{ij}\|$ of the coefficients of the equations (2), we have $p_{ii}\neq 0$ $(i=1,2,\ldots\ldots)$, $p_{ij}=0$ $(i>j)$, and we see at once that P has the inverse matrix $P^{-1}=\|p'_{ij}\|$ such that $p'_{ii}\neq 0$ $(i=1,2,\ldots\ldots)$, $p'_{ij}=0$ $(i>j)$.

Now, we transform also the normal base ξ_i of Λ using the transposed matrix of p^{-1} into the normal base

$$(3) \qquad \eta_i \textstyle\sum_{j=1}^{\infty} \xi_j \, p'_{ji} \qquad (i=1,2,\ldots\ldots).$$

Under these contragredient transformations (2), (3), the form of (1) is invariant: namely, if $f=\sum a_i x_i=\sum b_i y_i$ $(a_i, b_i \in K)$, $\varphi=\sum \alpha_i \xi_i=\sum \beta_i \eta_i$ $(\alpha,\beta \in K)$ are the expressions of f, φ of L, Λ respectively with regard to the old and the new normal beses of L, Λ, then we get

$$\varphi[f]=f[\varphi]=\sum a_i \alpha_i=\sum b_i \beta_i.$$

Therefore, Λ (A) is constituted of all finite linear combinations of $\eta_i (i\neq r_1, r_2,\ldots\ldots)$, and hence $L(\Lambda(A))$ is equal to the totality of all linear combinations of y_i $(i=r_1,r_2,\ldots\ldots)$, i.e. $L(\Lambda(A))=A$.

We can similarly show the equality $\Lambda(L(\Gamma))=\Gamma$ for any linear subspace Γ of Λ. Thus we get the Pontrjagin's theorem of duality in our case:

If A is a closed linear subspace of L, then we get $L(\Lambda(A))=A$. If Γ is a linear subspace of Λ, we get $\Lambda(L(\Gamma))=\Gamma$.

Since the sum-space and the intersection of any two closed linear subspaces of L are also closed, *the lattice $\{A\}$ of all closed linear subspaces A of L and the lattice $\{\Gamma\}$ of all linear subspaces Γ of Λ are dual-isomorphic by the reversible correspondence $A\to\Lambda(A), \Gamma\to L(\Gamma)$.*

REMARK. The statements of §§2–3 hold true in more general cases. Let L be a complete vector space over K with a base x_λ (λ running over a well-ordered set M of indices), i.e. the totality of all finite or enumerably infinite linear combinations of the x_λ over K; the topology being defined in our sense. On the other hand, let Λ be a vector space over K with a base ξ_λ (λ running over the same set M as above), i.e. the totality of all finite linear combinations of the ξ_λ over K. Then, we see that L and Λ are related in a same manner as it was stated above.

4. *Base conditions at a point.*

Let Δ be the vector space over K generated by the reciprocals x_i^{-1} of the power products x_i $(i=1,2,\ldots\ldots)$, i.e. the totality of all finite linear combinations of x_i^{-1} over K. Now, Δ can be considered as an L-module. Namely, for each $x_i \in L$ and for each $x_j^{-1} \in \Delta$, we define the multiplication

$x_i \times x_j^{-1}$ by setting

$$(4) \quad x_i \times x_j^{-1} = \begin{cases} x_k^{-1} (\text{if the usual product } x_i\, x_j^{-1} \text{ is equal to } x_k^{-1} \text{ for a value of } k) \\ 0 \ (\text{if otherwise}). \end{cases}$$

And, if $f=\sum_{i=1}^{\infty} a_i\, x_i \in L$, $\varphi'=\sum_{i=1}^{m} a_i x_i^{-1}=\sum_{i=1}^{\infty} a_i x_i^{-1}$ $(a_i=0\,(i>m)) \in \varDelta$, we set

$$(5) \qquad f\times\varphi'=\sum_{i,j=1}^{\infty} a_i\, a_j\, (x_i\times x_j^{-1}).$$

Since the right-hand side consists effectively of a finite number of non-zero terms, $f\times\varphi'$ is always an element of $\varDelta$, and we have

$$(6) \qquad f\times\varphi'=\sum_{k=1}^{\infty}\Big(\sum_{i,j}^{(k)} a_i a_j\Big)\, x_k^{-1},$$

where the summation $\sum_{i,j}^{(k)}$, for each k, means the summation over all pairs (i, j) such that $x_i x_j^{-1}=x_k^{-1}$. We have clearly

$$(fg)\times\varphi'=f\times(g\times\varphi')=g\times(f\times\varphi')=(gf)\times\varphi',$$
$$(f+g)\times\varphi'=f\times\varphi'+g\times\varphi',\ f\times(\varphi'+\psi')=f\times\varphi'+f\times\psi',$$
$$(cf)\times\varphi'=c(f\times\varphi'),\quad f\times(\gamma\ \varphi')=\gamma(f\times\varphi),$$
$$(f,\ g\in L;\ \varphi',\ \psi'\in\varDelta;\quad c, \gamma\in K).$$

If we define an L-multiplication on $\varLambda$ by replacing ξ_j, ξ_k for x_j^{-1}, x_k^{-1} respectively in (4), (5), then $\varLambda$ becomes an L-module isomorphic to the L-module $\varDelta$.

While, if $f=\sum a_i\, x_i \in L$, then

$$x_k f=\sum_{i=1}^{\infty} a_i(x_i\ x_k)=\sum_{j=1}^{\infty} a_i\ x_j,$$

in the last summation i being, for each j, such that $x_i\, x_k=x_j$ i.e. $x_i x_j^{-1}=x_k^{-1}$ and if $\varphi=\sum a_j\, \xi_j \in \varLambda$, then

$$x_k\times\varphi=\sum_{j=1}^{\infty} a_j(x_k\xi_j)=\sum_{i=1}^{\infty} a_j\ \xi_i,$$

in the last summation j being, for each i, such that $x_k\times\xi_j=\xi_i$ i.e. $x_i x_k^{-1}=x_k^{-1}$. Consequently we get

$$(x_k\, f)[\varphi]=(x_k\times\varphi)[f]=\sum_{i,j}^{(k)} a_i a_j,$$

$\sum_{i,j}^{(k)}$ meaning the same as in (6). Hence, if $f=\sum a_i x_i \in L$, $\varphi'=\sum a_i\, x_i^{-1}\in \varDelta$, $\varphi=\sum a_i\ \xi_i \in \varLambda$, we get from (6)

$$(7) \qquad \begin{cases} f\times\varphi'=\sum_{k=1}^{\infty}\ (x_k\ f)[\varphi]x_k^{-1}=\sum_{k=1}^{\infty}(x_k\times\varphi)[f]x_k^{-1}, \\ f\times\varphi=\sum_{k=1}^{\infty}(x_k\ f)[\varphi]\ \xi_k=\sum\,(x_k\times\varphi)[f]\ \xi_k. \end{cases}$$

Thus every element $\varphi\in\varLambda$ can be considered as an L-homomorphism of the L-module L into the L-module $\varDelta$ (i.e. a $\varDelta$-character of the L-module L). And $\varLambda$ can be considered as the $\varDelta$-character group of L.

According to (7), we get $f\times\varphi=0$ if and only if $(x_k f)[\varphi]=(x_k\times\varphi)[f]=0$ for all k i.e. if and only if $\varphi\in\Lambda\ (Lf)$ or $f\in L(L\times\varphi)$. Hence, we obtain *the duality between ideals (i.e. L-submoduli) A of L and L-submoduli Γ of Λ by the reversible correspondences* $A\rightarrow\Lambda(A)$, $\Gamma\rightarrow L(\Gamma)$:

THEOREM 1. *If Γ is an L-submodule of Λ, then* $L(\Gamma)$ *is an ideal of L and equal to the set of all* $f\in L$ *such that* $f\times L=\{0\}$.[3] *If A is an ideal of L, then* $\Lambda(A)$ *is an L-submodule of Λ and equal to the set of all* $\varphi\in\Lambda$ *such that* $A\times\varphi=\{0\}$.

If an L-submodule Γ is not expressible as a sum of two L-submoduli both of which are properly contained in Γ, then Γ will be called an *irreducible L-submodule.*

We get by the duality:

THEOREM 2. *Let Γ be an L-submodule and A the ideal (of L) defined by Γ, i.e.* $A=L(\Gamma)$. *Then,* 1°) *A is an irreducible ideal if and only if Γ is irreducible as L-submodule*; 2°) *a decomposition of A into an intersection of irreducible ideals induces a decomposition of Γ into a sum of irreducible L-submoduli, and vice versa.*

Now we add a remarkable theorem:

THEOREM 3. *The ideal* $A=L(\Gamma)$ *is o-dimensional if and only if the L-submodule Γ has a finite K-module-base.*

Proof. Let $P=(X_1,\ldots\ldots,X_n)$ be the maximal prime ideal of the power series ring L. Then, A is 0-dimensional if and only if there exists a positive integer m such that $P^m\subset A$. Suppose that Γ has a finite module-base φ_k $(k=1\ldots\ldots,s)$ and that m_0 is the maximum of the orders of $\varphi^k(k=1,\ldots\ldots,s)$. Then, $\varphi_k[x_i]=0$ for all $i>m_0$ and for all k, and hence $x_i\in L(\Gamma)=A$ for all $i>m_0$. This implies that $P^m\subset A$ for sufficiently large m.

Conversely, suppose that $P^m\subset A$ for an integer m. Then, $x_i\in P^m\subset A$ for all sufficiently large i(say$>m_0$). Accordingly, if $\varphi=\sum a_i x_i^{-1}\in\Gamma$, we shall have $a_i=\varphi[x_i]=0$ for all $i>m_0$. Hence Γ is contained in the module $Kx_1^{-1}+\ldots\ldots+Kx_m^{-1}$ of finite rank over K.

REMARK. An L-submodule has a finite K-module base if it has a finite L-generators (cf. § 5), and vice versa.

5. *Irreducible L-submoduli.*

Let Φ be a subset of Λ. The minimum L-submodule containing Φ will be denoted by (Φ), and Φ will be called a set of *L-generators* of the L-submodule (Φ).

LEMMA. *An L-submodule Γ is irreducible if and only if, for each set Φ of L-generators of Γ and for each separation $\Phi=\{\Phi_1,\Phi_2\}$ of Φ into two subsets Φ_1, Φ_2, either (Φ_1) or (Φ_2) coincides with Γ.*

If we separate a set Φ of L-generators of Γ into two nonvacuous subsets Φ_1, Φ_2, we get clearly $\Gamma=(\Phi_1)+(\Phi_2)$, and so the proof of the lemma is immediate.

Now, given any element $\varphi \in \Lambda$, (φ) will be called a *principal L-submodule*; $(\varphi)=\{f\times\varphi | f \in L\}$.

THEOREM 4. *An L-submodule L defines a 0-dimensional irreducible ideal of L if and only if Γ is principal.*

Proof. If $A=L(\Gamma)$ is a 0-dimensional irreducible ideal, then Γ has a finite K-module-base, say $\varphi_1,\ldots\ldots, \varphi_s$ (by Theorem 3), consequently $\Gamma=(\varphi_1,\ldots\ldots, \varphi_s)$, and is irreducible as L-submodule. (Theorem 2). It follows (by the preceding lemma) that Γ coincides with one of $(\varphi_1),\ldots\ldots,(\varphi_s)$.

Conversely, let $\Gamma=(\varphi)$, φ being of order m. Since every element of $(\varphi)=\{f\times\varphi | f \in L\}$ is of order$\leqq m$, (φ) will have a finite K-module-base, and consequently A is a 0-dimensional ideal. Furthermore, each set of generators of Γ must contain at least one element of order m, and such an element is necessarily of the form $f\times\varphi$ with a unit f of the ring L. Hence, Γ satisfies the condition of irreducibility of the preceding lemma. Thus A is irreducible. q.e.d.

It is easy to prove the following theorems:

THEOREM 5. *If A_1, A_2 are ideals defined by L-submoduli Γ_1, Γ_2 respectively, then $\Gamma_1:\Gamma_2=A_2:A_1$ where $\Gamma_1:\Gamma_2$ means the set $\{f | f\times\Gamma_2\subset\Gamma_1\}$ and $A_2:A_1$ the quotient-ideal.*

THEOREM 6. *Let A_1, A_2 be 0-dimensional irreducible ideals defined by principal L-submoduli (φ_1), (φ_2) respectively. Then, A_1 contains A_2 if and only if $\varphi_2=f\times\varphi_1$ for an element f $(\in L)$. When that is so, 1°) A_1 contains A_2 properly if and only if f is a non-unit of L; 2°) $A_2:A_1$ is equal to the ideal (f, A_2); 3°) there exists no 0-dimensional irreducible ideal between A_1 and A_2 if and only if f is irreducible mod A_2.*

THEOREM 7. *If A is a 0-dimensional irreducible ideal, then so is $A:f$ for any non-zero $f \in L$.*

REMARK 1. Let $R=K[X_1,\ldots\ldots, X_n]$ be the ring of polynomials of $X_1,\ldots\ldots, X_n$ over K. The 0-dimensional ideal a of R belonging to the point O and the 0-dimensional ideal A of L correspond one to one by the reversible correspondences $a\rightarrow L\cdot a$, $A\rightarrow R\cap A$. Hence, we have a one-to-one correspondence between L-submoduli with finite module-bases and such 0-dimensional ideals of R. Theorems 4—7 hold true for 0-dimensional ideals of R.

REMARK 2. Assume that the ground field K is algebraically closed and of characteristic 0. Let $\boldsymbol{K}=K(X_1,\dots\dots, X_n)$ be the quotient field of R. Let $\boldsymbol{m}$ be the 0-dimensional prime ideal of R belonging to the point O. Given a 1-dimensional prime ideal $\boldsymbol{p}$ $(\subset \boldsymbol{m})$ of R definig an irreducible algebraic curve C through O, let $\bar{B}$ be a valuation along C with centre O. Namely, $\bar{B}$ is a valuation of the rest-class field $Q(R/\boldsymbol{p})$ such that every element of $\bar{K}$ has $\bar{B}$-value 0 and the rest-classes mod $\boldsymbol{p}$ of $X_1,\dots\dots X_n$ have positive $\bar{B}$-values. By means of $\bar{B}$ we shall define a "valuation" B of $\boldsymbol{K}$. If z is any element of the quotient-ring R_p, the $\bar{B}$-value of the rest-class of z mod $\boldsymbol{p}R_p$ will be denoted by $v_B(z)$; if $z \in \boldsymbol{K}$, $z \in R_p$, then we do not define its B-value $v_B(z)$. Such an evaluation B will be called a *valuation* of $\boldsymbol{K}$ with centre O (along C).

Let B be such a valuation of $\boldsymbol{K}$ with centre O. The set $\boldsymbol{B}$ of all elements of R_P with non-negative B-values is called the *valuation-ring* of B. The intersection of an ideal of $\boldsymbol{B}$ with R will be called the *v-ideal* of R belonging to the valuation B. Then, similarly as in Zariski (loc. cit.), we see the followings: 1°) all the v-ideals $\boldsymbol{q}_i$ of R belonging to B form a Jordan sequence $\{\boldsymbol{q}_i\}$ 2°) $\boldsymbol{q}_i$ are primary ideals for $\boldsymbol{m}$ and $\cap \boldsymbol{q}_i=\boldsymbol{p}$, 3°) for each non-zero element $a \in R$ and for each i, $\boldsymbol{q}_i : a=\boldsymbol{q}_j$ (j being an integer $\geqq i$). Furthermore, we can prove that any Jordan sequence $\{\boldsymbol{q}_i\}$ of ideals of R belongs to a valuation of $\boldsymbol{K}$ with centre O if all $\boldsymbol{q}_i$ contain a same 1-dimensional prime ideal of R and if the condition 3°) is satisfied.

Let $\{\boldsymbol{q}_i\}$ be such a Jordan sequence of ideals in R. We can prove that if an ideal $\boldsymbol{q}_i$ is irreducible, then the set of irreducible ideals among $\boldsymbol{q}_1,\dots$ $\dots, \boldsymbol{q}_{i-1}$ is completely determined by $\boldsymbol{q}_i$. Namely, they are the set of all distinct ideals $\boldsymbol{q}_i : a$ ($a \in R$) (necessarily irreducible by Theorem 7), arranged in accordance with inclusion relation.

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Notes.

1) Cf. F. S. Macaulay, Algebraic theory of modular systems. Cambridge Tracts, No. 19. Cambridge (1916); O. Zariski, Polynomial ideals defined by infinitely near base points. Amer. J. of Math., vol. 60 (1938), pp. 151—204.

2) If there are infinitely many f_k, all infinite linear combinations of f_k over K are convergent, since f_k are of ascending ranks.

3) Cf. Macaulay (loc. cit.).