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An operator-theoretical treatment of temporally homogeneous Markoff process

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1. *Introduction.* Let $\{U_t\}$, $0 \leqq t < \infty$, be a one-parameter semi-group of linear (=everywhere defined additive, continuous) operators from a complex Banach space E to E:

(1.1) $\quad U_t\, U_s = U_{t+s}.\quad U_0 = I$ (=the identity operator).

(1.2) $\quad \sup_t \| U_t \| \leqq 1,$

(1.3) $\quad \lim_{t \to t_0} U_t\, x = U_{t_0}\, x,\ 0 \leqq t_0 < \infty$ (lim=strong limit).

In a preceding note[1], the author obtained the following results. i) If D is the totality of x for which

(1.4) $\quad \text{weak limit}_{h \downarrow 0}\ h^{-1}\,(U_h - I)x = Ax$

exists, then D coincides with the totality of x for which

(1.4)′ $\quad \lim_{h \downarrow 0} h^{-1}(U_h - I)x = Ax$

exists and D is dense in E. The differential quotient operator (d.q.o.) A is a closed additive operator from D to E with the properties:

(1.5) $\quad U_t x - x = \int_0^t U_s\, Ax\, ds \quad$ for $x \in D$,

(1.6) for any positive integer n, $I_n = (I - n^{-1} A)^{-1}$ exists and $\| I_n \| \leqq 1$, $AI_n = n(I_n - I)$, $\lim_{n \to \infty} AI_n x = Ax$ for $x \in D$,

(1.7) $\quad I_n x = \int_0^\infty n \exp(-nt) U_t x\, dt$ and $\lim_{n \to \infty} I_n x = x$ for $x \in E$.

(1.8) $U_t x = \lim_{n \to \infty} \exp(tAI_n)\ x,\ x \in E$, uniformly in t for any finite interval of t.[2]

ii) Let conversely A be an additive operator from a dense linear subset D of E such that (1.6) is satisfied for any positive integer n, then there

1) On the differentiability and the representation of the one-parameter semi-group of linear operators, the Journal of the Math. Soc. of Japan, 1 (1948).

2) We may obtain, similarly as (1.8), another representation of U_t:

(1.8)′ $\quad U_t\, x = \lim_{n \to \infty} (I - n^{-1} t\, A)^{-n}\, x.$

exists a uniquely determined one-parameter semi-group $\{U_t\}$ which satisfies (1.1) – (1.5). This $\{U_t\}$ is given by (1.8).

The purpose of the present note is to give, as an application of these results, a characterisation of the *temporally homogeneous Markoff process*. By virtue of the composition rules for the d.q.o.'s and the differentiability theorem (1.4)', we may determine the explicite form of the d.q.o. A in the special case where the Markoff process is not only temporally but also spatially homogeneous. The result may be considered as an operator-theoretical interpretation of the infinitely divisible law.[3] The results in **2** are also applied to the integration of the Fokker-Planck's equation.

2. *A characterisation of the d.q.o. of the temporally homogeneous Markoff process.* Let E be an *abstract-L-space*[4] and let, for $t \geqq 0$,

(2.1) U_t be a positive operator ($U_t x \geqq 0$ for $x \geqq 0$) isometric on positive elements ($\|U_t\, x\| = \|x\|$ for $x \geqq 0$.)

Such operator may be called a *transition-operator*, and the semi-group $\{U_t\}$ may be considered as an abstract form of the temporally homogeneous Markoff process. In this case,

(2.2) $I_n = (I - n^{-1}\, A)^{-1} (= \int_0^\infty n \exp(-nt) U_t\, dt)$ is, for each $n = 1, 2, \ldots\ldots$, a transition operator.

Conversely it is easy to see that if (2.2) is satisfied for large n then

$$\begin{aligned} U_t x &= \lim_{n\to\infty} \exp\ (tAI_n)x = \lim_{n\to\infty} \exp(tn\ (I_n - I))x \\ &= \lim_{n\to\infty} \exp\ (-nt)\ \exp\ (tnI_n)x \end{aligned}$$

is also a transition operator. Thus we may construct all the temporally homogeneous Markoff process satisfying the continuity condition (1.3).

Let, in particular, E be the space $L_1(-\infty, \infty)$ and let $x \geqq 0$ mean $x(t) \geqq 0$ almost everywhere on $(-\infty, \infty)$. Then the additive operators

$$\begin{aligned} (2.3) \qquad (Ax)(s) &= \gamma x'(s) \qquad (\gamma\ real \gtrless 0), \\ &= \sigma x''(s) \qquad (\sigma > 0), \\ &= \lambda(x(s-u) - x(s)) \quad (\lambda > 0,\ u \gtrless 0). \end{aligned}$$

satisfy (1.6) aud (2.2). For the proof see the examples below.

Example 1. Let us consider the translation:

$$(2.4) \qquad (U_t x)(s) = x(s+t), \quad x(s) \in L_1(-\infty, \infty).$$

We have

3) P. Lévy: Théorie de l'addition des variables aléatoires, Paris (1937).

$$y_n(s)=(I_nx)(s)=\int_s^\infty n \exp\ (n(s-k))\ x(k)\ dk,$$

and hence, if $x(s)$ is continuous,

$$y_n'(s)-ny_n(s)=-n\ x(s).$$

Thus

$$(2.5)\qquad (AI_nx)(s)=(Ay_n)(s)=(n(I_n-I)x)(s)=y_n'(s)$$
$$=\frac{d}{ds}\int_s^\infty n \exp\ (+n(s-k))\quad x(k)dk$$

and hence the operator $A=A_T$ is the differential operator $(\frac{d}{ds})$
We have, by (1.8) and (1.8)' two expansions of Taylor's type[5].

Example 2. Consider the integral

$$(2.6)\qquad (U_tx)(s)=\frac{1}{\sqrt{\pi t}}\int_{-\infty}^\infty exp\Big(-\frac{(s-k)^2}{t}\Big)x(k)dk,\ x(k)\in L_1(-\infty,\infty)$$

corresponding to the *Gaussian distribution.* We have

$$y_n(s)=(I_nx)(s)=\int_0^\infty x(k)\,dk\int_{-0}^\infty\frac{1}{\sqrt{\pi t}}\ n\ \exp\ (-nt-\frac{(s-k)^2}{t})\ dt$$
$$=\int_{-\infty}^\infty\sqrt{n}\ \exp\ (-2\sqrt{n}|s-k|)\ x(k)dk,$$

and hence, if $x(s)$ is continuous,

$$y_n''(s)=4n\ y_n(s)-4nx(s).$$

Thus

$$(2.7)\qquad (AI_n\ x)(s)=(Ay_n)(s)=(n(I_n-I)x)(s)=4^{-1}\ y_n''(s)$$
$$=\frac{1}{4}\ \frac{d^2}{ds^2}\int_{-\infty}^\infty\sqrt{n}\exp\ (-2\sqrt{n}\ |s-k|)\ x(k)\ dk.$$

Therefore $A=A_G$ is the differential operator $(\frac{1}{4}\ \frac{d^2}{ds^2})$, and we have, by (1.8) and (1.8)', two expansions, the first of which improves Eddington's formal expansion.[6]

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4) G. Birkhoff: Lattice Theory, New York (1940). S. Kakutani; Concrete representation of abstract (L)-spaces and the mean ergodic theorem, Ann, of Math., **42** (1941).

5) Cf. N. Dunford and I. E. Segal: Semi-groups of operators and the Weierstrass theorem, Bullet. Amer. Math. Soc., **52** (1946).

6) A. A. Eddington: On a formula for correcting statistics for the effect of a known probable error of observations, Monthly Notice R. Astr. Soc., **73** (1914).

Example 3. Let $\lambda>0$, $u \neq 0$ and consider

(2.8) $$(U_t\, x)(s) = \exp\,(-\lambda\, t) \sum_{k=0}^{\infty} \frac{(\lambda t)^k}{k\,!}\, x(s-ku),\ x(s) \in L_1(-\infty,\infty)$$

corresponding to the *Poisson distribution*. We have

$$y_n(s) = (I_n x)(s) = \int_0^{\infty} n\ \exp(-(n+\lambda)t) \sum_{k=0}^{\infty} \frac{(\lambda\ t)^k}{k\,!}\, x(s-ku)\ dt$$

$$= \sum_{k=0}^{\infty} \frac{n\lambda^k}{(n+\lambda)^{k+1}}\ x\ (s-ku)\ ,$$

and therefore, when $n \to \infty$.

(2.9) $$(Ay_n)(s) = (n(I_n - I)x)(s) \longrightarrow (Ax)(s) = \lambda\ (x(s-u) - x(s)).$$

$A = A_P$ is thus the difference operator.

Example 4. Let A be a linear operator defined on the abstract-L-space E satisfying the condition:

(2.10) $P = k^{-1}\,(A+kI)$ is a transition operator for a certain positive number k.

In this case we may show that $(I-n^{-1}\,A)^{-1}$ satisfies, for $n>0$, (2.2) and (1.6).

Proof. We have

$$I - n^{-1}A = (1+\sigma)(I - \frac{\sigma}{1+\sigma}\, P),\ \sigma = n^{-1}k > 0,$$

and hence, by $\left\| \frac{\sigma}{1+\sigma}\, P \right\| = \frac{\sigma}{1+\sigma} < 1 = \| I \|$,

(2.11) $$(I - n^{-1}\, A)^{-1} = (1+\sigma)^{-1} \left\{ I + \sum_{m=1}^{\infty} \left(\frac{\sigma}{1+\sigma}\right)^m P^m \right\}$$

exists. It is easy to see that this expansion defines a transition operator with P. (Q.E.D.).

The example 3 surely satisfies (2.10). The criterion (2.10) may also be used to deduce

Kolmogoroff's theorem.[7] Let E be the space of the n-dimensional complex vectors $x = (\xi_1, \xi_2, \ldots\ldots \xi_n)$ with the norm $\| x \| = \sum_{i=1}^{n} | \xi_i |$, and let $x \geqq 0$ mean $\xi_i \geqq 0$ $(i = 1,2,\ldots\ldots,n)$. Then the transition operator P on E is represented by the matrix (p_{ij}) satisfying the condition:

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7) A. Kolmogoroff: Die analytische Methoden in der Wahrscheinlichkeitsrechnung, Math. Ann., **104** (1931).

(2.12) $$p_{ij} \geqq 0, \quad \sum_{j=1}^{n} p_{ij} = 1.$$

In this case, the d.q. matrix $A = (a_{ij})$ of the one-parameter semi-group of transition matrices $U(t) = (u_{ij}(t))$ is characterised by

(2.13) $$\sum_{j=1}^{n} a_{ij} = 0, \quad a_{ij} \geqq 0 \ (i \neq j), \quad a_{ii} \leqq 0.$$

3. *Composition of the d. q. o. 's.* We shall give two lemmas which enable us to construct another d.q.o. of the temporally homogeneous Markoff process from, for example, the d.q.o.'s in **2**.

Lemma 1. Let the intersection D of the domains of two additive operators A_1 and A_2 satisfying (1.6) and (2.2) be dense in E. Let, moreover, A_1 and A_2 be commutative in the sense that

(3.1) $$A_1 A_2 x = A_2 A_1 x,$$

if either A_1A_2x or A_2A_1x is well defined. Then

(3.2) $$A = A_1 + A_2$$

also satisfies (1.6) and (2.2).

Proof. Put

(3.3) $$I_{1n} = (I - n^{-1}A_1)^{-1}, \quad I_{2n} = (I - n^{-1}A)^{-1}.$$

Then

(3.4) $$A^{(n)} = A_1 I_{1n} + A_2 I_{2n} = n(I_{1n} - I) + n(I_{2n} - I)$$

satisfies (2.10) and hence (1.6) and (2.2) too. Thus the semi-group

(3.5) $$U_t^{(n)} = \exp(tA^{(n)})$$

constitutes a temporally homogeneous Markoff process satisfying (1.3). By (3.1), $A^{(m)}$ is commutative with $U_t^{(n)}$. Hence we have

$$\left\| (U_t^{(m)} - U_t^{(n)}) x \right\| = \left\| \int_0^t \frac{d}{ds} (\exp((t-s)A^{(n)}) U_s^{(m)} x) ds \right\|$$

$$= \left\| \int_0^t (\exp((t-s)A^{(n)}) U_s^{(m)} (A^{(n)} - A^{(m)})) x ds \right\|$$

$$\leqq \int_0^t \| (A^{(m)} - A^{(n)}) x \| \, ds$$

by $\| \exp(tA^{(n)}) \| \leqq 1$. Therefore, by (1.6),

$$U_t y = \lim_{n \to \infty} U_t^{(n)} y \qquad (y \in D)$$

exists uniformly in t for any finite interval of t. Since D is dense in E

and since $\| U_t^{(n)} \| \leqq 1$, we see that $U_t y$ exists for all $y \in E$ and satisfies (1.1) $-$(1.3). Surely U_t is a transition operator with $U_t^{(n)}$. We have, from (3.5)

$$U_t^{(n)} x - x = \int_0^t U_s^{(n)} A^{(n)} x \, ds, \ x \in E.$$

Hence, by letting $n \to \infty$,

$$U^t x - x = \int_0^t U_s Ax \, ds, \ x \in D,$$

in virtue of (1.6). Therefore $A = A_1 + A_2$ is the d.q.o. of U_t.

Similarly we may prove the

Lemma 2. Let $\{A^{(n)}\}$ be a sequence of mutually commutative linear operators satisfying (1.6) and (2.2), and let

(3.6) $$\lim_{n \to \infty} A^{(n)} x = Ax$$

exist for a dense linear subset D of E. Then

(3.7) $$U_t x = \lim_{n \to \infty} \exp \ (t A^{(n)}) x$$

exists uniformly in t for any finite interval of t. Thus $\{U_t\}$ defines a temporally homogeneous Markoff process satisfying (1.3) whose d.q.o. is given by A.

4. *Temporally and spatially homogeneous Markoff process.* As an application of the above results, we shall give an operator-theoretical interpretation of the infinitely divisible law, to the effect that the examples given in 2 exhaust, in a certain sense, the d. q. o. A of the temporally and spatially homogeneous Markoff process.

Let U_t be defined by

(4.1) $$(U_t x) \ (s) = \int_{-\infty}^{\infty} x(s-u) d_u \ F(t,u) \ , \ x(s) \in L_1(-\infty, \infty) \ ,$$

where $F(t,u)$ is, for any $t \geqq 0$, a distribution function of u. Then, for any $x(s)$ from the domain of the d.q.o. A,

(4.2) $$(Ax)(s) = \text{strong} \lim_{n \to \infty} n \left(\int_{-\infty}^{\infty} x(s-u) d_u \ F(n^{-1},u) - x(s) \right).$$

Hence, by the Fourier transformation,

(4.3) $$X(\lambda) \int_{-\infty}^{\infty} (\exp(i\lambda u) - 1) n \ d_u F(n,^{-1}u), \ (X(\lambda) = \frac{1}{\sqrt{2\pi}}$$

$$\int_{-\infty}^{\infty} \exp \ (i \lambda s) x(s) ds)$$

converges, as $n \to \infty$, uniformly in λ. Since the domain of A is dense in $L_1(-\infty, \infty)$, it is easy to see that

$$(4.4) \qquad \lim_{n\to\infty}\int_{-\infty}^{\infty} (\exp(i\lambda u)-1)n\, d_u F(n^{-1},u)$$

exists uniformly in any finite interval of λ. We put

$$(4.5) \qquad G_n(u) = \int_0^u n \frac{u^2}{1+u^2} d_u F(n^{-1},u) .$$

Then, following after A. Khintchine's argument[8], we may prove that the suquence $\{G_n(u)\}$ of the monotone increasing functions contains a subsequence $\{G_{n'}(u)\}$ such that

$$(4.6) \qquad \text{a bounded} \lim_{n'\to\infty} G_{n'}(u) = G(u) \text{ exists,}$$

$$(4.7) \qquad \lim_{a\to\infty} \int_{|v|>a} dG_{n'}(u) = 0 \quad \text{uniformly in } n',$$

$$(4.8) \qquad \text{a finite} \lim_{n'\to\infty} \int_{-\infty}^{\infty} u^{-1}\, dG_{n'}(u) = \gamma \quad \text{exists.}$$

Thus, by $G(0)=0$, we have, for continuous function $x(s) \in L_1(-\infty,\infty)$ whose continuous second derivative $x''(s)$ is also contained in $L_1(-\infty,\infty)$,

$$(4.9) \qquad \begin{aligned} &\operatorname{weak}\lim_{h\downarrow 0} (h^{-1}(U_h - I)x)(s) = (Ax)(s) \\ &= -\gamma x'(s) + \sigma x''(s) \\ &+ \lim_{\varepsilon\downarrow 0} \int_{|u|>\varepsilon} \left(x(s-u) - x(s) + \frac{ux'(s)}{1+u^2}\right) \frac{1+u^2}{u^2}\, dG(u), \end{aligned}$$

where

$$(4.10) \qquad \sigma = \lim_{\varepsilon\downarrow 0} (G(\varepsilon) - G(-\varepsilon)).$$

Conversely we see, by the two lemmas in **3**, that the operator A defined by (4.9) is the d.q.o. of a temporally and spatially homogeneous Markoff process. Here we make use of the fact that the operators (2.7) are all the d.q.o.'s of the temporally and spatially homogeneous Markoff processes.

5. *On the integration of the Fokker-Planck's equation.*[9] Consider

8) A. Khintchine: Déduction nouvelle d'une formule de P. Lévy, Bullet. de l'université d'état à Moscow, Sect. A, 1 (1937).

9) Cf. W. Feller: Zur Theorie der stochastischen Prozesse, Math. Ann., 113 (1936). K. Itô: On stochastic processes (II), to appear in the Mem. of the Am. Math. Soc. Our method of integration may be extended to the Fokker-Planck's equation in homogeneous Riemannian spaces. For example, we may determine the "Brownian motion" on the surface of the sphere. The details will be published elsewhere. Here I express my hearty thanks to Dr. K. Itô for his friendly criticism during the preparation of the present note.

(5.1) $$\frac{\partial y(s,t)}{\partial t}=\frac{\partial(a(s)y(s,t))}{\partial s}+\frac{\partial^2(b(s)y(s,t))}{\partial s^2}$$
$$=\delta(s)y(s,t)+\gamma(s)\frac{\partial y(s,t)}{\partial s}+\frac{\beta(s)}{4}\frac{\partial^2 y(s,t)}{\partial s^2}$$

where $t\geqq 0$, $-\infty<s<\infty$, with positive $b(s)$ and

(5.2) $\delta(s)=a'(s)+b''(s)$, $\gamma(s)=a(s)+2b'(s)$, $\beta(s)=4b(s)$.

If we assume

(5.3) $\delta(s)$, $\gamma(s)$, $\beta(s)^{-1}$ and $\beta'(s)$ are all bounded and continuous,

(5.4) $\bar{s}=\int_0^s\frac{ds}{\sqrt{\beta(s)}}\to\infty$ *as* $s\to\infty$, and $\bar{s}=\int_0^s\frac{ds}{\sqrt{\beta(s)}}\to-\infty$ *as* $s\to-\infty$,

then the additive Operator A defined by

(5.5) $$(Ay)(s)=(a(s)y(s))'+(b(s)y(s))''$$
$$=\delta(s)y(s)+\gamma(s)y'(s)+4^{-1}\beta(s)y''(s)$$

in L_1 $(-\infty,\infty)$ is the d.q.o. of a temporally homogeous Markoff process.

Proof. We have only to show that A satisfies (1.6) and (2.2) for large n.

The above example 2 suggests us that the solution $y_n(s)$ of

(5.6) $$y_n(s)-n\ (Ay_n)\ (s)=x(s)$$

will be given by the integral equation

(5.7) $$y_n(s)=\int_{-\infty}^{\infty}\sqrt{n}\ \exp\left(-2\sqrt{n}\left|\int_k^s\frac{du}{\sqrt{\beta(u)}}\right|\right)\left(\frac{8\gamma(k)-\beta'(k)}{8n}y_n'(k)\right.$$
$$\left.+\frac{\delta(k)}{n}y_n\ (k)+x(k)\right)\frac{dk}{\sqrt{\beta(k)}}\ .$$

That (5.7) admits solution $y_n(s)$ for continuous $x(s)\in L_1$ $(-\infty,\infty)$ will be seen as follows. Put

(5.8) $$C=\sup_s\left(8^{-1}|8\gamma(s)-\beta'(s)|,|\delta(s)|,\frac{1}{\sqrt{\beta(s)}}\right).$$

Then, for the successive approximations

$$y_{n1}(s)=\int_{-\infty}^{\infty}\sqrt{n}\ \exp\left(-2\sqrt{n}\left|\int_k^s\frac{du}{\sqrt{\beta(u)}}\right|\right)x(k)\frac{dk}{\sqrt{\beta(k)}}\ ,$$

$$y_{n,m}(s)=\int_{-\infty}^{\infty}\sqrt{n}\ \exp\left(-2\sqrt{n}\left|\int_k^s\frac{du}{\sqrt{\beta(u)}}\right|\right)\left(\frac{8\gamma(k)-\beta'(k)}{8n}\right.$$

$$y'_{n,m-1}(k)+\frac{\delta(k)}{n}\,y_{n,m-1}(k))\frac{dk}{\sqrt{\beta(k)}}$$

we easily obtain

$$\sup_s |y_{n,1}(s)|\leq \sup_s |x(s)|, \quad \sup_s |y'_{n,1}(s)\leq 2\sqrt{n}\,C\sup_s |x(s)|,$$
$$\sup_s |y_{n,m}(s)|\leq n^{-1}C\,(\sup_s |y'_{n,m-1}(s)|+\sup_s |y_{n,m-1}(s)|,$$
$$\sup |y'_{n,m}(s)|\leq \frac{2\sqrt{n}\,C^2}{n}(\sup_s |y'_{n,m-1}(s)|+\sup_s |y_{n,m-1}(s)|).$$

Hence, for large n, the two series

$$\sum_{m=1}^{\infty} y_{n,m}(s), \quad \sum_{m=1}^{\infty} y'_{n,m}(s)$$

are uniformly and absolutely convergent to bounded continuous functions. Therefore

$$y_n(s)=\sum_{m=1}^{\infty} y_{n,m}(s) \tag{5.9}$$

satisfies (5.7) and hence (6.6).

That this $y_n(s)$ belongs to $L_1(-\infty, \infty)$ with $x(s)$ will be seen as follows. If $\int_{-\infty}^{\infty} x(s)|\,\overline{ds}<\infty$, then

$$\int_{-\infty}^{\infty}|y_{n,1}(s)|\overline{ds}\leq\int_{-\infty}^{\infty}|x(s)|\,\overline{ds} \quad \int_{-\infty}^{\infty}|y'_{n1}(s)|\,\overline{ds} \leq 2\sqrt{n}\,C\int_{-\infty}^{\infty}|x(s)|\,\overline{ds},)$$
$$\int_{-\infty}^{\infty}|y_{n,m}(s)|\overline{ds}\leq n^{-1}C(\int_{-\infty}^{\infty}|y'_{n,m-1}(s)|\overline{ds}+\int_{-\infty}^{\infty}|y_{n,m-1}(s)|\,\overline{ds}),$$
$$\int_{-\infty}^{\infty}|y'_{n,m}(s)|\,\overline{ds}\leq\frac{2\sqrt{n}\,C^2}{n}\quad(\int_{-\infty}^{\infty}|y'_{n,m-1}(s)|\overline{ds}+\int_{-\infty}^{\infty}|y_{n,m-1}(s)|\,\overline{ds}).$$

Hence we easily see that $y_n(s)=\sum_{m=1}^{\infty} y_{n,m}(s) \in L_1(-\infty,\infty)$. Moreover, the above inequalities show that

$$y_n(s)-y_{n,1}(s)=\sum_{m=2}^{\infty} y_{n,m}(s)$$

converges to zero strongly in $L_1(-\infty, \infty)$. Since the strong $\lim_{n\to\infty} y_{n1}=x$, we see that strong $\lim_{n\to\infty} y_n=x$.

On the other hand we see that, for large n,

$$y(s)-n^{-1}\,(Ay)(s)\geq 0$$

implies $y(s) \geqq 0$, because such $y(s)$ cannot have negative minimum by the positivity of $b(s)$. Thus if $x(s) \geqq 0$ satisfies $\int_{-\infty}^{\infty} |x(s)| \overline{ds} < \infty$ and $\int_{-\infty}^{\infty} |x(s)| ds < \infty$, then the solution $y_n(s) \in L_1(-\infty, \infty)$ obtained above of (5.6) satisfies

$$\int_{-\infty}^{\infty} |y_n(s)| ds = \int_{-\infty}^{\infty} y_n(s) \ ds = \int_{-\infty}^{\infty} |x(s)| ds,$$

because

$$\int_{-\infty}^{\infty} (Ay_n)(s) ds = [a(s)y_n(s)]_{-\infty}^{\infty} + [(b(s)y_n(s))']_{-\infty}^{\infty} = 0.$$

Since $x(s) \in L_1(-\infty, \infty)$ satisfying $\int_{-\infty}^{\infty} |x(s)| \overline{ds} < \infty$ are dense in $L_1(-\infty, \infty)$, the above results shows that $I_n = (I - n^{-1}A)^{-1}$ exists and satisfies (2.2). Since $AI_n x = I_n Ax$ if x is in the domain D of A, we have (1.6) by strong lim $y_n = x$, *viz.* lim $I_n \ x = x$.

Thus we may integrate the original equation (5.1) by virtue of (1.8) or (1.8)′.

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Added during the proof. On reading the manuscript of the present note, Prof. E. Hille kindly remarked me that essentially the same results as stated in **1** was already obtained by him by a different method. See his book: Functional Analysis and Semi-groups, New York (1948). He also kindly sent to me his manuscript "On the integration problem for Fokker-Planck's equation in theory of stochastic processes" which, replacing my analysis by a simpler argument, extends the results in **5.** After the present note was presented to the M. S. of Japan, I published two notes concerning the integration of F—P equation: Brownian motion on the surface of the 2–sphere, Ann. of Math. Stat., **20,** No. 2 (1949) ; Integration of Fokker-Planck's equation in a compact Riemannian space, Arkiv för Math. **1,** No. 9 (1949).