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On the cohomology theory of rings.

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Recently G. Hochschild has developed the theory of cohomology groups of associative algebras[1]. We shall consider in this paper some problems concerning the cohomology groups of rings. Especially we shall be able to characterize the vanishing of $\mathbf{H}_n(\mathbf{R}, \mathbf{m})$ for every $\mathbf{R}$-$\mathbf{R}$-module $\mathbf{m}$ in the case $n=1$ and 3 by the extension properties (Theorem 6 and 8).

In § 1 necessary definitions from Hochschild's theory are given. § 2 concerns the extensions of $\mathbf{R}$-$\mathbf{R}$-modules. In § 3 we define a useful mapping $F_{\beta\gamma}$ of $\mathbf{H}_n(\mathbf{R}, \mathbf{m})$ to $\mathbf{H}_{n+1}(\mathbf{R}, \mathbf{n})$ for any $\mathbf{R}$-$\mathbf{R}$-modules $\mathbf{m}$ and $\mathbf{n}$, which is a generalization of the fundamental isomorphism of Hochschild. In § 4 we consider the special extension problem, which corresponds to the Teichmüller's theory for simple algebras[2]. These considerations can also be applied to Lie algebras, as I shall show in another paper.

§ **1.** *Definitions of the cohomology groups of rings.*

Let $\mathbf{R}$ be a ring and $\mathbf{m}$ an $\mathbf{R}$-$\mathbf{R}$-module. Namely, we suppose that am, mb $(m \in \mathbf{m},\ a, b \in \mathbf{R})$ belong to $\mathbf{m}$, are linear, distributive in a, b, m, and satisfy the associative law $a(bm)=(ab)m$, $(ma)b=m(ab)$, $(am)b=a(mb)$. We call an element $f_0 \in \mathbf{m}$ a *0-cochain*, and $f_n(a_1, \ldots\ldots, a_n) \in \mathbf{m}$ $(a_i \in \mathbf{R})$, which is linear with respect to $a_1, \ldots\ldots, a_n$, a *n-cochain* $(n \geqq 1)$. We denote the module of all n-cochains by $\mathbf{L}_n(\mathbf{R}, \mathbf{m})$. Moreover, we define the *coboundary operator* $\delta f_n = f_{n+1}$ $(f_n \in \mathbf{L}_n(\mathbf{R}, \mathbf{m}),\ f_{n+1} \in \mathbf{L}_{n+1}(\mathbf{R}, \mathbf{m}))$ by

$$(\delta f_n)(a_1, \ldots\ldots, a_{n+1}) = a_1 f_n(a_2, \ldots\ldots, a_{n+1}) + \sum_{k=1}^{n}(-1)^k f_n(a_1, \ldots\ldots, a_k a_{k+1}, \ldots\ldots, a_{n+1}) + (-1)^{n+1} f_n(a_1, \ldots\ldots, a_n) a_{n+1}. \quad (1)$$

Then δ is a linear mapping and satisfies the relation $\delta(\delta f_n)=0$ for any f_n. We call an element f_n with $\delta f_n = 0$ an *n-cocycle* $(n \geqq 0)$ and an element f_n with $f_n = \delta g_{n-1}$ $(n \geqq 1)$ an *n-coboundary*. We denote the module of all n-cocycles (n-coboundaries) by $\mathbf{C}_n(\mathbf{R}, \mathbf{m})$ $(\mathbf{B}_n(\mathbf{R}, \mathbf{m}))$. And we define the *n-cohomology group* $\mathbf{H}_n(\mathbf{R}, \mathbf{m}) = \mathbf{C}_n(\mathbf{R}, \mathbf{m}) / \mathbf{B}_n(\mathbf{R}, \mathbf{m})$ $(n \geqq 1)$.

§ **2.** *Extension of* $\mathbf{R}$-$\mathbf{R}$-*module and* 1-*cohomology group* $\mathbf{H}_1(\mathbf{R}, \mathbf{m})$.

Def. Let $\mathbf{m}$, $\mathbf{n}$ be two $\mathbf{R}$-$\mathbf{R}$-modules. We call an another $\mathbf{R}$-$\mathbf{R}$-module $\mathbf{M}$ an *extension of* $\mathbf{m}$ *by* $\mathbf{n}$, if (i) $\mathbf{M} \supseteqq \mathbf{n}$, (ii) $\mathbf{M}/\mathbf{n} \cong \mathbf{m}$ (as $\mathbf{R}$-$\mathbf{R}$-module), (iii)

$\mathbf{M} \cong \mathbf{m}+\mathbf{n}$ (direct sum as module) hold. If $\mathbf{M} \cong \mathbf{m}+\mathbf{n}$ (direct sum as **R**-**R**-module) we say that **M** *splits*.

Now we consider an extension **M** of **m** by **n**. We denote elements of **m** by a, b, ..., and of **n** by α, β, ... For an element $m \in \mathbf{m}$ take linear representatives $B_m \in \mathbf{M}$ from the class corresponding to $m \in \mathbf{m}$ by the relation $\mathbf{M}/\mathbf{n} \cong \mathbf{m}$. Then

$$aB_m = B_{am} + \beta(a, m), \quad B_m a = B_{ma} + \gamma(m, a) \quad (a \in \mathbf{R},\ m \in \mathbf{m},\ \beta, \gamma \in \mathbf{n}). \qquad (2)$$

$\beta(a, m)$ and $\gamma(m, a)$ are linear in a, m. By the associative law $a(bB_m) = (ab)B_m$, $(B_m a)b = B_m(ab)$, $(aB_m) = a(B_m b)$ we have

$$a\beta(b, m) + \beta(a, bm) - \beta(ab, m) = 0, \quad \gamma(m, a)b + \gamma(ma, b) - \gamma(m, ab) = 0,$$
$$\beta(a, mb) - \beta(a, m)b = \gamma(am, b) - a\gamma(m, b). \qquad (3)$$

If we choose another linear representatives

$$B_m^* = B_m + \mu(m) \qquad (m \in \mathbf{m},\ \mu \in \mathbf{n}), \qquad (4)$$

we have

$$\beta^*(a, m) = aB_m^* - B_{am}^* = \beta(a, m) + \{a\mu(m) - \mu(am)\},$$
$$\gamma^*(m, a) = B_m^* a - B_{ma}^* = \gamma(m, a) + \{\mu(m)a - \mu(ma)\}. \qquad (5)$$

We call $\{\beta, \gamma\}$ satisfying the relations (3) a *factor system*, and two factor systems $\{\beta, \gamma\}$ and $\{\beta^*, \gamma^*\}$ satisfying (5) for some $\{\mu\}$ *associated*. The structure of an extension **M** is completely determined by $\{\beta, \gamma\}$. Hence, we write $\mathbf{M} = (\mathbf{m}, \mathbf{n}, \beta, \gamma)$. Conversely, for any factor system $\{\beta, \gamma\}$, there exists an extension $\mathbf{M} = (\mathbf{m}, \mathbf{n}, \beta, \gamma)$ satisfying the relation (2). Two extensions $\mathbf{M}_i(\mathbf{m}, \mathbf{n}, \beta_i, \gamma_i)$ $(i = 1, 2)$ are isomorphic (as **R**-**R**-module, each element of $\mathbf{n} \subseteq \mathbf{M}_i$ $(i = 1, 2)$ corresponding to itself) if and only if $\{\beta_1, \gamma_1\}$ and $\{\beta_2, \gamma_2\}$ are associated. We identify these $\mathbf{M}_1$ and $\mathbf{M}_2$.

We define $\{\beta_1, \gamma_1\} + \{\beta_2, \gamma_2\} = \{\beta_1 + \beta_2, \gamma_1 + \gamma_2\}$, then all the factor systems make a module $\mathbf{F}(\mathbf{m}, \mathbf{n})$. Splitting factor systems

$$\beta(a, m) = a\mu(m) - \mu(am), \quad \gamma(m, a) = \mu(m)a - \mu(ma) \qquad (6)$$

make a submodule $\mathbf{S}(\mathbf{m}, \mathbf{n})$ of $\mathbf{F}(\mathbf{m}, \mathbf{n})$. Then we have obviously

Theorem 1. *Each element of* $\mathbf{F}(\mathbf{m}, \mathbf{n})/\mathbf{S}(\mathbf{m}, \mathbf{n})$ *corresponds one to one to the extension* **M** *of* **m** *by* **n**.

Now we consider the relation to the cohomology groups. We shall first assume that we can select the linear representatives $\{B_m\}$ with

$$B_m a = B_{ma} \qquad (m \in \mathbf{m},\ a \in \mathbf{R}). \tag{7}$$

Such extension has factor system $\{\beta^*, 0\}$. The condition for β^* is

$$a\beta^*(b, m) + \beta^*(a, bm) - \beta^*(ab, m) = 0, \quad \beta^*(a, m)b - \beta^*(a, mb) = 0 \tag{8}$$

Let $\mathbf{r}$ be the set of all the linear mappings λ of $\mathbf{m}$ into $\mathbf{n}$ with

$$\lambda(m)a = \lambda(ma) \qquad (m \in \mathbf{m},\ a \in \mathbf{R}). \tag{9}$$

If we define the left and right operations by an element $a \in \mathbf{R}$ to $\mathbf{r}$ by

$$(a*\lambda)(m) = a\lambda(m), \quad (\lambda * a)(m) = \lambda(am) \qquad (m \in \mathbf{m}), \tag{10}$$

then $\mathbf{r}$ is an $\mathbf{R}$-$\mathbf{R}$-module. We denote this $\mathbf{r}$ by $\mathbf{r} = \mathbf{1}(\mathbf{m}, \mathbf{n})$.

Each 1-cochain $f(a) \in \mathbf{L}_1(\mathbf{R}, \mathbf{r})$ is representable by

$$f(a) = \varphi(a, m) \in \mathbf{n} \qquad (a \in \mathbf{R},\ m \in \mathbf{m})$$

with $\varphi(a, m)b = \varphi(a, mb)$. The condition $\delta f = 0$ amounts to $\delta\varphi = a*\varphi(b, m) - \varphi(ab, m) + \varphi(a, m)*b = a\varphi(b, m) - \varphi(ab, m) + \varphi(a, bm) = 0$. This is exactly the condition (8). The condition $f = \delta f_0$ is

$$\varphi(a, m) = \delta\psi(m) = a*\psi(m) - \psi(m)*a = a\psi(m) - \psi(am).$$

This means that $\{\beta^*, 0\}$ splits.

Now let $\mathbf{F}^*(\mathbf{m}, \mathbf{n})$ be the set of all factor systems $\{\beta^*, 0\}$, and put $\mathbf{S}^*(\mathbf{m}, \mathbf{n}) = \mathbf{F}^*(\mathbf{m}, \mathbf{n}) \cap \mathbf{S}(\mathbf{m}, \mathbf{n})$. Then we have

Theorem 2. *For* $\mathbf{r} = \mathbf{1}(\mathbf{m}, \mathbf{n})$, *it holds*

$$\mathbf{H}_1(\mathbf{R}, \mathbf{r}) \cong \mathbf{F}^*(\mathbf{m}, \mathbf{n}) / \mathbf{S}^*(\mathbf{m}, \mathbf{n})$$

Corollary 1. *If* $\mathbf{R}$ *contains the left unit* e $(ea = a$ *for every* $a \in \mathbf{R})$ *and if* $u\mathbf{R} \neq 0$ for $u \neq 0$ $(u \in \mathbf{n})$, *then*

$$\mathbf{H}_1(\mathbf{R}, \mathbf{n}) \cong \mathbf{F}^*(\mathbf{R}, \mathbf{n}) / \mathbf{S}^*(\mathbf{R}, \mathbf{n}).$$

For, any linear mapping λ from $\mathbf{R}$ to $\mathbf{n}$ with (9) can be represented as $\lambda(a) = ua$ $(u = \lambda(e))$. Hence, it holds $\mathbf{r} = \mathbf{1}(\mathbf{R}, \mathbf{n}) \cong \mathbf{n}$.

Corollary 2. (Hochschild [1], § 4) *If* $\mathbf{H}_1(\mathbf{R}, \mathbf{n}) = 0$ *for every* $\mathbf{n}$, *then any representation of* $\mathbf{R}$ *is completely reducible.*

For, any $\mathbf{R}$-left modules $\mathbf{m}, \mathbf{n}$ can be made to $\mathbf{R}$-$\mathbf{R}$-modules by $ma = 0$, $na = 0$ $(a \in \mathbf{R})$, and any linear mapping λ from $\mathbf{m}$ to $\mathbf{n}$ satisfies the relation

(9). Therefore, any **R**-submodule **n** of **R**-module **M** is a factor of direct sum.

If we consider **R**-right modules instead of **R**-left module , then we can also conclude that any **R**-right module **M** which is an extension of **R**-right module **m** by **n** splits: $\mathbf{M} \cong \mathbf{m} + \mathbf{n}$ (direct sum as **R**-right module).

Now if we consider an **R**-**R**-module $\mathbf{M} = (\mathbf{m}, \mathbf{n}, \beta, \gamma)$ merely as **R**-right module, then we can choose linear representatives B_m with the property (7). This means $\mathbf{F}(\mathbf{m}, \mathbf{n})/\mathbf{S}(\mathbf{m}, \mathbf{n}) \cong \mathbf{F}^*(\mathbf{m}, \mathbf{n})/\mathbf{S}^*(\mathbf{m}, \mathbf{n}) = 0$. Hence we have

Corollary 3. *If* $\mathbf{H}_1(\mathbf{R}, \mathbf{m}) = 0$ *for every* **m**, *then every extension* **M** *of* **m** *by* **n** *splits* (the converse will be proved in § 3).

§ 3. *Mapping* $F_{\beta\gamma}$ *from* $\mathbf{H}_n(\mathbf{R}, \mathbf{m})$ *to* $\mathbf{H}_{n+1}(\mathbf{R}, \mathbf{n})$.

Let **m**, **n** be two **R**-**R**-modules and $\mathbf{M} = (\mathbf{m}, \mathbf{n}, \beta, \gamma)$ an extension of **m** by **n**. We define the linear mapping $F = F_{\beta\gamma}$ from $\mathbf{L}_n(\mathbf{R}, \mathbf{m})$ into $\mathbf{L}_{n+1}(\mathbf{R}, \mathbf{n})$ $(n \geqq 0)$:

$$F_{\beta\gamma}(g) = f_{n+1} \in \mathbf{L}_{n+1}(\mathbf{R}, \mathbf{n}) \qquad (g_n \in \mathbf{L}_n(\mathbf{R}, \mathbf{m}))$$

by

$$f_{n+1}(a_1, \ldots\ldots, a_{n+1}) = \beta(a_1, g_n(a_2, \ldots\ldots, a_{n+1})) + (-1)^{n+1}\gamma(g_n(a_1, \ldots, a_n), a_{n+1}). \qquad (11)$$

Lemma 1. $F_{\beta\gamma}(\delta g_n) + \delta(F_{\beta\gamma}(g_n)) = 0 \qquad (n \geqq 0)$.

Proof. $F_{\beta\gamma}(\delta g_n) + \delta(F_{\beta\gamma}(g_n)) = \{\beta(a_0, a_1 g_n(a_2, \ldots\ldots, a_{n+1})) + \sum_{k=1}^{n}(-1)^k \beta(a_0, g_n(a_1, \ldots\ldots, a_k a_{k+1}, \ldots\ldots, a_{n+1})) + (-1)^{n+1}\beta(a_0, g_n(a_1, \ldots\ldots, a_n)a_{n+1})\} + (-1)^{n+2}\{\gamma(a_0 g_n(a_1, \ldots\ldots, a_n), a_{n+1}) + \sum_{k=1}^{n}(-1)^k\gamma(g_n(a_0, \ldots\ldots, a_{k-1}a_k, \ldots\ldots, a_n), a_{n+1}) + (-1)^{n+1}\gamma(g_n(a_0, \ldots\ldots, a_{n-1})a_n, a_{n+1})\} + \{a_0\beta(a_1, g_n(a_2, \ldots\ldots, a_{n+1}) - \beta(a_0 a_1, g_n(a_2, \ldots\ldots, a_{n+1})) + \sum_{k=1}^{n}(-1)^{k+1}\beta(a_0, g_n(a_1, \ldots\ldots, a_k a_{k+1}, \ldots\ldots, a_{n+1})) + (-1)^n\beta(a_0, g_n(a_1, \ldots\ldots, a_n))a_{n+1}\} + (-1)^{n+1}\{a_0\gamma(g_n(a_1, \ldots\ldots, a_n), a_{n+1}) + \sum_{k=1}^{n}(-1)^k\gamma(g_n(a_0, \ldots\ldots, a_{k-1}a_k, \ldots\ldots, a_n), a_{n+1}) + (-1)^{n+1}\gamma(g_n(a_0, \ldots\ldots, a_{n-1}, a_n a_{n+1}) + (-1)^n\gamma(g_n(a_0, \ldots\ldots, a_{n-1}), a_n)a_{n+1}\} = \{\beta(a_0, a_1 g_n(a_2, \ldots\ldots, a_{n+1})) + a_0\beta(a_1, g_n(a_2, \ldots\ldots, a_{n+1}) - \beta(a_0 a_1, g_n(a_2, \ldots\ldots, a_{n+1})\} + (-1)^{n+1}\{-\beta(a_0, g_n(a_1, \ldots\ldots, a_n)a_{n+1}) + \beta(a_0, g_n(a_1, \ldots\ldots, a_n))a_{n+1} - \gamma(a_0 g_n(a_1, \ldots\ldots, a_n), a_{n+1}) + a_0\gamma(g_n(a_1, \ldots\ldots, a_n), a_{n+1})\} - \{\gamma(g_n(a_0, \ldots\ldots, a_{n-1})a_n, a_{n+1}) - \gamma(g_n(a_0, \ldots\ldots, a_{n-1}), a_n a_{n+1}) + \gamma(g_n(a_0, \ldots\ldots, a_{n-1}), a_n)a_{n+1}\} = 0$ (by (3)), Q.E.D.

Lemma 2. *For two associated factor systems* $\{\beta, \gamma\}$, $\{\beta^*, \gamma^*\}$ *with* (5), *we have for* $h_n(a_1, \ldots\ldots, a_n) = \mu(g_n(a_1, \ldots\ldots, a_n))$

$$F_{\beta\gamma}(g_n) - F_{\beta^*\gamma^*}(g_n) = (\delta h_n) - \mu(\delta g_n), \qquad (n \geqq 0).$$

Proof. $\{F_{\beta\gamma}(g_n)-F_{\beta^*\gamma^*}(g_n)\}(a_0,\ldots\ldots,a_n)=a_0\mu(g_n(a_1,\ldots\ldots,a_n))-\mu(a_0 g_n(a_1,\ldots\ldots,a_n))+(-1)^{n+1}\{\mu(g_n(a_0,\ldots\ldots,a_{n-1})a_n)-\mu(g_n(a_0,\ldots\ldots,a_{n-1})a_n)\}$ $=a_0\mu(g_n(a_1,\ldots\ldots,a_n))+\sum_{k=1}^{n}(-1)^k\mu(g_n(a_0,\ldots\ldots,a_{k-1}a_k,\ldots\ldots,a_n)+(-1)^{n+1}$ $\mu(g_n(a_0,\ldots\ldots,a_{n-1}))\,a_n-\mu(\delta g_n(a_0,\ldots\ldots,a_n))=(\delta h_n)\,(a_0,\ldots\ldots a_n)-\mu(\delta g_n(a_0,\ldots\ldots,a_n))$, Q.E.D.

From Lemma 1 we have $\delta(Fg_n)=0$ for $\delta g_n=0$ $(n\geqq 0)$ and $F(g_n)=\delta F(h_{n-1})$ for $-g_n=\delta h_{n-1}$ $(n\geqq 1)$. Hence we have

Theorem 3. *The mapping $F_{\beta\gamma}$ of (11) induces a linear mapping $F_{\beta\gamma}'$ of $\mathbf{H}_n(\mathbf{R},\mathbf{m})$ into $\mathbf{H}_{n+1}(\mathbf{R},\mathbf{n})$ $(n\geqq 1)$.*

From Lemma 2 we have

Theorem 4. *If factor systems $\{\beta,\gamma\}$ and $\{\beta^*,\gamma^*\}$ are associated, then $F_{\beta\gamma}$ and $F_{\beta^*\gamma^*}$ induce the same mapping $F_{\beta\gamma}'=F_{\beta^*\gamma^*}'$ of $\mathbf{H}_n(\mathbf{R},\mathbf{m})$ into $\mathbf{H}_{n+1}(\mathbf{R},\mathbf{n})$ $(n\geqq 1)$.*

Corollary. *If $\{\beta,\gamma\}$ splits, then $F_{\beta\gamma}'=0$, that is, $F_{\beta\gamma}$ maps $\mathbf{H}_n(\mathbf{R},\mathbf{m})$ into 0 $(n\geqq 1)$.*

Now we show that the fundamental mapping defined by Hochschild [1] is a special case of $F_{\beta\gamma}$. Let $\mathbf{n}$ be any given $\mathbf{R}$-$\mathbf{R}$-module. Then we take as $\mathbf{m}$ the set of all linear mappings m of $\mathbf{R}$ into $\mathbf{n}$ $(\mathbf{m}=\mathbf{L}_1(\mathbf{R},\mathbf{n}))$ with the relations

$$a*m(b)=am(b),\quad (m*a)(b)=m(ab)-m(a)b \qquad (a,b\in\mathbf{R},\ m\in\mathbf{m}),$$

which is also an $\mathbf{R}$-$\mathbf{R}$-module. We take then factor system

$$\beta(a,m)=0,\quad \gamma(m,a)=m(a),$$

which evidently satisfies the condition (3). Then the mapping $F_{\beta\gamma}$ is defined by

$$F_{\beta\gamma}(g_n)(a_0,\ldots\ldots,a_n)=(-1)^{n+1}g_n(a_0,\ldots\ldots,a_{n-1})(a_n). \tag{13}$$

Conversely for any given $f_{n+1}\in\mathbf{L}_{n+1}(\mathbf{R},\mathbf{n})$, take $g_n\in\mathbf{L}_n(\mathbf{R},\mathbf{m})$ with

$$g_n(a_0,\ldots\ldots,a_{n-1})(a_n)=(-1)^{n+1}f_{n+1}(a_0,\ldots\ldots,a_n),$$

then we have $F(g_n)=f_{n+1}$. This shows the isomorphism $\mathbf{L}_n(\mathbf{R},\mathbf{m})\simeq\mathbf{L}_{n+1}(\mathbf{R},\mathbf{n})$ $(n\geqq 0)$. Lemma 1, 2 show also $\mathbf{C}_n(\mathbf{R},\mathbf{m})\simeq\mathbf{C}_{n+1}(\mathbf{R},\mathbf{n})$ $(n\geqq 0)$ and $\mathbf{B}_n(\mathbf{R},\mathbf{m})\simeq\mathbf{B}_{n+1}(\mathbf{R},\mathbf{n})$ $(n\geqq 1)$. Thus we have

Theorem 5. (Hochschild) *For the special* $\mathbf{R}$-$\mathbf{R}$-*module* $\mathbf{m}$ *defined above*

$$\mathbf{H}_n(\mathbf{R},\mathbf{m})\simeq\mathbf{H}_{n+1}(\mathbf{R},\mathbf{n}) \qquad (n\geqq 1).$$

Now take $\mathbf{C}_0(\mathbf{R}, \mathbf{m}) = \{m;\ \delta m=0,\ m\epsilon\mathbf{m}\}$, and $\mathbf{B}_0(\mathbf{R}, \mathbf{m}) = \{F_{\beta\gamma}^{-1}f_1;\ f_1\epsilon\mathbf{B}_1(\mathbf{R}, \mathbf{n})\}$, then $\mathbf{C}_0(\mathbf{R}, \mathbf{m})/\mathbf{B}_0(\mathbf{R}, \mathbf{m}) \cong \mathbf{H}_1(\mathbf{R}, \mathbf{n})$. This isomorphism is given by $F_{\beta\gamma}'$. But if $\{\beta, \gamma\}$ splits, then $F_{\beta\gamma}'=0$, and we have $\mathbf{H}_1(\mathbf{R}, \mathbf{n})=0$. Hence we have the converse of Corollary 1 of Theorem 2.

Theorem 6. *A necessary and sufficient condition for the vanishing of* $\mathbf{H}_1(\mathbf{R}, \mathbf{n})$ *for every* **R-R**-*module* **n** *is that every extension* **M** *of* **m** *by* **n** *splits.*

§ **4.** *3-cohomology group.*

2-cohomology group $\mathbf{H}_2(\mathbf{R}, \mathbf{m})$ was related by Hochschild [1] to the extension **R** by **m** as follows. We call the *extension* $\mathbf{A} = (\mathbf{R}, \mathbf{m})$ *of* **R** *by* **m** the following ring: (i) **A** contains **m** as two sided ideal, (ii) $\mathbf{m}^2=0$, (iii) $\mathbf{A}/\mathbf{m} \cong \mathbf{R}$ (iv) the linear representatives $A_a\epsilon\mathbf{A}$ corresponding to $a\epsilon\mathbf{R}$ by $\mathbf{A}/\mathbf{m} \cong \mathbf{R}$ satisfies $A_a m = am$, $mA_a = ma$ ($a\epsilon\mathbf{R}$, $m\epsilon\mathbf{m}$). The structure of **A** is completely determined by

$$A_aA_b = A_{ab} + g(a, b) \qquad (a, b\epsilon\mathbf{R},\ g\epsilon\mathbf{m}), \tag{14}$$

where g satisfies the condition corresponding to $A_a(A_bA_c) = (A_aA_b)A_c$

$$ag(b, c) + g(a, bc) = g(ab, c) + g(a, b)c. \tag{15}$$

This is, $g\epsilon\mathbf{C}_2(\mathbf{R}, \mathbf{m})$. Conversely, for any given $g\epsilon\mathbf{C}_2(\mathbf{R}, \mathbf{m})$ there exists an extension **A** with this g. We denote this extension **A** by $(\mathbf{R}, \mathbf{m}, g)$. If we take another system of representatives $A_a^* = A_a + h(a)$ ($h\epsilon\mathbf{m}$), the corresponding $g^*(a, b)$ is given by $g^*(a, b) = g(a, b) + \{ah(b) - h(ab) - h(a)b\}$, namely $g^* \equiv g$ (mod. $\mathbf{B}_2(\mathbf{R}, \mathbf{m})$). This shows that the vanishing of $\mathbf{H}_2(\mathbf{R}, \mathbf{m})$ for every **m** means the splitting of all extensions $\mathbf{A} = (\mathbf{R}, \mathbf{m})$.

Now we consider the meaning of $\mathbf{H}_3(\mathbf{R}, \mathbf{m})$ in relation to the Teichmüller's theory of factor systems of higher degree. Let $\mathbf{A} = (\mathbf{R}, \mathbf{m}, g)$ be an extension of **R** by **m**, and let $\mathbf{M} = (\mathbf{m}, \mathbf{n}, \beta, \gamma)$ be an extension of **m** by another **R-R**-module **n**. We shall consider the problem to construct an extension $\mathbf{B} = (\mathbf{R}, \mathbf{M})$, for which $\mathbf{B}/\mathbf{n} \cong \mathbf{A}$ holds.

Suppose that we have such an extension. We take linear representatives $A_a\epsilon\mathbf{B}$ ($a\epsilon\mathbf{R}$), $B_m\epsilon\mathbf{B}$ ($m\epsilon\mathbf{m}$) corresponding to $\mathbf{B}/\mathbf{M} \cong \mathbf{R}$ and $\mathbf{M}/\mathbf{n} \cong \mathbf{m}$, then in **B** hold the relations

$$\begin{cases} \lambda\mu = 0,\ B_m\lambda = \lambda B_m = 0,\ (\lambda, \mu\epsilon\mathbf{n});\ A_a u = au,\ uA_a = ua\ (u\epsilon\mathbf{n}), \\ A_aB_m = B_{am} + \beta(a, m),\ B_mA_a = B_{ma} + \gamma(m, a),\ (\beta, \gamma\epsilon\mathbf{n}), \\ A_aA_b = A_{ab} + B_{g(a,b)} + \alpha(a, b),\ (\alpha\epsilon\mathbf{n}). \end{cases} \tag{16}$$

By the associative law we have the conditions (3) for β, γ and from $A_a(A_bA_c)=(A_aA_b)A_c$

$$\beta(a, g(b, c))-\gamma(g(a, b), c)+\{a\alpha(b, c)-\alpha(ab, c)+\alpha(a, bc)-\alpha(a, b)c\}=0, \quad (17)$$

that is.

$$F_{\beta\gamma}(g_2)+\delta\alpha_2=0. \quad (17^*)$$

Conversely, if we have $\alpha(a, b)\in \mathbf{n}$ with (17) or (17*), then we can construct an extension $\mathbf{B}=(\mathbf{R}, \mathbf{M})$ by (16) with the desired properties. Thus we have

Theorem 7. *For any given extensions* $\mathbf{A}=(\mathbf{R}, \mathbf{m}, g)$ *and* $\mathbf{M}=(\mathbf{m}, \mathbf{n}, \beta, \gamma)$ *the necessary and sufficient condition for the existence of another extension* $\mathbf{B}(\mathbf{R}, \mathbf{M})$ *with* $\mathbf{B}/\mathbf{n}\cong\mathbf{A}$ *is that 3-cocycle* $F_{\beta\gamma}(g)$ *is a coboundary.*

If $\mathbf{H}_3(\mathbf{R}, \mathbf{n})=0$, *then there is always such an extension.*

Now for any given **R**-**R**-module **n** take **m** and β, γ as in Theorem 5. And let f_3 be any 3-cocycle which is not a 3-coboundary. Then there is g_2 such that $F_{\beta\gamma}(g_2)=f_3$. For such $\mathbf{A}=(\mathbf{R}, \mathbf{m}, g_2)$ and $\mathbf{M}=(\mathbf{m}, \mathbf{n}, \beta, \gamma)$ we cannot construct the desired extension. Hence we have

Theorem 8. *A necessary and sufficient condition for the vanishing of* $\mathbf{H}_3(\mathbf{R}, \mathbf{n})$ *for any* **R**-**R**-*module* **n** *is the possibility of an extension* $\mathbf{B}=(\mathbf{R}, \mathbf{m})$ *with* $\mathbf{B}/\mathbf{n}\cong\mathbf{A}$ *for any* $\mathbf{A}=(\mathbf{R}, \mathbf{m}, g)$ *and* $\mathbf{M}=(\mathbf{m}, \mathbf{n}, \beta, \gamma)$.

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