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Determination of Function by its Fourier Series. Notes on Fourier Analysis (XII)

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§ **1.** Introduction. Let $f(x)$ be an integrable function with period 2π, and $\bar{s}(x)$ and $\bar{\sigma}_n(x)$ be the $(n+1)$-th conjugate partial sum and arithmetic mean of the Fourier series of $f(x)$, respectively. If x is a point of discontinuity of $f(t)$ of the first kind, we put

$$l(x)=f(x+0)-f(x-0).$$

Fejer[1] has proved that

(1) $$\lim_{n\to\infty} \bar{s}_n(x)/\log n=-l(x)/\pi.$$

Later Lukacs[2] proved that, if there is an $l(x)$, such that

$$\int_0^t |\psi_x(t)-l(x)|\,dt=o(t),$$

$$\psi_x(t)=f(x+t)-f(x-t),$$

as $t\to 0$, then (1) nolds. In this case x need not be the point of discontinuity of the first kind.

Recently 0. Szász[3] proved that, if there is an $l(x)$ snch that

(2) $$\int_0^t \{\psi_x(t)-l(x)\}\,dt=o(t)\,\bullet$$

and

(3) $$\int_0^t |\psi_x(t)-l(x)|\,dt=O(t),$$

then

(4) $$\lim_{n\to\infty}(\bar{\sigma}_{2n}(x)-\bar{\sigma}_n(x))=l(x).\,(\log 2)/\pi.$$

Mr. Matsuyama proposed the problem: do (2) and (3) imply (1)? This problem seems to be negative. In this paper we prove that (2) and

(5) $$\int_{\pi/n}^{\pi}\frac{|\psi_2(t)-\psi_x(t-\pi/n)|}{t}\,dt=o\,(\log n)$$

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1) L. Fejér, ibidem, 142 (1913).

2) F. Lucács, Jour. für Math., 150 (1920).

3) O. Szász, Duke Math. Journ., 4 (1938).

imply (1), and that (2) and

(6) $$\int_{\pi/n}^{\pi} \frac{|\psi_x(t)-\psi_x(t-\pi/n)|}{t}\,dt=O\ (\log n)$$

imply (4).

§ **2.** Theorem 1. *If there is an* $l\ (x)$ *such that*

(2) $$\int_0^t \{\psi_x(t)-l(x)\}\,dt=o(t)$$

and

(7) $$\int_{\pi/n}^{\pi}\left|\frac{\psi_x(t)}{t}-\frac{\psi_x(t+\pi/2n)}{t+{}^{\pi}/_{2n}}\right|dt=o(\log n),$$

then

(1) $$\lim_{n\to\infty} \bar{s}_n(x)/\log n=-l(x)/\pi.$$

Proof. If we put

$$a(t/2)\equiv\psi_x(t)-l(x),\ A(t)\equiv\int_0^t a(u)\ du,$$

then we have

$$\bar{s}_n^*(x)+l(x)/n=\int_0^{\pi} a(t/2)\bar{D}^*(t)\,dt$$

$$=\int_0^{\pi} a\ (t)\frac{\sin^2 nt}{t}\ dt+O(1).$$

Hence it is sufficient to prove that

$$\mathrm{I}\equiv\int_0^{\pi} a(t)\ \frac{\sin^2 nt}{i}\ dt=o(\log n).$$

Dividing the intsgral into two parts we have

$$\mathrm{I}=\left(\int_0^{\pi/n}+\int_{\pi/n}^{\pi}\right)a(t)\frac{\sin^2 nt}{t}\ dt\equiv\mathrm{I}_1+\mathrm{I}_2$$

say. Firstly we have by integration by parts

$$\mathrm{I}_1=\int_0^{\pi/n} a\ (t)\frac{\sin^2 nt}{t}\ dt$$

$$=\left[A(t)\frac{\sin_2 nt}{t}\right]_0^{\pi/n}-\int_0^{\pi/n} A(t)\left[\frac{n\sin 2nt}{t}-\frac{\sin^2 nt}{t^2}\right]dt$$

$$=o(1)-O\left(\int_0^{\pi/n} A(t)n^2\,dt\right)$$

$$=o(1)-o\left(n^2\int_0^{\pi/n} t\log\frac{1}{t}dt\right)=o(\log n).$$

by (1). For the estimation of I_2, we use the argument due to Lebesgue.

$$I_2=\int_{\pi/n}^{\pi} a(t)\frac{\sin^2 nt}{t}dt$$

$$=\frac{1}{2}\int_{\pi/n}^{\pi}\frac{a(t)}{t}dt-\frac{1}{2}\int_{\pi/n}^{\pi}a(t)\frac{\cos 2nt}{t}dt$$

$$=\frac{1}{2}(J_1+J_2)$$

say. We have by integration by parts

$$J_1=\int_{\pi/n}^{\pi}\frac{a(t)}{t}dt=\left[\frac{A(t)}{t}\right]_{\pi/n}^{\pi}+\int_{\pi/n}^{\pi}\frac{A(t)}{t^2}dt$$

$$=o(1)+o\left(\int_{\pi/n}^{\pi}\frac{dt}{t}\right)=o(\log n).$$

$$2J_2=2\int_{\pi/n}^{\pi}a(t)\frac{\cos 2nt}{t}dt$$

$$=\int_{\pi/n}^{\pi}a(t)\frac{\cos 2nt}{t}dt-\int_{3\pi/n}^{\pi+\pi/2n}a\left(t+\frac{\pi}{2n}\right)\frac{\cos 2nt}{t-\pi/2n}dt$$

$$=\int_{\pi/n}^{3\pi/2n}a(t)\frac{\cos 2nt}{t}dt-\int_{\pi}^{\pi+\pi/2n}a\left(t-\frac{\pi}{2n}\right)\frac{\cos 2nt}{t-\pi/2n}dt$$

$$+\int_{3\pi/2n}^{\pi}\left[\frac{a(t)}{t}-\frac{a(t-\pi/2n)}{t-\pi/2n}\right]\cos 2nt\,dt$$

$$=K_1+K_2+K_3$$

say. We have

$$K_1=\int_{\pi/n}^{3\pi/2n}a(t)\frac{\cos 2nt}{t}dt$$

$$=\left[\frac{A(t)}{t}\cos 2nt\right]_{\pi/n}^{3\pi/2n}+\int_{\pi/n}^{3\pi/2n}A(t)\frac{\cos 2nt}{t^2}dt+2n\int_{\pi/n}^{3\pi/2n}A(t)\frac{\sin 2nt}{t}dt$$

$$=o(1).$$

$$K_2=O(1).$$

$$|K_3| \leqq \int_{\pi/n}^{\pi} \left| \frac{a(t)}{t} - \frac{a(t-\pi/2n)}{t-\pi/2n} \right| dt$$

$$= \int_{\pi/n}^{\pi} \left| \frac{\psi_x(t)}{t} - \frac{\psi_x(t-\pi/2n)}{t-\pi/2n} \right| dt + O(1).$$

Thus we get the theorem.

§ **3.** Theorem 2. *If*

(1) $$\int_0^t \{\psi_x(t) - l(x)\} = o(t)$$

and

(5) $$\int_{\pi/n}^{\pi} \frac{|\psi_x(t) - \psi_x(t-\pi/2n)|}{t} dt = o(\log n) ,$$

then we get (3).

Proof. It is sufficient to prove that $K_3 = o(\log n)$.

$$K_3 = \int_{\pi/n}^{\pi} \left[\frac{a(t)}{t} - \frac{a(t-\pi/2n)}{t-\pi/2n} \right] \cos 2nt\, td + o(\log n)$$

$$= \int_{\pi/n}^{\pi} \frac{a(t) - a(t-\pi/2n)}{t} \cos 2nt\, dt - \frac{\pi}{2n} \int_{\pi/n}^{\pi} \frac{a(t-\pi/2n)}{t(t-\pi/2n)}$$

$$\cos 2nt\, dt + o(\log n) \equiv L_1 + \frac{\pi}{2} L_2 + o(\log n).$$

say. By integration by parts

$$L_2 = \frac{1}{n} \int_{\pi/n}^{\pi} \frac{a(t-\pi/2n)}{t(t-\pi/2n)} \cos 2nt\, dt = -\frac{1}{n} \int_{\pi/n}^{\pi} \frac{a(t) \cos 2nt}{t(t+\pi/2n)} dt$$

$$= -\frac{1}{n} \left\{ \left[\frac{A(t) \cos 2nt}{t(t+\pi/2n)} \right]_{\pi/n}^{\pi} + \int_{\pi/n}^{\pi} A(t) \frac{2n \sin nt}{t\,(t+\pi/2n)} dt \right.$$

$$\left. + \int_{\pi/n}^{\pi} A(t) \frac{\cos 2nt\, dt}{t^2(t+\pi/2n)} + \int_{\pi/n}^{\pi} A(t) \frac{\cos 2nt\, dt}{t(t+\pi/2n)^2} \right\}$$

$$= o(\log n)$$

by the condition (2) and

$$|L_1| \leqq \int_{\pi/n}^{\pi} \frac{|a(t) - a(t-\pi/2n)|}{t} dt = o(\log n)$$

by (5). Thus we get the theorem.

§ **4.** Theorem 3. *If*

(2) $$\int_0^t \{\psi_x(t) - l(x)\}\, dt = o(t)$$

and

(6) $$\int_{\pi/n}^{\pi} \frac{|\psi_x(t) - \psi_x(t-\pi/2n)|}{t}\, dt = O\ (\log n)$$

then we get

(4) $$\lim_{n\to\infty} (\bar{\sigma}_{2n}(x) - \bar{\sigma}_n(x)) = l(x)(\log 2)/\pi\ .$$

Proof. If we put

$$\omega_n \equiv \frac{1}{\pi n} \int_0^{\pi} \sin nt \left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt,$$

then $\omega_n \to 2\ (\log 2)/\pi$. Thus we have

$$\bar{\sigma}_{2n}(x) - \bar{\sigma}_n(x) - \frac{\omega_n}{2}\cdot l\ (x)$$

$$= \frac{1}{2n\pi} \int_0^{\pi} a(t)\ \sin nt \left(\frac{\sin nt/2}{\sin t/2}\right)^2 dt$$

$$= \frac{1}{2n\pi} \int_0^{\pi} a(t)\ \frac{2 \sin nt - \sin 2nt}{t^2}\ dt + o(1)$$

$$\equiv \mathrm{I} + o(1),$$

say. Now

$$\mathrm{I} = \frac{1}{2n\pi}\left(\int_0^{kn\pi/x} + \int_{k_n\pi/n}^{\pi}\right) a(t)\ \frac{2 \sin nt - \sin 2nt}{t^2}\ dt$$

$$= (\mathrm{I}_1 + \mathrm{I}_2)/2\pi$$

say, where k_n will be determined later.

$$\mathrm{I}_1 = \frac{1}{n}\int_0^{k_n\pi/n} a(t)\ \frac{2 \sin nt - \sin 2nt}{t^2}\ dt$$

$$= \frac{1}{n}[A(t)\,(2 \sin nt - \sin 2nt)/t^2]_0^{k_n\pi/n}$$

$$-2\int_0^{k_n\pi/n} A(t) \left\{n\ \frac{\cos nt - \cos 2nt}{t^2} - \frac{2 \sin nt - \sin 2nt}{t^3}\right\} dt$$

$$= O\left(\frac{1}{n}\left[\frac{A(t)}{t}\cdot\frac{(nt)^3}{t}\right]_0^{k_n\pi/n}\right) + O\left(\frac{1}{n}\int_0^{k_n\pi/n} \left|\frac{A(t)}{t}\right| \frac{(nt)^3}{t^2}\, dt\right)$$

$$= O\left(\left|\frac{A(k_n\pi/n)}{k_n\pi/n}\right| k_n^2\right) + O\left(\sup_{t\le k_n\pi/n} \left|\frac{A(t)}{t}\right| \cdot k_n^2\right)$$

We can take k_n such that the right hand side tends to zero and $k_n \to \infty$. On the other hand

$$I_2 = \frac{1}{n}\int_{k_n \pi/n}^{\pi} a(t) \frac{2 \sin nt - \sin 2nt}{t^2}\, dt$$

$$= \frac{2}{n}\int_{k_n \pi/n}^{\pi} a(t) \frac{\sin nt}{t^2}\, dt - \frac{1}{n}\int_{k_n \pi/n}^{\pi} a(t) \frac{\sin 2nt}{t^2}\, dt = J_1 + J_2$$

say. We shall estimate J_1 only, since the remaining can be treated quite similarly.

$$J_1 = \frac{2}{n}\int_{k_n \pi/n}^{\pi} a(t) \frac{\sin nt}{t^2}\, dt$$

$$= \frac{1}{n}\int_{k_n \pi/n}^{\pi} a(t) \frac{\sin nt}{t^2}\, dt - \frac{1}{n}\int_{\frac{k_n \pi}{n}+\frac{\pi}{n}}^{\pi+\pi/n} a\left(t-\frac{\pi}{n}\right) \frac{\sin nt}{t-\frac{\pi}{n}}\, dt$$

$$= \frac{1}{n}\left\{\int_{k_n \pi/n}^{\frac{k_n\pi}{n}+\frac{\pi}{n}} a(t) + \frac{\sin nt}{t^2}\, dt - \int_{\pi}^{\pi+\frac{\pi}{n}} a\left(t-\frac{\pi}{n}\right) \frac{\sin nt}{t-\frac{\pi}{n}}\, dt\right.$$

$$\left. + \int_{(k_n+1)\pi/n}^{\pi} \left[\frac{a(t)}{t^2} - \frac{a(t-\pi/n)}{(t-\pi/n)^2}\right] \sin nt\, dt\right\}$$

$$= \frac{1}{n}\int_{k_n \pi/n}^{\pi} \left\{\frac{a(t)}{t^2} - \frac{a(t-\pi/n)}{(t-\pi/n)^2}\right\} \sin nt\, dt + o(1)$$

$$= \frac{1}{n}\int_{k_n \pi/n}^{\pi} \frac{a(t) - a(t-\frac{\pi}{n})}{t^2} \sin nt\, dt$$

$$+ \frac{\pi}{n^2}\int_{k_n \pi/n}^{\pi} a\left(t-\frac{\pi}{n}\right) \frac{2t-\pi/n}{t^2\left(t-\frac{\pi}{n}\right)^2} \sin nt\, dt + o(1).$$

Hence we have

$$|J_1| \leqq \frac{1}{k_n}\int_{\pi/n}^{\pi} \frac{|(t) - a(t-\pi/n)|}{t}\, dt + o(\log n).$$

Thus we get the theorem.

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