Journal of the Mathematical Society of Japan Vol. 2, No. 3, Dec. 1949.

A Theorem on Riemann Sum. Notes on Fourier Analysis (XIII)

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Let us consider the series

$$f(x)=\sum_{n=1}^{\infty}\frac{\cos nx}{n^{a}} \quad \left(0<a<\frac{1}{2}\right),$$

and let $f_n(x)$ be its n-th Riemann sum, i.e.

$$f_n(x)=\frac{1}{n}\sum_{k=1}^{n} f\left(x+2\pi\frac{k}{n}\right).$$

Since $f(x)$ is of order $1/x^{1-a}$ in the neighbourhood of the origin, we have

$$\lim_{n\to\infty}\sup f_n(x)=\infty$$

for almost all x, by the theorem due to J. Marcinkiewicz, A. Zygmund[1] and H. Ursell[2]

Connected with this fact it may be of some interest to prove the following

Theorem. *Let $f(x)$ be a function integrable in $(0,2\pi)$ and of period 2π. Let $f_n(x)$ be its Riemann sum and its Fourier series be*

$$(1) \qquad f(x)\sim\frac{1}{2}a_0+\sum_{n=1}^{\infty}(a_n\cos nx+b_n\sin nx)\ .$$

If the Fourier coefficients satisfy the condition

$$(2) \qquad \lim_{n\to\infty}\sum_{\nu=1}^{\infty}\left(|a_{n\nu}-a_{n(\nu+1)}|+|b_{n\nu}-b_{n(\nu+1)}|\right)=0,$$

in particular if

$$\sum_{n=1}^{\infty}\left(|a_n-a_{n+1}|+|b_n-b_{n+1}|\right)<\infty,$$

or if $\{a_n\}$, $\{b_n\}$ are non-increasing sequences, then for almost all x there exists a sequence of integers $\{m_k\}$ (depending on x) such that

$$\lim_{k\to\infty} f_{m_k}(x)=\int_0^{2\pi} f(x)\,dx$$

1) Mean values of trigonometrical polynomials, Fund. Math., **28** (1937), p. 131—166, spec., p. 157.

2) On the behaviour of a certain sequence of functions derived from a given one, Jour. London Math. Soc., **12** (1937).

and

$$\lim_{k\to\infty} k/m_k=1.$$

Proof. Without any loss of generality we can suppose that $a_0=0$ (if $a_0\neq 0$, it is sufficient to consider the function $f(x)-a_0/2$ in stead of $f(x)$), and that the series (1) is the cosine one, since we can treat the cosine part and the sine part of the series similarly.

For sufficiently large n, by (2) and the relation $a_n\to 0$ as $n\to\infty$, we have

$$f_n(x)=\sum_{\nu=1}^{\infty} a_{n\nu}\cos n\nu x.$$

By Abel's transformation

$$f_n(x)=\sum_{\nu=1}^{\infty}(a_{n\nu}-a_{n(\nu+1)})\sum_{\mu=1}^{\infty}\cos n\mu x+\lim_{\nu=\infty} a_{n\nu}\sum_{\mu=1}^{\nu}\cos n\mu x,$$

if the right-hand side exists. As easily seen,

$$\Big|\sum_{\mu=1}^{\nu}\cos\mu y\Big|<1/\sin\frac{\delta}{2}\quad(\nu=1,2,\ldots\ldots)$$

if $y\in(\delta,2\pi-\delta)$.

By the well known theorem due to Weyl, for almost all x there exists an infinite sequence of integers $\{n_k\}$ such that

$$n_k x\in(\delta,2\pi-\delta)\quad(k=1,2,\ldots\ldots)$$

and

$$k/n_k\to\frac{(2\pi-\delta)-\delta}{2\pi}=1-\frac{\delta}{\pi}\qquad\text{as } k\to\infty.$$

For such n_k and x, we have

$$|f_{n_k}(x)|\leqq\frac{1}{\sin\frac{\delta}{2}}\sum_{\nu=1}^{\infty}|a_{n_k\nu}-a_{n_k(\nu+1)}|$$

which implies $f_{n_k}(x)\to 0$ $(k\to\infty)$ by the condition (2).

We will now take sequences $\{\delta_\nu\}$ and $\{\varepsilon_\nu\}$ such that $\delta_\nu\downarrow 0$, $\varepsilon_\nu\downarrow 0$ as $\nu\to\infty$. Let $\{n_k^{(\nu)}\}_k$ be a sebuence of integers which correspond to $\pi\delta_\nu$ similarly as $\{n_k\}$ to δ. Then by the above result we have for almost all x

(3) $$k_k^{(\nu)}/n\to 1-\delta_\nu\quad(k\to\infty)$$

$$(\nu=1,2,\ldots\ldots)$$

(4) $$f_{n_k^{(\nu)}}(x)\to 0\quad(k\to\infty)$$

where we can take the exceptional x-set E of measure zero irrelevant to ν.

From (3) and (4) there exists a number N'_1 such that

$$\left|1-\frac{k}{n_k^{(1)}}\right|<\delta_1+\varepsilon_1 \qquad (k\geqq N'_1)$$

and

$$|f_{n_k^{(1)}}(x)|<\varepsilon_1 \qquad (k\geqq N'_1).$$

Nextly we can take $N_2>N'_1$ such that

$$|f_{n_k^{(2)}}(x)|<\varepsilon_2 \qquad (k\geqq N_2).$$

If we put $N_1''=\max k$, for $n_k^{(1)}>n_{N_2}^{(2)}$ then by (3) there exists $N_2'>N_2$ such that

$$\left|1-\frac{(k-N_2+1)+N''_1}{n_k^{(2)}}\right|<\delta_2+\varepsilon_2 \quad (k\geqq N_2')$$

Thirdly we take $N_3>N_2'$ such

$$|f_{n_k^{(3)}}(x)|<\varepsilon_3 \qquad (k\geqq N_3).$$

If we put $N''_2=\max k$, (for $n_k^{(2)}<n_{N_3}^{(3)}$) then there exists (by (3)) $N'_3<N_3$ such that

$$\left|1-\frac{(k-N_3+1)+(N_2''-N_2+1)+N_1''}{n_k^{(3)}}\right|<\delta_3+\varepsilon_3 \ (k>N_3')$$

Thus proceeding we get an infinite sequence of integers

$$n_1^{(1)},\ n_2^{(1)},\ldots\ldots,n_{N_1''}^{(1)},\ n_{N_2}^{(2)},\ n_{N_2+1}^{(2)},\ldots\ldots,n_{N_2''}^{(2)},\ n_{N_3}^{(3)},\ldots\ldots,$$

which we denote by $\{m_k\}$.

Then we can easily see that the sequence $\{m_k\}$ is the required one. q.e.d.

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