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On the harmonic prolongation.

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In the customary proofs of the theorems of the harmonic prolongation, it is usual to make use of Green's formula and so it is necessary to assume the functions to have the first derivatives continuous on a part or whole of the bonndary. In this note we prove them without using Green's formula and show that the theorems hold without the assumption of the continuity of the first derivatives.

Theorem. I. *Let $v(x,y)$ be harmonic in the domain, $(x-a)^2+y^2<r^2$, $y>0$, and continuous on its closure, $(x-a)^2+y^2\leqq r^2$, $y\geqq 0$, and suppose that its normal derivative vanishes on the real axis, $y=0$, $a-r<x<a+r$. If we define the function $v^*(x,y)$ by setting*

(1) $$v^*(x, y)=v(x, y) \quad \text{when } (x-a)^2+y^2\leqq r^2,\ y\geqq 0,$$

$$v^*(x, y)=v(x,-y) \quad \text{when } (x-a)^2+y^2\leqq r,\ y\leqq 0,$$

then $v^*(x, y)$ is a harmonic prolongation of $v(x,y)$ in the circle $(x-a)^2+y^2<r^2$.

Without loss of generality we can assume that $r=1$ and $a=0$. We consider the Poisson integral $V(x,y)$ for the boundary value of $v^*(x,y)$,

$$V(x,y)=\frac{1}{2\pi}\int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\varphi-\theta)+r^2}\,v^*(\xi,\eta)\,d\varphi,$$

where $\xi+i\eta=e^{i\varphi}$ and $x+iy=re^{i\theta}$, $0\leqq r<1$. $V(x,y)$ is harmonic in $x^2+y^2<1$, continuous on $x^2+y^2\leqq 1$, and its normal derivative vanishes on the real axis, $y=0$, $-1<x<1$, because of the continuity and symmetry of $v^*(x,y)$. Then $u(x, y)=V(x, y)-v(x, y)$ has the same properties as $v(x, y)$ and vanishes on the peripherie $x^2+y^2=1$, $y\geqq 0$. Hence if we can prove the following lemma, we obtain the theorem I.

Lemma. *If $u(x, y)$ is harmonic in the domain $x^2+y^2<1$, $y>0$ and continuous on its closure $x^2+y^2\leqq 1$, $y\geqq 0$ and satisfies the following boundary conditions*:

(2) $$u(x, y)=0 \qquad \text{for } x^2+y^2=1,\ y\geqq 0\,;$$

(3) $$\frac{\partial u}{\partial n}=0 \qquad \text{for } -1<x<1, \quad y=0,$$

then $u(x, y)$ is identically zero.

The proof of Lemma. Let (ξ, η) be an inversion of (x, y) with respect to the unit circle and put

(4) $$u^*(\xi, \eta)=-u(x, y) \qquad (x^2+y^2\leqq 1, \ y\geqq 0)$$

then, from (2), we have the harmonic prolongation ot $u(x, y)$ on the upper half plane. We denote again by $u(x, y)$ this prolongation. $u(x, y)$ is continuous and bounded ; hence we have the Poisson representation of $u(x, y)$ in the upper half planc,

(5) $$u(x, y)=\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{y}{(t-x)^2+y^2}u(t)\,dt \quad (y>0),$$

where $u(t)=u(t, 0)$. By the boundary condition (3), for $-1<x<1$.

(6) $$0=\frac{\partial u}{\partial n}=\lim_{y\to 0}\frac{u(x, y)-u(x)}{y}$$
$$=\lim_{y\to 0}\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{u(t)-u(x)}{(t-x)^2+y^2}\,dt.$$

From (4) follows

$$\int_{|t|>1}\frac{u(t)-u(x)}{(t-x)^2+y^2}\,dt=-\int_{-1}^{1}\frac{u(t)+u(x)}{(1-tx)^2+(ty)^2}\,dt.$$

Therefore by (6) we have

(7) $$\lim_{y\to 0}\int_{-1}\frac{u(t)-u(x)}{(t-x)^2+y^2}\,dt=\int_{-1}^{1}\frac{u(t)+u(x)}{(1-tx)^2}\,dt.$$

Suppose that $u(t)$ is not identically zero in $-1\leqq t\leqq 1$, then the maximum M of $|u(t)|$ in $-1\leqq t\leqq 1$ is positive, and there exists an x_0 such that $u(x_0)=M$ or $u(x_0)=-M$. If $u(x_0)=M$, from the continuity of $u(t)$ and $u(1)=u(0)=0$, we can casily derive the following inequelities

(8) $$\lim_{y\to 0}\int_{-1}^{1}\frac{u(t)-u(x_0)}{(t-x_0)^2+y^2}\,dt<0,$$
$$\int_{-1}^{1}\frac{u(t)+u(x_0)}{(1-tx_0)^2}\,dt>0.$$

This contradicts the equation (7). When $u(x_0)=-M$, we are similarly led to the contradiction. Hence it results that $u(t)$ and therefore, by (4), $u(x, y)$ also is identically zero.

From the above theorem we can easily obtain the following

Theorem. Let D_1 and D_2 be two domains without common inner point, but whose boundaries contain a common isolated analytic curve C. Let $u_1(x, y)$ be harmonic in D_1 and $u_2(x, y)$ in D_2, and be continuous and

$$u_1(x, y)=u_2(x, y), \tag{10}$$

$$\frac{\partial u_1}{\partial n}=\frac{\partial u_2}{\partial n} \tag{11}$$

on the inner point of C, where $\partial/\partial n$ being the normal derivative in the same sense. Then each is the harmonic prolongation of the other.

Without loss of generality we can assume that C is a segment on the real axis. Let $(a, 0)$ be any inner point on the common boundary. Suppose that the upper semi-circle about $(a, 0)$ with sufficiently small radius belongs to D_1.

In order to prove the theorem it is sufficient to show that for every sufficiently small $r>0$.

$$\begin{aligned}\frac{1}{2\pi}\Big[\int_0^{\pi} & u_1\,(a+r\cos\varphi, r\sin\varphi)\,d\varphi \\ & +\int_{\pi}^{2\pi} u_2(a+r\cos\varphi, r\sin\varphi)\,d\varphi\Big] \\ &=u_1(a, 0)(=u_2(a, 0)).\end{aligned} \tag{12}$$

If we define the function $U(x,y)$ by setting

$$U(x, y)=u_1\,(x,-y)+u_2(x,-y)$$
$$\text{for } y\geqq 0,\quad (x-a)^2+y^2>r,$$

then $U(x, y)$ is harmonic for $(x-a)^2+y^2\leqq r$, $y>0$, and by (11) its normal derivative vanishes on the $a-r<x<a+r$, $y=0$.

Hence, denoting by $U^*(x, y)$ the hamonic prolongation of $U(x, y)$ in the circle $(x-a)^2+y^2<r^2$ in the theorem 1, we have

$$\frac{1}{2\pi}\int_0^{2\pi} U^*(a+r\cos\varphi, r\sin\varphi)\,d\varphi=U^*(a, 0)$$
$$=u_1(a,0)+u_2(a,0)=2u_1(a,0).$$

On the other hand, from the definition of U^*, we have

$$\frac{1}{2\pi}\int_0^{2\pi} U^*(a+r\cos\varphi, r\sin\varphi))\,d\varphi$$
$$=\frac{1}{2\pi}\int_0^{\pi}\{u_1(a+r\cos\varphi, r\sin\varphi)$$

$$+u_2(a+r\cos\varphi,-r\sin\varphi)\}\,d\varphi$$

$$+\frac{1}{2\pi}\int_{\pi}^{2\pi}\{u_1(a+r\cos\varphi,\ -r\sin\varphi)$$

$$+u_2\,(a+r\cos\varphi,r\sin\varphi)\}d\varphi$$

$$=\frac{1}{\pi}\left[\int_0^{\pi}u_1(a+r\cos\varphi,\ r\sin\varphi)d\varphi\right.$$

$$\left.+\int_{\pi}^{2\pi}u_2(a+r\cos\varphi,\ r\sin\varphi)d\varphi\right].$$

Thus we have (12), as was to be proved.

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