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On Integral Invariants and Betti Numbers of Symmetric Riemannian Manifolds, II.

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Chapter III.

Formation of invariant differentials and the Schubert varieties.

I.

1. We have already seen that the manifold $S(n)$ can be considered as the set of all *null-systems* $x \rightarrow y = Sx$ such that the skew matrix S is at the same time orthogonal. Let us show that this manifold can also be considered as the set of all isotropic m-planes in P_n.

In order that a subspace $\mathfrak{M} \epsilon P$ be isotropic, it is neccesary and sufficient that $(x, y) = 0$ for all $x, y \epsilon \mathfrak{M}$. For any $\mathfrak{M}$ there exists the conjugate $\overline{\mathfrak{M}}$ of $\mathfrak{M}$. $\overline{\mathfrak{M}}$ is namely the set of vectors $\bar{x}$, where $x \epsilon \mathfrak{M}$. The correspondence $\mathfrak{M} \rightarrow \overline{\mathfrak{M}}$ is invariant under the group $O(n)$. In $\mathfrak{M}$ we take m vectors $x_1, \ldots\ldots, x_m$ such that $(x_i\ \bar{x}_j) = \delta_{ij}$. The vectors $e_1, \ldots\ldots, e_n$ with

$$(1) \qquad e_i = (x_i + \bar{x}_i)/\sqrt{2}\ , \ e_{m+i} = (x_i - \bar{x}_i)/\sqrt{-2}$$

constitute a real coordinate system such that $(e_i\ e_j) = \delta\ (ij)$. This shows at once that the manifold $\Sigma(n)$ is tronsformed transitively by the group $OL(n.)$ Now the manifold $\Sigma(n)$ consists of two different continuous families, each being transformed transitively by the group $O(n)$. We denote one of them by $\Sigma(n)$. Then there exists a one to one correspondence between the elements of the manifolds $S(n)$ and $\Sigma(n)$ invariant with respect to the group $O(n)$. In fact, let S be an element of $S(n)$. We consider a set $\mathfrak{M}$ of all complex vectors of the form

$$\mathfrak{x} = x + \sqrt{-1}\ Sx \qquad x \epsilon R_n\ .$$

The vector x is isotropic. To show that $\mathfrak{M}$ is isotropic m-dimensional we take a special coordinate system such that $S = I$. $\mathfrak{M}$ is then the set of all vectors of the form $(z_1, \ldots\ldots, z_m, \sqrt{-1}\ z_1, \ldots\ldots, \sqrt{-1}\ z_m)$, where $z_i \epsilon K$.

The group of desplacements of $S(n)$ is $S \rightarrow TS^*T$, $T \epsilon O(n)$. We denote by $\Sigma(n)$ the set of all isotropic m-planes of P_n , where $n = 2m$.

$\mathfrak{M}$ is thus an m-dimensional linear subspace and is defined dy the equation $x_i+\sqrt{-1}\ x_{m+i}=0$. Conversely, let $\mathfrak{M}$ be an isotropic m-plane of P. Let us show that for any vector $x \in R_n \subset P$ there exists a uniquely defined vector y such that $y \in R_n$. $x+\sqrt{-1}\ p \in \mathfrak{M}$ and that the correspondence $x \rightarrow y=Sx$ gives a null-system in which S is orthogonal. In fact, if we take the orthogonal frame $e_1, \ldots\ldots, e_n$ defined by the equation (1), where $x_1, \ldots\ldots, x_m \in \mathfrak{M}$, then the equation of $\mathfrak{M}$ is $x_i+\sqrt{-1}\ x_{m+i}=0$. Now let $x \in R_n$. In order that $x+\sqrt{-1}\ y$ belongs to $\mathfrak{M}$, it is necessary and sufficient that $(x_i+\sqrt{-1}\ y_i)+\sqrt{-1}\ (x_{m+i}+\sqrt{-1}\ y_{m+i})=0$. That is, $x_i=-y_{m+i}$, $x_{m+i}=y_i$. The correspondence $x \rightarrow y$ is given by $y=Ix$.

Theorem. 3. 1. *Suppose* $\mathfrak{M} \in \sum(n)$; *let* S *be the element of* $S(n)$ *corresponding to* $\mathfrak{M}$. *Then* $\mathfrak{M}$, $\overline{\mathfrak{M}}$ *are the totality of the vectors* P_n *such that*

$$S\mathfrak{x}=\sqrt{-1}\ \bar{\mathfrak{x}}\ ,$$

and

$$S\bar{\mathfrak{x}}=-\sqrt{-1}\ \mathfrak{x}$$

respectively.

2. We know that $S(n)$ is a symmetric Riemannian manifold. The group of rotation with centre I is the totality of orthogonal transformations leaving invariant the skew product $[x_1\ x_{m+1}]+\cdots\cdots+[x_m x_n]$. We call *transvection* an element θ of the group $O(n)$ such that $I^{-1}\theta\ I=\theta^{-1}$. There exists a one to one correspondence between the elements of the space $S(n)$ and the set of transvections invariant with respect to the group g. This correspondence is defined by

$$\theta=SI, \qquad S=-\theta I.$$

Now, any transvection θ can be written as $E(\theta)\ E(\theta)$, where

$$E(\theta)=\begin{cases} D(\theta_1)+\cdots\cdots+D(\theta_k) & m=2l, \\ D(\theta_1)+\cdots\cdots+D(\theta_k)+1, & m=2l+1, \end{cases}$$

$D(\theta)$ being the matrix of two dimensional rotation with angle θ. In fact, let x, y be a pair of eigenvectors corresponding to the angle θ:

$$\theta\ \|x\ y\|=\|x\ y\|=D(\theta).$$

Then we have by the relation $I^{-1}\theta I=\theta^{-1}$:

$$\theta\ \|Ix\ Iy\|=\|Ix\ Iy\|\ D(\theta).$$

That is, Ix, Iy are also a pair of eigenvectors with angle $-\theta$. We can thus take l mutualy orthogonal vectors $x_1, y_1, \ldots\ldots x_l, y_l$ such that the vectors x_j, y_j, Ix_j, Iy_j constitute a complete system of eigenvectors of the matrix

θ, that is

$$\theta R \,\|\, R. \,\|\, E(\theta)+E(\theta) \,\| , \quad R= \| \, x_1 \, y_1 \ldots\ldots (z), \, Ix_1, \, Iy_1 \ldots\ldots (Iz) \, \| .$$

The S corresponding to θ is

$$S=R^{-1} \left\| \begin{matrix} & E(\theta) \\ -E(-\theta) & \end{matrix} \right\| R.$$

The m-dimensional subspaces corresponding to $R^{-1}\ S\ R$ and I are the rows of the matrices

$$M: \| -\sqrt{-1}\ E(\theta), \quad E(-\theta) \, \| , \text{ and } M: \| -\sqrt{-1}\,, 1 \, \| .$$

The matrices of the scalar products in both senses (unitary and orthogonal) are

$$\frac{1}{2} M\bar{M}^* = \|E(\theta)+E(-\theta)\| , \quad \frac{1}{2} M\ M^* = \| E(\theta)-E(-\theta) \, \| .$$

Theorem 3.2. *For any two isotropic subspaces* $\mathfrak{M}$, $\mathfrak{N}$ *there exist unitary frames in* $\mathfrak{M}$ *and* $\mathfrak{N}$ *:* $x_1, \ldots\ldots, x_l, \ y_1 \ldots\ldots, y_l, \ (z)\ ;\ x_1', \ldots\ldots, x_l', \ y_1', \ldots\ldots y_l', \ (z)$ *such that*

$$(x_i \bar{x}'_i) = (y_i \bar{y}_i') = \cos\ \theta_i, \quad (x_i,\ y_i') = -(x_i',\ y_i) = \sin\ \theta_i, \quad ((z\ \bar{z}')=1)$$

and that all other scalar products between x, y, z *and* x', y', z' *vanish.*

3. In $\mathfrak{M} \in \sum(n)$ we take a unitary frame $x_1, \ldots\ldots, x_m$. x' are determined up to the unitary transformation between them. Let $x_i + dx_i$ be a unitary frame of a $\mathfrak{M}+d\mathfrak{M}$ near $\mathfrak{M}$. If we put

$$\omega_{ij} = (x_i,\ dx_j)$$

then ω_{ij} are skew symmetric and are transformed according to $(\omega) \to U(\omega) U^*$ by the transformations of x, $x+dx$. The positive-definite quadratic form

$$(ds)^2 = \sum_{i<j} \omega_{ij} \omega_{ij}$$

gives thus the invariant Riemannian metric of the manifold $\sum(n)$. We construct $m-1$ exterior forms

$$\Omega_s = \omega_{i_1 i_2}\ \omega_{i_2 i_3} \ldots\ldots \omega_{i_{2s-1}\ i_{2s}}\ \omega_{i_{2s}\ i_1}, \quad (s=1,2,\ldots\ldots,m-1)$$

Then :

Theorem. 3.3. *The forms*

$$\Omega_{h1} \ldots\ldots \Omega_{hp}$$

with $h_1 + \ldots\ldots + h_1 = s$ *are all linearly indepependent and constitute a basis of* *the invariant differentials of rank s.*

Indeed. if we put $\theta_i^{\;\gamma}=\omega_{ik}\ \omega_{kj}$, then it is possible to show from the first main theorem of the group $U(n)$ or $GL(n)$ that any invariant is a polynomial in $\Omega_s=\theta_{i1}^{i2}\cdots\cdots\theta_{is}^{i1}$. But we readily see that $\tilde{\Omega}_s=\Omega_s$, $(s<m)$. So that it remains to show that the terms $\tilde{\Omega}_s$ with $s\geqq m$ can be omitted. But this can be carried out in the same manner as in H. Weyl, The classical groups, p. 236–237. *The independence of these forms follows from our knowledge on Betti numbers.* (chap. II.)

The Schübert varieties of the manifold $\sum(n)$ were given by Ehresmann. Let $\mathfrak{M}\in\sum(n)$ and $\mathfrak{M}_1,\ \mathfrak{M}_2,\cdots\cdots\in P_n$ be linear subspaces such that

$$0<\mathfrak{M}_1<\mathfrak{M}_2<\cdots<\mathfrak{M}_n<1\ ;\quad \mathfrak{M}_m=\mathfrak{M}\quad ,$$

Let $f_1,\cdots\cdots,f_l$ be natural numbers such that $n>f_1>f_2>\cdots>f_l>0$. We consider the diagram $h_1',\ h_2',\cdots\cdots h_m'$ where

$$h_i=h_i+m-i+1$$

$$h_1=f_1+1,\quad h_2=f_2+2,\ldots,\quad h_l=f_l+l$$

and $(h_{l+1},\cdots\cdots h_m)$ is the conjugate of $(h_1-l,\ h_2-l,\ldots,h_l-l)$. We then denote by $V(f_1,\cdots\cdots,\ f_l)$ the set of all isotropic m-spaces such that $\dim\,(\mathfrak{M}\wedge\mathfrak{M}'_{hi}\ )\geqq i$. If we denote by $\omega(f_1,\cdots\cdots,f_l)$ the invariant differential corresponding to the partition $(f_1, f_2,\cdots, f_l)$ constructed in Chap. II, The product of $V(f'_1,\cdots\cdots,\ f'_l)$ with the cocycle $\omega(f_1',\cdots\cdots,f'_{l'})$ does not vanish if and only if $f_1=f'_1,\cdots\cdots,f'_l=f'_l$; $l=l'$.

II.

The manifolds $\tilde{A}^+(n)$, $A^+(n)$.

1. We denote by $OSp\ (2n)$ the set of all orthogonal matrises leaving invarlant the skew form $I: T^{-1}\ I\ T=I$. If we transform the coordinate axis by the matrix

$$P=\frac{1}{\sqrt{2}}\begin{Vmatrix} E & E \\ \sqrt{-1}E & \sqrt{-1}E \end{Vmatrix}$$

then the result reads

$$P^{-1}T\ P=U+\bar{U},\ U\in U(n).$$

The group $OSp\ (2n)$ is not absolutely irreducible. In K it decomposes into two parts $U,\ \bar{U}$, conjugate with each other and isomorphic with the group $U(n)$. The transformation

$$T\longrightarrow J\ T\ J\qquad\qquad J=E_n+-E_n$$

is an involutive automorphism of the group $OSp(n)$. In $U(n)$ it corresponds to the automorphism $U \rightarrow \overline{U}$.

Let x be a vector of R_{2n}. Consider the correspondence $x \rightarrow x = xI$ which is invariant under the group $OSp(2n)$. If we denote by $\mathfrak{A}^+(n)$ the set of all m-dimensional linear subspaces of R_{2n} such that $\mathfrak{M} \wedge \hat{\mathfrak{M}} = 0$, where $\hat{\mathfrak{M}}$ is the trasform of $\mathfrak{M}$ by $\wedge$, then $\mathfrak{A}^+$ is a symmetric Riemannian manifold which is isomorphic with $A^+(n)$.

2. Let $x_1, \ldots\ldots, x_n$ be n vectors of $\mathfrak{M} \in \mathfrak{A}^+(n)$ such that $(x_i\ x_j) = \delta_{ij}$, and let $x_1 + dx_1, \ldots\ldots, x_n + dx_n$ be an orthogonal frame of a $\mathfrak{M} + d\mathfrak{M} \in \mathfrak{A}^+(n)$ near $\mathfrak{M}$. Consider the differential forms

$$\omega_{ij} = [x_i,\ dx_j]$$

where we define $[x, y] = x^* I\ y$ for any x, y (x, y are columns). ω_{ij} are invariant with respect to the group $OSp(n)$ and transforms as

$$(\omega) \rightarrow T^* (\omega) T \qquad T \in U(n)$$

by the transformation of $x_1, \ldots\ldots x_n$, $x_1 + dx_1, \ldots\ldots, x_n + dx_n$. These forms satisfy the condition $\omega_{ij} = \omega_{ji}$. The quadratic form

$$(ds)^2 = \sum_{i<j} \omega_{ij}\ \omega_{ij}$$

gives thus the invariant Riemannian metric of $\mathfrak{A}^+(n)$. The exterior forms be

$$\Omega_s = \omega_{i_1 i_2}\ \omega_{i_2 i_3} \ldots\ldots, \omega_{i_s\ i_1}; \quad s \equiv 1 \pmod 4), \quad 1 \leqq s < 2n.$$

and

$$\Omega = \sum \epsilon(P) P \omega_{i_1\ i_2}\ \omega_{i_3\ i} \ldots\ldots \omega_{i_{2v-1} i_{2v}}. \quad (n = 2v)$$

constitute a basis of the invariant differentials of the manifold $\mathfrak{A}^+(n)$.

In R_{2n} we consider a system of linear subspaces $\mathfrak{M}_1, \mathfrak{M}_2, \ldots\ldots, \mathfrak{M}_{2n}$, such that

$$0 < \mathfrak{M}_1 < \mathfrak{M}_2 < \ldots\ldots < \mathfrak{M}_{2n}\ ; \qquad \mathfrak{M}_n = \mathfrak{M} \in \mathfrak{A}^+(n).$$

We denote by $V(f_1 \ldots\ldots, f_l)$ the set of all linear subspaces $\mathfrak{M}$ of $\mathfrak{A}^+(n)$ such that

$$\dim (\mathfrak{M} \wedge \mathfrak{M}_{k'_i}) \geqq i\ ; \quad (n \geqq f_1 > f_2 > \ldots\ldots f_l > 0, \quad l \leqq n)$$

where $k', \ldots\ldots, k_u'$ are defined as follows:

$$k_i' = k_i + n - i + 1, \qquad (i = 1, 2, \ldots\ldots, n)$$

$$k_1 = f_1, \quad k_2 = f_2, \ldots\ldots, \quad k_l = f_l + l - 1$$

where $(k_{l+1}, \ldots\ldots k_n)$ is the conjugate of $(k_1 - l, \ldots\ldots k_l - l)$. The manifolds $V(f_1, \ldots\ldots, f_l)$ with all $f_1, \ldots\ldots, f_l$ satisfying the condition $n \geqq f_1 > f_2 > \ldots\ldots >$

$f_l > 0$, $1 \leqq n$ are the Schübert varieties of $\mathfrak{A}^+(n)$. The product of $V(f_1, \ldots \ldots, f_l)$ with the cocycle $\omega(f_1', \ldots\ldots, f_{l'}')$ of Chap. II does not vanish if and only if $f_i = f_i'$ for all i, and $l = l'$.

The manifold $\mathfrak{A}^+(n)$ may be considered as a subspace of $\mathfrak{A}^+(n)$ and can be treated in like manner. The basis of the invariant differentials coincides with that of $\mathfrak{A}^+(n)$ except that the form Ω_1 must be omitted.

III.

The manifolds $A^-(n)$, $'A^-(n)$.

1. Consider the involutive automorphism of the group $OSp(n)$ defined by:

$$T \longrightarrow J^{-1}\ T\ J$$

with $J = I \dotplus -I$, $I = n$-dimensional skew matrix, $n = 2m$. In $U(n)$ this corresponds to the automorphism $U \longrightarrow I^{-1}\overline{U}I$. The symmetric Riemannian manifold corresponding to this automorphism is thus isomorphic with $A(n)$. A transformation θ of $OSp(n)$ is a transvection if $J^{-1}\theta\ J = \theta^{-1}$. If we put $S = \theta\ J$, then

$$S^2 = \theta\ J\ \theta\ J = \theta J^{-1}\ \theta\ J = -1.$$

that is, $S \in S(2n)$. Morever, we have

$I^{-1}S\ I = I^{-1}\theta J\ \ I = I^{-1}\ \theta\ I\ \ I^{-1}\ J\ I$, $I = 2d$-dimensional skew matrtx.

Because any element of $OSp(n)$ satisfy the relation $I^{-1}\ T\ I = T$, we see

$$I^{-1}\ S\ I = \theta\ I\ J\ I = -\theta\ J = -S.$$

or $-SI = IS$. We know by Chap. I that there exists a one to one correspondence of the elements of $\tilde{A}^-(n)$ with the set of transvections. $A^-(n)$ is thus the totality of the elements of the manifold $S(2n)$ satisfying the relation $SI = -IS$. Now the elements of $\sum(2n)$ corresponding to S is the set of all isotropic vectors $x \in P_n$ such that

$$Sx = \sqrt{-1}\ \bar{x}\ .$$

By using the relation $SI = -IS$ we see

$$S\hat{x} = \sqrt{-1}\ \hat{x},$$

where $\hat{x} = I\ \bar{x}$:

Theorem. 3.4. *The manifold $\tilde{A}^-(n)$ may be considersd as the set of all n-dimensional isotropic subspaces of P_n such that $\mathfrak{M} = \hat{\mathfrak{M}}$, or $\mathfrak{M} \in S(2n, n)$.*

2. The manifold $\tilde{A}^-(n)$ is thus the set $S(2n) \wedge S(2n, n)$. We denote this by $\mathfrak{A}^-(n)$. In $\mathfrak{M}$ of $\mathfrak{A}^-(n)$ we consider n vectors $x_1, \ldots\ldots x_n$ such that $(x_i\ x_j)=\delta(i\ j)$, $x_{n+i}=x_i$. This is possible because of the relation $\mathfrak{M}=\hat{\mathfrak{M}}$ and the vectors $x_1, \ldots\ldots, x_n$ are determined up to the group $USp(n)$. Consider the differential forms ω_{ij} which are skew-symmetric with respect to the indices i, j:

$$\omega_{ij}=(x_i,\ dx_j).$$

ω_{ij} are invariant with respect to the group $OSp(n)$ and transform according to $(\omega)\longrightarrow T^*(\omega)T$, $T\in USp(n)$ by the transformations of $x_1, \ldots\ldots, x_n$. Thus

$$(ds)^2=\sum_{i<j}\omega_{ij}\ \bar{\omega}_{ij}$$

gives the invariant Riemannian metric of $\mathfrak{A}^-(n)$. The basic differential forms of $\mathfrak{A}^-(n)$ are defined by

$$\Omega_s=\omega_{i1}{}^{i2}\ \omega_{i2}{}^{i3}\ldots\ldots\omega_{is}{}^{i1},\quad \omega_i{}^j=\varepsilon^{jk}\ \varepsilon_{ik}\ ;\quad s\equiv1\ (mod\ 4),\ 1\leqq s<2n$$

Let $f_1, \ldots\ldots, f_l$ $(l<n)$ be natural numbers such that $n>f_1>f_2>\ldots\ldots>f_l>0$,

$$f_i+i\equiv0\ (mod.\ 2),\quad f_{2i-1}\geqq f_{2i}=1\ (mod\ 4),\ i\geqq1.$$

(chap. II, Lemma II.). We construct the diagram $k_1', \ldots\ldots, k_n'$:

$$k_i'=k_i+n-i+1,\quad (i=1,2,\ldots\ldots,n)$$
$$k_1=f_1+1,\quad k_2=f_2+2,\ldots\ldots,\ k_l=f_l+l$$

where $(k_{l+1}, \ldots\ldots, k_n)$ is the conjugate of $(k_1-t, \ldots\ldots,\ k_l-l)$. Consider a system of linear subspaces $0<\mathfrak{M}_1<\ldots\ldots<\mathfrak{M}_{2n}=1$, $\mathfrak{M}_n\in\mathfrak{A}^-$, where $\mathfrak{M}_{2j}\in S(2n\ 2j)$ and denote by $V(f_1, \ldots\ldots, f_l)$ the set of linear subspaces contained in $\mathfrak{A}^-(n)$ satisfying the relation

$$\dim\ (\mathfrak{M}\wedge\mathfrak{M}_{k'i})\geqq i.$$

The product of $V(f_1, \ldots\ldots, f_l)$ with $\omega(f_1', \ldots\ldots, f_l')$ defined in Chap. II is not zero if and only if $f_1=f_1'$, $\ldots\ldots$, $f_l=f_l'$, $l=l'$,.

IV.

The manifold $C(n)$.

1. $C(n)$ is the set of all linear subspaces of P_n such that $\mathfrak{M}\wedge\hat{\mathfrak{M}}=0$. We take in $\mathfrak{M}\in C(n)$ m vectors such that $(x_i\ x_j)=\delta_{ij}$. The forms

$$\omega_{ij}=[x_i\ dx_j]$$

are forms of $C(n)$ which are symmetric with respect to i, j and transform according to $(\omega)\rightarrow T^*(\omega)T$ $T\in U(m)$ by the transformations of x, $x+dx$. The invariant Riemannian metric of $C(n)$ is thus $(ds)^2=\sum\omega_{ij}\ \bar{\omega}_{ij}$. The forms

$$\Omega_s=\theta_{i1}{}^{i_2}\ldots\ldots\theta_{is}{}^{i1}\ ,\quad \theta_i{}^j=\omega_{ik}\ \overline{\omega}_{jk}\ ;\quad 1\leqq S\leqq m$$

are basic forms of $O(n)$. Let $f_1,\ldots\ldots,f_l$ be natural numbers such that

$$n+1>f_1>f_2>\ldots\ldots>f_l>0$$

and we consider the diagram $k_1',\ldots\ldots k_m'$, :

$$k_i'=k_i+m-i+1,\qquad (i=1,2,\ldots\ldots,\ m)$$
$$k_1=f_1,\quad k_2=f_2+1,\ldots\ldots,\quad k_l=f_l+l-1$$

where $(k_{l+1},\ldots\ldots,\ k_m)$ is the conjugate of $(k_1-l,\ldots\ldots k_l-l)$. Consider the system of linear subspaces $0<\mathfrak{M}_1<\mathfrak{M}_2<\ldots\ldots\mathfrak{M}_n<1$, $\mathfrak{M}_m\in C(n)$, and denote by $V(f_1,\ldots\ldots,f_l)$ the set of all $\mathfrak{M}\in C(n)$ such that $\dim(\mathfrak{M}\wedge\mathfrak{M}_{k'i})\geqq i$. The product of $V(f_1,\ldots\ldots,f_l)$ with the form $\omega(f_1',\ldots\ldots,f_{l'}')$ defined in Chap. II. does not vanish if and only if $f_1=f_1',\ldots\ldots,\ f_l=f_l',\ l=l'$.

V.

The manifolds $A(n,k)$, $S(n,k)$, $R(n,k)$.

1. Consider a sequence of linear subspaces

$$\begin{array}{lll} \mathrm{A}: & 0<\mathfrak{M}_1<\mathfrak{M}_2<\ldots\ldots<\mathfrak{M}_n=1\ , & \mathfrak{M}_k\in A(n,k)\\ \mathrm{B}: & 0<\mathfrak{M}_2<\mathfrak{M}_4<\ldots\ldots<\mathfrak{M}_n=1\ , & \mathfrak{M}_k\in S(n,k)\\ \mathrm{R}: & 0<\mathfrak{M}_1<\mathfrak{M}_2<\ldots\ldots<\mathfrak{M}_n=1\ , & \mathfrak{M}_k\in R(n,k)\end{array}$$

and denote by $V(f_1,\ldots\ldots,\ f_k)$ the set of all k-dimensional subspaces of P_n or R_n such that dim $(\mathfrak{M}\wedge\mathfrak{M}_{f_j+k-j+1})\geqq j$*. Then the integral

$$\int_{V(f_1,\ldots\ldots,\ f_k)}\omega(f_1'\ldots\ldots f'_k)$$

is not zero if and only if $f_1=f_1',\ldots\ldots,\ f_k=f'_k$.

Let $x_1,\ldots\ldots,x_k$ be k vectors of $\mathfrak{M}$ such that:

A: $(x_i\ \bar{x}_j)=\delta_{ij}$, S: $(x_i\ \bar{x}_j)=\delta_{ij}$, $x_{(k/2)+i}=I\bar{x}_i$, and R: $(x_ix_j)=\delta_{ij}$,

and let $y_1,\ldots\ldots,y_h$ be $n-k$ vectors contained in the orthogonal complement of $\mathfrak{M}$ satisfying the relations

A: $(y_i\ \bar{y}_j)=\delta_{ij}$, S: $(y_i\ \bar{y}_j)=\delta_{ij}$, $y_{(h/2)+i}=I\bar{y}_i$ and R: $(y_iy_j)=\delta_{ij}$.

Then the forms

$$\omega_{i\lambda}=(y_i\ dx_\lambda),\ \text{or}\ (\bar{y}_i\ dx_\lambda)$$

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* For $S\ (n,\ k)$ the nnmbers $f_1,\ldots f_k$ are

$$f_1=f_2,\ldots\ldots,\ f_{k-1}=f_k;\quad f_j\equiv 0(\mathrm{mod}\ 2)$$

and $V(f_1\ldots\ldots,\ f_k)$ is the set of $\mathfrak{M}$ such that dim $(\mathfrak{M}\wedge\mathfrak{M}f_{2j+2k-2j-2})\geqq 2j$.

are forms of A, S, R, and transforms according to

$$(\omega)\rightarrow T_k^* \ (\omega) \ T_k$$

by the transformations of x, y. Thus the differentials of A, S, R may be explicitely obtained by the process indicated in Chap. II.

Remark. We have seen that the Poincaré Polynomials of the manifolds $\tilde{A}^+(n)$ and $\tilde{A}^-(n)$ are $P(z)=(1+z)P_0(z)$, where $P_0(z)$ are the Poincaré Polynomials of $A^+(n)$ and $A^-(n)$ respectively. This has following geometric significance. *Not only topologically, but also gemetrically, the manifolds $\tilde{A}^\pm(n)$ are the direct prodocts of the circumference with the manifolds $A^\pm(n)$.* Indeed, the adjoint group of the manifolds $A^\pm(n)$ are

$$\Omega\rightarrow A^*\Omega A$$

where $\Omega=\Omega^*$ for $\tilde{A}^+$ and $\Omega+\Omega^*=0$ for $\tilde{A}^-$. This group is not irreducible and decomposes into the direct sum of the unit representation and Ω_0, the representation induced in the symmetric or skew symmetric tensor spaces of zero traces. Geometrically this means that the same thing holds for the groups in vector spaces induced by the groups of holonomy of $A^\pm(n)$. So that there exists a system of ∞^1 totally geodesic hypersurfaces of $A^\pm$ whose orthogonal trajectories are geodesics. In fact let $A=e^{\sqrt{-1}\theta} A_0$, where $A_0 \in A^\pm(n)$. If we vary A_0 in $A^\pm(n)$, then the matrices A varies within some totally geodesic hypersurface. There exists ∞^1 such surfaces corresponding to the values of θ, $0\leqq\theta\leqq 2\pi$. If we fix A and let θ vary then the set of A forms a geodesic orthogonal to these surfaces. *$\tilde{A}^\pm$ are thus the „pythagorean product" of the circumference with the manifolds $A^\pm$.*

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