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Notes on Fourier Analysis (XI): On the absolute summability of Fourier series.

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In the present note the author discusses three different problems concerning the absolute Cesàro summability of Fourier series. Firstly we prove a series theorem and as corollaries we get some analoga of the absolute convergence theorems of Fourier series (in §1). In §2 we prove theorems concerning absolute summability factors. Finally, in §3, we prove a localization theorem of the absolute summability and show that the analogue of the Denjoy-Lusin theorem does not hold in general.

§1. **Theorem 1.** *If* $\sum_{n=1}^{\infty} |u_n|$, *then* $\sum_{n=1}^{\infty} u_n/A_n^{(\gamma)}$ $(0<\gamma<1)$ *is* $|C,-\gamma|$-*summable, where* $A_n^{(\gamma)}=\binom{n+\gamma}{n}$.

Proof. By $s_n^{(\delta)}$ we denote the n-th Cesàro mean of order δ (>-1) of the series $\sum_{n=1}^{\infty} x_n$. Then[1]

$$x_n^{(-\gamma)} \equiv s_n^{(-\gamma)} - s_{n-1}^{(-\gamma)} = \frac{1}{nA_n^{(-\gamma)}} \sum_{k=1}^{n} kA_{n-k}^{(-\gamma-1)} x_k.$$

Putting $x_n=u_n/A_n^{(\gamma)}$, we have

$$|x_n^{(-\gamma)}| \leq \frac{-1}{nA_n^{(-\gamma)}} \sum_{k=1}^{n-1} A_{n-k}^{(-\gamma-1)} k \frac{|u_k|}{A_k^{(\gamma)}} + |u_n|,$$

$$\sum_{n=2}^{N+1} |x_n^{(-\gamma)}| \leq \sum_{p=1}^{N} \frac{-1}{(p+1)A_{p+1}^{(-\gamma)}} \sum_{k=1}^{p} A_{p-k+1}^{(-\gamma-1)} \frac{k|u_k|}{A_k^{(\gamma)}} + \sum_{n=2}^{N} |u_n|$$

$$\leq \sum_{k=1}^{N} \frac{k|u_k|}{A_k^{(\gamma)}} \sum_{p=k}^{N} \frac{-A_{p-k+1}^{(-\gamma-1)}}{(p+1)A_{p+1}^{(-\gamma)}} + \sum_{n=2}^{N} |u_n|,$$

where

$$\sum_{p=k}^{N} \frac{-A_{p-k+1}^{(-\gamma-1)}}{(p+1)A_{p+1}^{(-\gamma)}} = \sum_{i=1}^{N-k+1} \frac{-A_i^{(-\gamma-1)}}{(k+i)A_{k+i}^{(-\gamma)}}$$

$$\leq \frac{1}{k^{1-\gamma}} \sum_{i=1}^{N-k+1} (-A_i^{(-\gamma-1)}) \leq \frac{1}{k^{1-\gamma}} \sum_{i=1}^{\infty} (-A_i^{(-\gamma-1)}).$$

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1) This formula is due to Kogbetlianz [6].

Since $(1-x)^{\gamma}=\sum_{i=0}^{\infty} A_i^{(-\gamma-1)}x^i$, and then $\sum_{i=0}^{\infty} A_i^{(-\gamma-1)}=0$, we have

$$\sum_{n=2}^{N-1} | x_n^{(-\gamma)} | \leqq \sum_{k=1}^{N} \frac{k}{k^{\gamma}} | u_k | \frac{1}{k^{1-\gamma}} \leqq \sum_{k=1}^{N} | u_k |,$$

which is the required.

Combining this with the following theorems[2]:

(i) If $f \epsilon Lip\, \alpha, 0 < \alpha \leqq 1$, then $\sum n^{\beta-\frac{1}{2}}(| a_n |+| b_n |) < \infty$ for every $\beta < \alpha$,

(ii) If f is, in addition, of bounded variation, then $\sum n^{\frac{\beta}{2}}(| a_n |+| b_n |) < \infty$,

(iii) If $f \epsilon Lip\, (\alpha, p)$, $0 < \alpha \leqq 1$, $1 \leqq p \leqq 2$, then $\sum n^{\gamma}(| a_n |+| b_n |) < \infty$ for every $\gamma < \alpha - 1/p$,

we get the following corollaries:

Corollary i. *If $f(x) \epsilon Lip\, \alpha$, then the Fourier series of $f(x)$ is $| C, \frac{1}{2}-\beta |$-summable for $\frac{1}{2} \leqq \beta < \alpha$.*

Corollary ii. *If $f(x) \epsilon Lip\, \alpha$ and is of bounded variation, then the Fourier series of $f(x)$ is $| C, -\beta/2 |$-summable for $\beta < \alpha$*[3].

Corollary iii. *If $f(x) \epsilon Lip\, (\alpha, p)$ for $0 < \alpha \leqq 1$, $1 \leqq p \leqq 2$, then the Fourier series of $f(x)$ is $| C, -\gamma |$-summable for $0 < \gamma < \alpha - 1/p$*[4].

These are generalizations of theorems due to Bernstein, Zygmund and Hardy-Littlewood, respectively.

§ **2.** We will begin by stating the theorems.

Theorem 2. *If $\{\lambda_n\}$ is a positive, bounded and convex sequence such that the series*

$$(1) \qquad \sum_{n=1}^{\infty} \lambda_n / n$$

converges, then $\{\lambda_n\}$ is a $| C, 1 |$-summability factor of the Fourier series of an integrable function.

For example each of the sequences:

$$\{ (\log n)^{-(1+\delta)} \}, \; \{ (\log n)^{-1} (\log_2 n)^{-(1+\delta)} \}, \ldots\ldots \; (\delta > 0)$$

are the $| C, 1 |$-summability factors of Fourier series.

Theorem 3. *If $\{\lambda_n\}$ is a positive sequence such that series*

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2) See Zygmund, Trigonometrical series, p. 143, problem 6.

3) I learned from Mathematical Review that this theorem had already been proved by K. K. Chen [2].

4) This is already proved by Chow [3].

(2) $$\sum_{n=1}^{\infty} n(\Delta\lambda_n)^2$$

and

(3) $$\sum_{n=1}^{\infty} \lambda_n^2/n$$

converge, then $\{\lambda_n\}$ *is a* $|C, 1|$*-summability factor of the Fourier series of functions in the class* $H^p(p \geq 1)$.

For example

$$\{(\log n)^{-(\frac{1}{2}+\delta)}\},\ \{(\log n)^{-\frac{1}{2}}(\log_2 n)^{-(\frac{1}{2}+\delta)}\}, \ldots\ldots, (\delta>0)$$

are $|C, 1|$-summability factors of functions in $H^p(p \geq 1)$.

These are generalizations of theorems due to B.N. Prasad [8] and Izumi-Kawata [4], [5]. They have proved that the $\{\lambda_n\}$ in the theorem is the absolute Abel summability factor.

We will now prove Theorem 2. Since $\{\lambda_n\}$ is bounded and convex[5],

(4) $\{\lambda_n\}$ is decreasing,

(5) $$n\Delta\lambda_n \to 0, \text{ as } n \to \infty,$$

(6) $$\sum_{n=0}^{\infty} (n+1)\Delta^2\lambda_n < \infty.$$

Let us put

$$f(x) \sim \frac{1}{2}a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx) \equiv \sum_{n=0}^{\infty} A_n(x),$$

$$u_n^{(1)}(x) \equiv s_n^{(1)}(x) - s_{n-1}^{(1)}(x) \equiv \frac{1}{n^2}\sum_{k=1}^{n} ku_k(x), \quad u_k(x) \equiv \lambda_k A_k(x).$$

For the proof of the theorem it is sufficient to prove that the series $\sum_{n=1}^{\infty} u_n^{(1)}(x)$ converges almost everywhere. By Abel's transformation, we have

$$\sum_{k=1}^{n} k\lambda_k A_k(x) = \sum_{k=1}^{n-1}\left(\sum_{m=1}^{k} mA_m(x)\right)\Delta\lambda_k + \lambda_n \sum_{m=1}^{n} mA_m(x)$$
$$= \sum_{k=1}^{n-1}\{k(s_k(x) - s_k^{(1)}(x)\Delta\lambda_k\} + n(s_n(x) - s_n^{(1)}(x))\lambda_n$$
$$= P_n(x) + Q_n(x),$$

5) Cf. Zygmund, loc. cit., p. 58.

say. Then

$$\sum_{n=1}^{\infty} | u_n^{(1)}(x) | \leqq \sum_{n=1}^{\infty} \frac{1}{n^2} | P_n(x) | + \sum_{n=1}^{\infty} \frac{1}{n^2} | Q_n(x) |$$
$$= P'(x) + Q'(x).$$

Since we have

$$\sum_{m=1}^{n} | s_m(x) - s_m^{(1)}(x) | = O(n), \text{ a.e.}$$

by a theorem due to J. Marcinkiewicz [7], we have

$$P'(x) = \sum_{n=1}^{\infty} \frac{1}{n^2} \sum_{k=1}^{n-1} k | s_k(x) - s_k^{(1)}(x) | \Delta\lambda_k$$
$$= \sum_{k=1}^{\infty} k | s_k(x) - s_k^{(1)}(x) | \Delta\lambda_k \sum_{n=k}^{\infty} \frac{1}{n^2}$$
$$= \sum_{k=1}^{\infty} | s_k(x) - s_k^{(1)}(x) | \Delta\lambda_k = \lim_{n\to\infty} \sum_{k=1}^{\infty} | s_k - s_k^{(1)} | \Delta\lambda_k$$
$$= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n-1} \left(\sum_{m=1}^{k} | s_m - s_m^{(1)} | \right) \Delta^2\lambda_k + \left(\sum_{m=1}^{n} | s_m - s_m^{(1)} | \right) \Delta\lambda_n \right\}$$
$$= \lim_{n\to\infty} \left\{ \sum_{k=1}^{n-1} O(k) \Delta^2\lambda_k + O(n) \cdot \Delta\lambda_n \right\}$$
$$= O\left(\sum_{k=1}^{\infty} k\Delta^2\lambda_k\right) + O\left(\lim_{n\to\infty} n\Delta\lambda_n\right) < +\infty, \quad \text{a.e.,}$$

by (5) and (6).

$$Q' = \sum_{n=1}^{\infty} \frac{| s_n - s_n^{(1)} |}{n} \lambda_n$$
$$= \lim_{k\to\infty} \left\{ \sum_{n=1}^{k} \left(\sum_{m=1}^{n} | s_m - s_m^{(1)} | \right) \Delta\left(\frac{\lambda_n}{n}\right) + \frac{\lambda_k}{k} \sum_{m=1}^{k} | s_m - s_m^{(1)} | \right\}$$
$$= \lim_{k\to\infty} \left\{ \sum_{n=1}^{k} \Delta\lambda_n + \sum_{n=1}^{k} \lambda_n/n + \lambda_k \right\}$$
$$< +\infty, \quad \text{a.e.,}$$

by (1) and (4). Thus we get the Theorem.

We will now turn to the proof of Theorem 3. By Abel's transformation, we have

$$n(n+1)\, u_n^{(1)}(x) = \sum_{k=1}^{n} k\lambda_k A_k(x)$$

$$=\sum_{k=1}^{n-1}\Big(\sum_{m=1}^{k} mA_m(x)\Big)\Delta\lambda_k+\lambda_n\sum_{m=1}^{n} mA_m(x)$$
$$=\sum_{k=1}^{n-1}(k+1)(s_k(x)-s_k^{(1)}(x))\Delta\lambda_k+\lambda_n(n+1)(s_n(x)-s_n^{(1)}(x))$$
$$=P_n(x)+Q_n(x),$$

say. Summing up by n we get

$$\sum_{n=1}^{\infty}|u_n^{(1)}(x)|\leqq\sum_{n=1}^{\infty}\frac{1}{n(n+1)}|P_n(x)|+\sum_{n=1}^{\infty}\frac{1}{n(n+1)}|Q_n(x)|$$
$$=P'(x)+Q'(x),$$

say. Then

$$P'(x)\leqq\sum_{n=1}^{\infty}\frac{1}{n(n+1)}\sum_{k=1}^{n-1}(k+1)|s_k(x)-s_k^{(1)}(x)|\,|\Delta\lambda_k|$$
$$\leqq\sum_{k=1}^{\infty}(k+1)|s_k(x)-s_k^{(1)}(x)|\,|\Delta\lambda_k|\sum_{n=k+1}^{\infty}\frac{1}{n^2}$$
$$\leqq\sum_{k=1}^{\infty}|s_k(x)-s_k^{(1)}(x)|\,|\Delta\lambda_k|$$
$$\leqq\Big(\sum_{k=1}^{\infty}\frac{|s_k(x)-s_k^{(1)}(x)|^2}{k}\Big)^{\frac{1}{2}}\Big(\sum_{k=1}^{\infty}k(\Delta\lambda_k)^2\Big)^{\frac{1}{2}}$$
$$<+\infty,\quad \text{a.e.},$$

by (2) and Zygmund's theorem [10].

$$Q'(x)\leqq\Big(\sum_{n=1}^{\infty}\frac{|s_n(x)-s_n^{(1)}(x)|^2}{n}\Big)^{\frac{1}{2}}\Big(\sum_{n=1}^{\infty}\frac{n\lambda_n^2}{n^2}\Big)<+\infty,\quad \text{a.e.},$$

by (3) and Zygmund's theorem. Thus we get Theorem 3.

§ **3.** Bosanquet and Kestleman [1] proved that the Fourier series of integrable functions have not local property for $|C,1|$. But we can prove that the Fourier series of functions in $L_p(p>1)$ have the local property for $|C,1|$. More precisely we can prove the following theorem.

Theorem 4. *If $f(x)\in L_p(p>1)$ and*

$$\Big(\frac{1}{t}\int_0^t|\varphi_x(u)|^s du\Big)^{1/s}=O(1/(\log 1/t)^{\varepsilon})\quad as\quad t\to 0$$

where $s>1$ and $\varepsilon>1$, then the Fourier series of $f(x)$ is $|C,1|$-summable at the point x.

For the proof we need a lemma, which is due to Mr. T. Tsuchikura [9]. For the sake of completeness we reproduce his proof here.

Lemma. *If* $f(x)\epsilon L_p(p>1)$ *and*

$$\left(\frac{1}{t}\int_0^t|\varphi_x(u)|^s du\right)^{1/s}=O(1/(\log 1/t)^{\varepsilon}) \quad as\ t\to 0,$$

where $s>1$ *and* $\varepsilon>0$, *then*

$$\left(\frac{1}{n}\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|^k\right)^{1/k}=O(1/(\log n)^{\varepsilon}), \quad k>0.$$

Proof. $$\left(\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|^k\right)^{1/k}=\left\{\sum_{\nu=1}^{n}\left|\frac{1}{\pi}\int_0^\pi\varphi_x(t)\frac{\sin(\nu+1/2)t}{2\sin t/2}dt\right|^k\right\}^{1/k}$$

$$\leqq\left\{\sum_{\nu=1}^{n}\frac{1}{\pi}\left|\int_0^{1/n}\varphi_x(t)D_\nu(t)\right|^k\right\}^{1/k}+\left\{\sum_{\nu=1}^{n}\left|\frac{1}{\pi}\int_{1/n}^{1/n^\alpha}\varphi_x(t)D_\nu(t)\right|^k\right\}^{1/k}$$

$$+\left\{\sum_{\nu=1}^{n}\frac{1}{\pi}\int_{1/n^\alpha}^{\pi}\varphi_x(t)D_\nu(t)\right|^k\right\}^{1/k}=I_1+I_2+I_3,$$

say, where $\alpha(<1)$ will be determined later. Then

$$I_1^k=O\left\{\sum_{\nu=1}^{n}\nu^k\left(\int_0^{1/n}|\varphi_x(t)|\,dt\right)^k\right\}=O\left\{n^{k+1}\left(\frac{1}{n(\log n)^{\varepsilon}}\right)^k\right\}$$
$$=O(n/(\log n)^{k\varepsilon}).$$

By the Hausdorff-Young theorem,

$$I_2=O\left[\left\{\sum_{\nu=1}^{n}\left|\int_{1/n}^{1/n^\alpha}\varphi_x(t)\frac{\sin\nu t}{t}dt\right|^k\right\}^{1/k}\right]$$
$$=O\left\{\left(\int_{1/n}^{1/n^\alpha}\left|\frac{\varphi_x(t)}{t}\right|^{k'}dt\right)^{1/k'}\right\}$$

where $1/k+1/k'=1$. We choose k so large that $k'\leqq s$ and $k'\leqq min(p,2)$, then

$$I_2=O\left\{\left([\Phi_{k'}(t)/t^{k'}]_{1/n}^{1/n^\alpha}+\int_{1/n}^{1/n^\alpha}\frac{\Phi_{k'}(t)}{t^{k'+1}}dt\right)^{1/k'}\right\}$$
$$=O\left\{\left(n^{k'-1}/(\log n)^{k'\varepsilon}+\int_{1/n}^{1/n^\alpha}\frac{dt}{t^{k'}(\log 1/t)^{k'\varepsilon}}\right)^{1/k'}\right\}$$
$$=O\{(n^{k'-1}/(\log n)^{k'\varepsilon}+n^{k'-1}/(\log n)^{k'\varepsilon})^{1/k'}\}$$
$$=O(n^{1/k}/(\log n)^{\varepsilon}).$$

Similarly we have

$$I_3=O\left\{\left(\int_{1/n^\alpha}^{\pi}\left|\frac{\varphi_x(t)}{t}\right|^{k'}dt\right)^{1/k'}\right\}=O\left\{\left(n^{k'\alpha}\int_0^{\pi}|\varphi_x(t)|^{k'}dt\right)^{1/k'}\right\}$$

which is $O(n^\alpha)$ for $\alpha<1/k$. Thus we have

$$I_1+I_2+I_3=O(n^{1/k}/(\log n)^\varepsilon).$$

We are now in a position to prove the theorem. By lemma and Hölder's inequality, we have

$$\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|=O(n/(\log n)^\varepsilon).$$

Since $\varepsilon>1$,

$$\sum_{n=1}^{k}|s_n(x)-f(x)|/n=\sum_{n=1}^{k-1}\left(\sum_{\nu=1}^{n}|s_\nu(x)-f(x)|\right)\Delta\frac{1}{n}+\frac{1}{k}\sum_{n=1}^{k}|s_n(x)-f(x)|$$

$$=\sum_{n=1}^{k-1}O\left(\frac{n}{(\log n)^\varepsilon}\frac{1}{n^2}\right)+\frac{1}{k}O\left(\frac{k}{(\log k)^\varepsilon}\right)$$

$$=O\left(\sum_{n=1}^{k}\frac{1}{n(\log n)^\varepsilon}\right)<+\infty.$$

Thus the series $\sum_{n=1}^{\infty}|s_n(x)-f(x)|/n$ converges at the point x. On the other hand we have

$$\sum_{n=1}^{\infty}\frac{|s_n^{(1)}(x)-f(x)|}{n}=\sum_{n=1}^{\infty}\left|\frac{s_1(x)+s_2(x)+\dots\dots+s_n(x)}{n}-f(x)\right|\Big/n$$

$$\leqq\sum_{n=1}^{\infty}\frac{|s_1(x)-f(x)|+|s_2(x)-f(x)|+\dots\dots+|s_n(x)-f(x)|}{n^2}$$

$$=\sum_{n=1}^{\infty}O\left(\frac{n}{n^2(\log n)^\varepsilon}\right)=O\left(\sum_{n=1}^{\infty}\frac{1}{n(\log n)^\varepsilon}\right)<+\infty.$$

Hence we have

$$\sum_{n=1}^{\infty}|s_n(x)-s_n^{(1)}(x)|/n\leqq\sum_{n=1}^{\infty}|s_n(x)-f(x)|/n+\sum_{n=1}^{\infty}|s_n^{(1)}(x)-f(x)|/n<\infty.$$

Thus the series $\sum_{n=1}^{\infty}|s_n^{(1)}(x)-s_{n-1}^{(1)}(x)|$ converges at x.

Theorem 5. *There exists a function in L_2 which is $|C,1|$-summable in (a,b) in $(0,2\pi)$, but not $|C,1|$-summable almost everywhere in the complementary interval.*

This theorem shows that the $|C, 1|$-analogue of the Denjoy-Lusin theorem does not hold in general.

Proof. After Zygmund [11], there is a function in L_2 which is not $|A|$-summable almost everywhere. Taking such $f(x)$ we define the function $\varphi(x)$ by

$$\varphi(x)=\begin{cases}0, & \text{if } x\epsilon(a,b)\\ f(x), & \text{if } x\bar{\epsilon}(a,b)\end{cases} \qquad \psi(x)=\begin{cases}f(x), & \text{if } x\epsilon(a,b),\\ 0, & \text{if } x\bar{\epsilon}(a,b).\end{cases}$$

Then $\varphi(x)$ is the required one. For, by Theorem 4, Fourier series of $\varphi(x)$ is $|C, 1|$-summable in (a, b), but the Fourier series of $f(x)=\varphi(x)+\psi(x)$ is not $|A|$-summable almost everywhere. Since the Fourier series of $\psi(x)$ is $|C, 1|$-summable in $(0, 2\pi)-(a, b)$, Fourier series of $\varphi(x)$ is not $|A|$-summable almost everywhere in $(0, 2\pi)-(a, b)$. Thus we get the theorem.

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