On the faithful representations of Lie groups

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It is well known that any Lie algebra may be represented faithfully by matrices. This result, which was first established by I. Ado, was by his powerful method proved by E. Cartan.[2] K. Iwasawa[3] has recently given purely algebraic proof; His result is most general in the sense that he proved the theorem for Lie algebras over an arbitrary field. But Cartan's analytical method of proof is a most direct one and has an advantage enabling us to discuss simultaneously the problem in the large. In the present paper we first attempt to simplify the Cartan's proof. Namely we construct a Lie algebra (which we shall call normal) containing the given algebra and having a more simple structure than the given one. Hence the problem is reduced to that of the faithful representations of normal Lie algebras and this may readily be reduced to nilpotent case by Cartan's method. Thus we may proceed without the rather complicated arguments in solvable and general cases. Further we prove some results on the faithful representations of Lie groups in the large. Some of them were already obtained by E. Cartan[4] and A. Malcev.[5]

§ 1. Let L be a Lie algebra over a field P. A linear mapping d on L will be called a derivation if

$$d[x, y] = [dx, y] + [x, dy].$$

The mapping $d_a : x \rightarrow [a, x]$ is, as we may readily verify, a derivation which we shall call inner derivation defined by an element $a \in L$.

In the following we assume that P is of characteristic 0.

Lemma 1.[6] Let d be a derivation of L. Then we may represent d uniquely in the form

(1) $$d = d^o + d^s, \qquad d^o\, d^s = d^s\, d^o,$$

1) Ado [1]
2) Cartan [3]
3) Iwasawa [1]
4) Cartan [6]
5) Malcev [3]. Malcev's papers [2] and [3] are not yet accessible to the writer. I knew his results on reading Mathematical Reveiews. Some were also obtained in M. Gotô [2] independently.
(6 Gantmacher [1]

where d^o and d^s are derivations which are, as the linear mappings on L nilpotent and semi-simple[7] respectively. If M is a linear subspace in L such that $d\,M \subseteq M$, then $d^o\,M \subseteq M$. Moreover if d' is a derivation such that $dd'=d'd$, then we have $d'd^o=d^od'$ and $d'd^s=d^sd'$.

Proof. First let P be algebraically closed. We decompose L into the eigen-spaces of d: $L=L_\alpha+L_\beta+\cdots\cdots$, where L_ρ mean the eigen-spaces corresponding to the eigen-values ρ ($\rho=\alpha,\beta,\cdots\cdots$). Then $[L_\rho\,L_\sigma]\subseteq L_{\rho+\sigma}$, where $L_{\rho+\sigma}=0$ if $\rho+\sigma$ is not an eigen-value. Now we define the linear mapping d^s by

$$d^s x = \rho x \,, \quad \text{for } x \in L_\rho \ \ (\rho=\alpha,\beta,\cdots\cdots).$$

We may easily see that d^s is a semi-simple derivation such that $dd^s=d^sd$ and $d'=d-d^s$ is nilpotent. Then $d=d'+d^s$, $d^od^s=d^sd^o$. The uniqueness and the other propositions may easily be seen from the above construction. Next let P be arbitrary and K be the finite Galois extension containing all the characteristic roots of d. Since d is also a derivation of L_K, we may decompose d as above into the form (1), where d^o and d^s are derivations of L_K. Let $e_1,\cdots\cdots,e_r$ be a basis of L over P and a a derivation of L_K defined by

$$a \cdot e_i = \sum_{k=1}^r e_k\, a_{ki}, \quad a_{ki} \in K.$$

Since the structural constants corresponding to the basis $e_1,\cdots\cdots,e_r$ belong to P, the linear mapping θa defined by

$$\theta\, a \cdot e_i = \sum_{k=1}^r e_k\, (\theta a_{ki})$$

is also a derivation, where θ denotes a substitution of the Galois group of K/P. Hence $\theta d=d=\theta d^o+\theta d^s$ and clearly $\theta d^o \cdot \theta d^s=\theta d^s \cdot \theta d$. Moreover, as we may easily see, θd^o and θd^s are nilpotent and semi-simple respectively. Hence we have $\theta d^o=d^o$ and $\theta d^s=d^s$ for any θ by the uniqueness of the decomposition (1) and this shows that d^o and d^s are derivations of L.

Lemma 2. Let R be a solvable ideal in L and N the maximal nilpotent ideal in R. Then $[L,R]\subseteq N$.

Proof. Let R_1 and N_1 be the radical and the maximal nilpotent ideal in L. Then $[L,R_1]\subseteq N_1$.[8] Hence $[L,R]\subseteq R\cap N_1\subseteq N$.

Definition. Let $L=S+R$, $S\cap R=0$ be a Levi decomposition of L, where S and R denote a semi-simple subalgebra and the radical of L respectively. We call L normal, if R contains a nilpotent ideal N in L and

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7) A linear transfomation is called semi-simple, if it has simple elementary divisors.

8) Cartan [1] P. 108 or Iwasawa [1], Matsushima [1]

an abelian subalgebra A such that $R=A+N$, $A\cap N=0$, $[S,A]=0$.[9]

Theorem 1. *We may construct a normal Lie algebra L_1 containing the given Lie algebra L such that every ideal in L is also an ideal in L_1 and*

$$L_1=A+L, \quad A\cap L=0 ,$$

where A is an abelian subalgebra.

Proof. Let S, R and N be a maximal semi-simple subalgebra, the radical and the maximal nilpotent ideal in L respectively. R is a completety reducible S-module by the well known complete reducibility of representations of semi-simple Lie algebras. Hence there exists an S-module M such that $R=M+N$. Since $[S,R]\subseteq N$ by Lemma 2, we have $[S,M]=0$. Let h be a regular element of R, i.e. an element which has as many different eigen-values as possible in the regular representation of R, and let $h=h_1 +h_2$, $h_1 \in M$, $h_2 \in N$. We may easily see that h_1 is also regular and $[h_1, S]=0$. Let $\varphi(x)=x^t \varphi_1(x)$ be the characteristic polynomial of the inner derivation $d=d_{h_1}$. where $(x,\varphi_1(x))=1$. Let R_0 and R_1 be the linear subspaces in R composed of all elements $x \in R$ such that $d^t x=0$ and $\varphi_1(d)x=0$ respectively. Then $R=R_0+R_1$ and since d induces in R a non-singular linear mapping, $dR_1=[h_1, R_1]=R_1$, whence R_1 is contained in N by Lemma 2. As is well known, R_0 is a nilpotent subalgebra. Furthermore since $dS=0$, we have $[S,R_0]\subseteq R_0$ and $[S,R_1]\subseteq R_1$. Let $N_0=R_0\cap N$. Then N_0 is an S-module and there exists an S-module M_0 such that $R_0=M_0+N_0$. We have as above $[S,M_0]=0$. As we may easily see, $N=N_0+R_1$. Let $a_1,...$ $...,a_m$ be a basis of M_0 over P. We decompose the inner derivations d_{a_i} into the form (1) and put $d_i=d_{a_i^s}$. Then we have $d_i x=0$, for $x \in S$ or $x \in R_0$, because d_{a_i} induce in $S+R_0$ nilpotent linear transformations. Then $d_i d_{a_j} x=d_i[a_j x]=[a_j, d_i x]=d_{a_j} d_i x$ for all $x \in L$, whence $d_i d_{a_j}=d_{a_j}\cdot d_i$. Therefore $d_i d_j=d_j d_i$ $(i,j=1,......, m)$ by Lemma 1. Let now A be the abelian Lie algebra spanned by the derivations $d_1,......,d_m$ and let $L_1=A+ L$ be the direct sum of two vector spaces A and L. We define the multiplication between elements of L_1 as follows:

$$[u_1+x_1, \quad u_2+x_2]=u_1 x_2-u_2 x_1+[x_1 x_2]$$

where $u_i \in A$ and $x_i \in L$ (i=1,2) and $u_i x_j$ mean the element of L which are the images of x_j under the derivations u_i. We may easily verify that L_1 becomes a Lie algebra. Furthermore A is an abelian subalgebra and

9) Normal Lie algebras have the similar structure to algebraic Lie algebras. For algebraic Lie algebras, see Chevalley and Tuan [1], Gotô [1], Matsushima [1]

every ideal in L is also an ideal in L by Lemma 1. Clearly $A+R$ is the radical of L_1 and $A+R=A+M_0+N$. Let $N_1=P(a_1-d_1)+\cdots\cdots+P(a_m-d_m)+N$. then $A+R=A+N_1$. We easily verify that N_1 is an ideal in L_1. Moreover, since N_1 is solvable and N is nilpotent and every eigenvalue of a_i-d_i in the adjoint representation is 0, N_1 is nilpotent. Further we have $[S,A]=0$. Thus L_1 is normal.

Remark. In the above proof, we used the semi-simplicity of S only in the point that R is a completely reducible S-module. Further we may prove that N_1 is the maximal nilpotent ideal and the derived algebra of L_1 coincides with that of L.

§ 2. In this section we apply Theorem 1 on the proof of faithful representability of Lie algebras. It is sufficient by Theorem 1 to prove the problem for normal Lie algebras. Let L be a normal Lie algebra over the field P of all real or complex numbers and let

$$L=S+A+N, \quad A+N=R, \quad [S,A]=0,$$

as in § 1. Then $T=S+A$ is a subalgebra in L, whose center equals A. Let $\mathfrak{N}$ and $\mathfrak{T}$ be local subgroups in $\mathfrak{L}$, which correspond to the subalgebras N and T respectively, where $\mathfrak{L}$ denotes the Lie group whose Lie algebra is L. Then $\mathfrak{N}$ is a nilpotent invariant subgroup and we may introduce canonical coordinates (x) in $\mathfrak{N}$ such that, if $x'=ax$, where a and x denote the elements of $\mathfrak{N}$ whose coordinates are (a) and (x) respectively, then $x'_i=g_i(a,x)$ are polynomials of (a) and (x). Moreover

$$\mathfrak{L}=\mathfrak{N}\mathfrak{T}, \qquad \mathfrak{N}\cap\mathfrak{T}=1.$$

Hence every element of $\mathfrak{L}$, which is sufficiently near the unit element, may be uniquely represented in the form ab, $a \in \mathfrak{N}$, $b \in \mathfrak{T}$. Then $(ab)(xy) = x'y'$, where

(2)　　　　$x'=a(b \times b^{-1})$, $y'=by$, x, $a \in \mathfrak{N}$; b, $y \in \mathfrak{T}$.

As we may readily verify, the transformations $(x)\to(x')$ of the coordinates in $\mathfrak{N}$ defined by (2) form a transformation group $\mathfrak{G}$ homomorphic to $\mathfrak{L}$. Moreover the elements of $\mathfrak{L}$, which correspond to the unit transformation of $\mathfrak{G}$, are contained in $\mathfrak{T}$. Hence

$$\mathfrak{G}\cong\mathfrak{L}/\mathfrak{D}$$

where $\mathfrak{D}$ is an invariant subgroup contained in $\mathfrak{T}$. We represent the transformation in $\mathfrak{G}$ defined by (2) by the equations

$$x'_i=F^i\ (ab;\ x).$$

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10)　Cartan [3]

Let $f(x) = b \, x \, b^{-1}$. Then, since the coordinates are canonical,

$$f_i(x) = \sum_k a_{ik}(b) \; x_k.$$

and we have $F_i(ab \; ; \; x) = g_i(a \; ; \; f(x))$. As $g_i(a \; ; \; f(x))$ are polynomials of (x). Hence we may faithfully represent $\mathfrak{G}$ by matrices by Cartan's lemma.[10] Hence there exists a representation R_1 of L such that $R_1 \cong L/D$, where D denotes the ideal in L which corresponds to the invariant subgroup $\mathfrak{D}$ in $\mathfrak{L}$. That is clearly a faithful representation R_2, because it is the direct sum of semi-simple ideal and its centre. Since $T = L/N$, R_2 is also a representation of D. Then, as D is contained in T, the representation R of L defined by

$$R = \begin{bmatrix} R_1 & \\ & R_2 \end{bmatrix}$$

is faithful. Thus we proved the

Theorem 2. *Any Lie algebra L over the field P of real or complex numbers may be represented faithfully by matrices.*

§ 3. Next we investigate the problem of faithful representability of Lie groups in the large. Let $\mathfrak{N}$ be a simply connected nilpotent Lie group. The parameters of $\mathfrak{N}$ may be real or complex. $\mathfrak{N}$ is homeomorphic with an euclidean space and we may introduce coordinates in the large in $\mathfrak{N}$ such that they are canonical in the neighbourhood of the unit element and if $x' = ax$, then the coordinates of x' are polynomials of a and x^{10}. Let θ be an antomorphism of $\mathfrak{N}$ and let $x_i' = \theta_i(x)$, where $\theta_i(x)$ are coordinates of $\theta(x)$. If (x) is sufficiently small, then, since the coordinates are canonical, $x_i' = \sum_j \theta_{ij} \, x_j$. Now we define the mapping $\bar{\theta}$ of $\mathfrak{N}$ onto itself by the equations

$$x_i' = \sum_j \theta_{ij} \, x_j$$

and show that $\theta = \bar{\theta}$. For, since $\theta = \bar{\theta}$ in the neighbourhood of (0). $\theta(ab) = \theta(a) \, \theta(b)$ for sufficiently small (a), (b). But as the coordinates of $\bar{\theta}$ (ab) and $\bar{\theta}(a) \cdot \bar{\theta}$ (b) are polynomials of (a) and (b) respectively, $\bar{\theta}(ab) = \bar{\theta}(a) \cdot \bar{\theta}(b)$ holds for any a and b. Thus $\bar{\theta}$ is an automorphism of the connected group $\mathfrak{N}$ and we have $\theta = \bar{\theta}$, since they coincide in the neighbourhood of the unit element. Thus every automorphism of $\mathfrak{N}$ is represented (in our coordinate) by linear transformation. Then we may prove the following Lemma by the similar argument as in the proof of Theorem 2.

Lemma 3. *Let $\mathfrak{L}$ be a Lie group and let $\mathfrak{T}$ and $\mathfrak{N}$ be a closed subgroup and a closed simply connected nilpotent invariant subgroup respe-*

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10) Cartan [3].

ctively such that

$$\mathfrak{L}=\mathfrak{T}\mathfrak{N} \qquad \mathfrak{T}\cap\mathfrak{N}=1$$

If $\mathfrak{T}$ is faithfully representable (f.r.), then $\mathfrak{L}$ is also f.r.

Now we may prove the following

Theorem 3. *Any simply connected complex Lie group $\mathfrak{L}$ is f.r.*

Proof. Let L be the Lie algebra of $\mathfrak{L}$ and L_1 the normal Lie algebra containing L constructed in § 1. Then $L_1=A+L$ and the simply connected Lie group $\mathfrak{L}_1$ corresponding to L_1 contains $\mathfrak{L}$ as a closed subgroup.[11] Hence we may assume that L itself is normal. Then $L=S+A+N$ as in § 1 and L contains simply connected closed subgroups $\mathfrak{T}$ and $\mathfrak{N}$ whose Lie algebras are $S+A$ and N respectively.[12] Moreover $\mathfrak{L}=\mathfrak{T}\mathfrak{N}$, $T\cap\mathfrak{N}=1$ and $\mathfrak{N}$ is nilpotent and invariant and $\mathfrak{T}$ is the direct product of a simply connected complex semi-simple group and a vector group. Since connected complex semi-simple groups are f.r. by a theorem of Cartan[12] and vector groups are also f.r., the same holds for $\mathfrak{T}$. Then $\mathfrak{L}$ is f.r. by Lemma 3.

Remark. If $\mathfrak{L}$ is solvable, as we may see from the above proof, the condition that $\mathfrak{L}$ is of complex parameters is unnecessary.

Theorem 4. *Let $\mathfrak{L}$ be a connected Lie group and L its Lie algebra. Suppose that $\mathfrak{L}$ contains connected closed subgroups $\mathfrak{T}$ and $\mathfrak{N}$ which satisfy the following conditions :*

   1) $\mathfrak{N}$ *is a simply connected solvable invariant subgroup and $\mathfrak{T}$ is f.r.*
   2) $\mathfrak{L}=\mathfrak{T}\mathfrak{N}$, $\mathfrak{T}\cap\mathfrak{N}=1$.
   3) *Denote by T and R the subalgebras of L corresponding to $\mathfrak{T}$ and $\mathfrak{N}$ respectively. Then R is a completely reducible T-module.[13] Then $\mathfrak{L}$ is f.r.*

Proof. By the remark in § 1. we may construct a Lie algebra L_1 containing L as an ideal such that

$$L_1=A+L=T+R+A. \qquad A\cap L=0, \quad [AT]=0.$$

where A is an abelian subalgebra. Moreover $A+R$ is a solvable ideal in L and $A+R=A+N$, where N is a nilpotent ideal in L. We construct a Lie group $\mathfrak{L}_1$ whose Lie algebra is L_1 such that it contains $\mathfrak{L}$ as a closed subgroup and satisfies the conditions of Lemma 3. Denote by $\mathfrak{A}$ the vector group whose Lie algebra is A. Let $\mathfrak{L}_1$ be the product space of $\mathfrak{T}\times\mathfrak{A}$ and $\mathfrak{N}$, where $\mathfrak{T}\times\mathfrak{A}$ denotes the direct product of the groups $\mathfrak{T}$ and $\mathfrak{A}$. We

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   11) Pontrjagin [1] Chap. IX § 54
   12) Cartan [2]
   13) The condition 3) is satisfied, if $\mathfrak{T}$ is compact or semi-simple.

will define a continuous automorphism φ_x of $\mathfrak{R}$ for each $x \in \mathfrak{T} \times \mathfrak{A}$ in such a way that $\varphi_{xy} = \varphi_x \varphi_y$ holds. Put $\varphi_t(r) = trt^{-1}$ for $t \in \mathfrak{T}$, $r \in \mathfrak{R}$. Let next $\mathfrak{L}_1'$ be the local Lie group corresponding to L_1. Then $\mathfrak{L}_1' = (\mathfrak{T}' \times \mathfrak{A}')\mathfrak{R}'$. $(\mathfrak{T}' \times \mathfrak{A}') \cap \mathfrak{R}' = 1$, where $\mathfrak{T}'$ etc. may be considered as the neighbourhoods of the unit element of $\mathfrak{T}$ etc. Let $\varphi_{a'}(r') = a'r'a'^{-1}$, where $a' \in \mathfrak{A}'$, $r' \in \mathfrak{R}'$. Since $\mathfrak{R}$ is simply connected, we may extend $\varphi_{a'}$ to a continuous automorphism of $\mathfrak{R}$. More over we may define a continuous automorphism φ_a for each $a \in \mathfrak{A}$ by the argument given in Pontrjagin [1] p. 275–276. Clearly $\varphi_{xy} = \varphi_x \varphi_y$ for x, $y \in \mathfrak{T}$ or $x, y \in \mathfrak{A}$. We show that

$$\varphi_t(\varphi_a(r)) = \varphi_a(\varphi_t(r)), \quad t \in \mathfrak{T}, \quad a \in \mathfrak{A}, \quad r \in \mathfrak{R}.$$

Since each group is connected, it is sufficient to prove the above relation for $t \in \mathfrak{T}'$, $a \in \mathfrak{A}'$, $r \in \mathfrak{R}'$. Then

$$\varphi_t(\varphi_a(r)) = t(ara^{-1})t^{-1} = (ta)r(ta)^{-1} = (at)r(at)^{-1} = \varphi_a(\varphi_t(r))$$

Put $\varphi_x(r) = \varphi_t(\varphi_a(r))$ for $x = t \; a \in \mathfrak{T} \times \mathfrak{A}$, where $t \in \mathfrak{T}$, $a \in \mathfrak{A}$. Then clearly $\varphi_{xy} = \varphi_x \varphi_y$. We now define the product of two pairs (r_1, x_1), (r_2, x_2) $(r_i \in \mathfrak{R}, x_i \in \mathfrak{T} \times \mathfrak{A})$ of the set $\mathfrak{L}_1$ as follows:

$$(r_1, x_1) \ (r_2, x_2) = (r_1 \; \varphi_{x_1}(r_2), \; x_1 x_2).$$

Then $\mathfrak{L}_1$ becomes a Lie group whose Lie algebra is $L_1^{[11]}$. Clearly $\mathfrak{L}$ is a closed invariant subgroup in $\mathfrak{L}_1$. Furthermore $\mathfrak{L}_1 = (\mathfrak{T} \times \mathfrak{A})\mathfrak{R}$, $(\mathfrak{T} \times \mathfrak{A}) \cap \mathfrak{R} = 1$ and, as we may see from the above construction $\mathfrak{A}\mathfrak{R}$ is a simply connected subgroup whose Lie algebra is $A + R$. Since $A + R = A + N$. $\mathfrak{A}\mathfrak{R}$ contains simply connected nilpotent subgroup $\mathfrak{N}$ such that $\mathfrak{A}\mathfrak{R} = \mathfrak{A}\mathfrak{N}$, Then $\mathfrak{L}_1 = (\mathfrak{T} \times \mathfrak{A})\mathfrak{N}$ and since $(\mathfrak{T} \times \mathfrak{A}) \cap \mathfrak{A}\mathfrak{R} = \mathfrak{A}$, $(\mathfrak{T} \times \mathfrak{A}) \cap \mathfrak{R} = (\mathfrak{T} \times \mathfrak{A}) \cap \mathfrak{A}\mathfrak{R} \cap \mathfrak{R} = \mathfrak{A} \cap \mathfrak{R} = 1$. Further $\mathfrak{T} \times \mathfrak{A}$ is f.r. Thus $\mathfrak{L}_1$ satisfies the conditions in Lemma 3 and is therefore f.r. Since $\mathfrak{L}_1 \supset \mathfrak{L}$, the same is true for $\mathfrak{L}$.

Remark. Theorem 3 follows also from this theorem.

From Theorem 4 and a theorem of A. Malcev[14] on the structure of linear solvable Lie groups we obtain immediately the following

Theorem 5.[15]   *A connected solvable Lie group $\mathfrak{L}$ is f.r. if and only if it contains a connected compact abelian group $\mathfrak{A}$ and a closed simply connected invariant subgroup $\mathfrak{R}$ such that*

$$\mathfrak{L} = \mathfrak{A}\mathfrak{R}, \qquad \mathfrak{A} \cap \mathfrak{R} = 1.$$

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14) Maldev [1], [2] Gotô [2]   15) Malcev [3], Gotô [2]

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