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Note on faithful modular representations of a finite group.

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In a recent note[1] the writer has studied the structure of finite groups possessing a faithful irreducible representation (i.r.), directly indecomposable representation (d.i.r.), or a faithful directly indecomposable component (d.i.c.) of the regular representation (r.r.) in a modular field of characteristic $p \neq 0$ (p-modular field). The result is similar to the case of groups with faithful non-modular i.r.[2] Namely: Let $\mathfrak{M}$ be the product of the totality of minimal abelian invariant subgroups of order prime to p in a finite group $\mathfrak{G}$, and let $\mathfrak{M} = \mathfrak{L}_1 \times \mathfrak{L}_2 \times \dots\dots \times \mathfrak{L}_g$ be its decomposition into subgroups of prime power orders with different primes $l_i (\neq p)$. *$\mathfrak{G}$ possesses a faithful p-modular i.r., if and only if $\mathfrak{G}$ has no invariant subgroup $\neq 1$ whose order is a power of p and moreover the following condition is satisfied:*

(*) *every $\mathfrak{L}_i$ possesses an invariant subgroup with cyclic factor group which contains no invariant subgroup $\neq 1$ of $\mathfrak{G}$.*

$\mathfrak{G}$ has a faithful d.i.c. of p-modular r.r. (or a faithful p-modular d.i.r. whatsoever), if and only if the condition (*) is satisfied.

(Furthermore, (*) is equivalent to that

(†) each $\mathfrak{L} = \mathfrak{L}_i$ is a product of c, say, mutually $\mathfrak{G}$-isomorphic minimal invariant subgroups of $\mathfrak{G}$ and the inequality $c \leq m/\lambda$ is satisfied, where l^m is the order of the minimal factor and l^λ is the number of elements in the $\mathfrak{G}$-automorphism quasifield of the minimal factor.)

As a corollary of the result we have: 1) *If $\mathfrak{G}$ has a faithful non-modular i.r. then it has a faithful d.i.c. of p-modular r.r. (for any p)*; 2) *If $\mathfrak{G}$ possesses faithful p-modular and q-modular d.i.r. with distinct p, q, then it has a faithful non-modular i.r.*

The present note is to supplement these by giving mutual relations between such modular and non-modular representations. We prove namely[3]:

I. *If a group[4] $\mathfrak{G}$ possesses a faithful non-modular i.r.[5] $M(\mathfrak{G})$, then any d.i.c. $V(\mathfrak{G})$ of a modular r.r. containing $M(\mathfrak{G})$, in the sense of R. Brauer-C. Nesbitt[6], is faithful.*

II. *If $\mathfrak{G}$ possesses a faithful non-modular i.r.[7], then an arbitrary faithful d.i.c. of a modular r.r. contains a faithful non-modular i.r.*

As to I we have somewhat stronger

I′. *If* $M(\mathfrak{G})$ *is a non-modular i.r. of a group* $\mathfrak{G}$, *and if* $V(\mathfrak{G})$ *is a d.i.c. of a modular r.r. of* $\mathfrak{G}$ *containing* $M(\mathfrak{G})$, *then the kernel of* $M(\mathfrak{G})$ *contains that of* $V(\mathfrak{G})$.

Proof is immediate, though the fact is not perfectly trivial, as it seems to the writer. Let namely $\mathfrak{K}$ be the kernel of $V(\mathfrak{G})$. The restricted representation $V(\mathfrak{K})$ is decomposed, directly, into a number of d.i.c. of r.r. of $\mathfrak{K}$, as it is the case with any subgroup. These d.i.c. are of course all 1-representation (=unit representation of degree 1). On the other hand, we may see easily that each irreducible constituents of the restriction $M(\mathfrak{K})$ is contained in some d.i.c. of $V(\mathfrak{K})$[8]. So the (non-modular) irreducible constituents of $M(\mathfrak{K})$ are all 1-representation, and the completely reducible representation $M(\mathfrak{K})$ itself is a unit representation. This shows that the kernel of $M(\mathfrak{G})$ contains $\mathfrak{K}$.

(We may also argue in the following manner by showing first that *the kernel* $\mathfrak{K}$ *of* $V(\mathfrak{G})$, *a d.i.c. of* p*-modular r.r., has an order prime to* p, which is perhaps of interest by itself. Namely, $\mathfrak{K}$ has a d.i.c. of p-modular r.r. which is 1-representation, as was seen by restricting $V(\mathfrak{G})$ on $\mathfrak{K}$. Such a group has an order prime to p, in virtue of a theorem of Brauer-Nesbitt concerning the first Cartan invariant[9]. Now the modular irreducible constituents of $M(\mathfrak{K})$ are all 1-representation. Since $\mathfrak{K}$ has an order prime to p, also the non-modular irreducible constituents of $M(\mathfrak{K})$ are 1-representation, and $M(\mathfrak{K})$ is a unit representation.)

Now we turn to II. Let $\mathfrak{N}$ be the product of all the minimal invariant subgroups of $\mathfrak{G}$, which is well known to be completely reducible by itself, and let

$$\mathfrak{N}=\mathfrak{H}_1\times\mathfrak{H}_2\times\cdots\cdots\times\mathfrak{H}_n\times\mathfrak{P}$$

where $\mathfrak{H}_1, \mathfrak{H}_2, \cdots\cdots, \mathfrak{H}_n$ are ideal factors with respect to $\mathfrak{G}$ and are not p-group, while $\mathfrak{P}$ is a such. We assume that $\mathfrak{G}$ possesses a faithful d.i.c. $V(\mathfrak{G})$ of p-modular r.r. Then each $\mathfrak{H}_i$ has an invariant subgroup $\mathfrak{K}_i$ with simple factor group and containing no invariant subgroup $\neq 1$ of $\mathfrak{G}$[10]. Let $\mathfrak{H}_i=\mathfrak{K}_i\times\mathfrak{S}_i$. The restricted representation $V(\mathfrak{N})$ is decomposed into a certain number of d.i.c. $v^{(\nu)}(\mathfrak{N})$ of r.r. of $\mathfrak{N}$. At least one of $v^{(\nu)}$ is faithful on (the simple group) $\mathfrak{S}_i$. Since $v^{(\nu)}$'s are $\mathfrak{G}$-conjugate[11], we may assume, perhaps by taking suitable $\mathfrak{G}$-conjugates of $\mathfrak{S}_i$, $\mathfrak{K}_i$, that $v(\mathfrak{S}_i)=v^{(1)}(\mathfrak{S}_i)$ is faithful. Let

$z_i(\mathfrak{S}_i)$, $w_i(\mathfrak{S}_i)$ be respectively the $\mathfrak{S}_i$-, $\mathfrak{K}_i$-component of $v(\mathfrak{N})$ with respect to the decomposition

$$\mathfrak{N}=\mathfrak{S}_1\times\mathfrak{K}_1\times\ldots\ldots\times\mathfrak{S}_n\times\mathfrak{K}_n\times\mathfrak{P}.$$

They are d.i.c. of p-modular r.r. of $\mathfrak{S}_i$, $\mathfrak{K}_i$ respectively. We assert that at least one non-modular irreducible constituent of $z_i(\mathfrak{S}_i)$ is not 1-representation (whence faithful). For, otherwise all the modular irreducible constituents are also 1-representation, and therefore $\mathfrak{S}_i$ has an invariant subgroup whose order is prime to p and whose index is a power of p, by a known result concerning primary decomposable group rings[12] which follows again from the theorem on the first Cartan invariant. Since $\mathfrak{S}_i$ is simple and is not a p-group, it has then necessarily an order prime to p. But then $z_i(\mathfrak{S}_i)$ itself is irreducible, and its unique non-modular component cannot be 1-representation. The contradiction proves our assertion. Let $s_i(\mathfrak{S}_i)$ be, for each i, such a faithful non-modular irreducible constituent of $z_i(\mathfrak{S}_i)$.

We assume now that $\mathfrak{G}$ has a faithful non-modular i.r. Then $\mathfrak{P}$ has a non-modular i.r. $s(\mathfrak{P})$ whose kernel contains no invariant subgroup $\neq 1$ of $\mathfrak{G}$. On taking for each i an arbitrary non-modular irreducibl constituent $r_i(\mathfrak{K}_i)$ of $w_i(\mathfrak{K}_i)$, we construct the Kronecker product

$$m(\mathfrak{N})=s_1(\mathfrak{S}_1)\times r_1(\mathfrak{K}_1)\times\ldots\ldots\times s_n(\mathfrak{S}_n)\times r_n(\mathfrak{K}_n)\times s(\mathfrak{P}).$$

It is of course irreducible, and its kernel contains no invariant subgroup $\neq 1$ of $\mathfrak{G}$, since this is contained in $\mathfrak{K}_1\times\ldots\ldots\times\mathfrak{K}_n\times$ (kernel of $s(\mathfrak{P})$). Moreover, it is contained in the d.i.c. $v(\mathfrak{N})$, because $s_i(\mathfrak{S})$, $r_i(\mathfrak{K})$ are contained respectively in $z_i(\mathfrak{S}_i)$, $w_i(\mathfrak{K}_i)$ and $s(\mathfrak{P})$ is contained in the $\mathfrak{P}$-component of $v(\mathfrak{N})$ which is nothing but the p-modular r.r. of $\mathfrak{P}$.

Let $f(\mathfrak{N})$, $F(\mathfrak{G})$ be the modular i.r. belonging to $v(\mathfrak{N})$, $V(\mathfrak{G})$ respectively. By a special case of the theorem of induced representations[13], the representation of $\mathfrak{G}$ induced by $f(\mathfrak{N})$ contains $F(\mathfrak{G})$ as an irreducible constituent. On the other hand, $f(\mathfrak{N})$ is contained in $m(\mathfrak{N})$, in virtue of the main theorem of modular representations. Hence $F(\mathfrak{G})$ is contained in some (non-modular) irreducible constituent $M(\mathfrak{G})$ of the representation of $\mathfrak{G}$ induced by $m(\mathfrak{N})$. Then $M(\mathfrak{G})$ is contained in $V(\mathfrak{G})$[14].

The restriction $M(\mathfrak{N})$ is decomposed into a number of mutually $\mathfrak{G}$-conjugate i.r., one of which is $m(\mathfrak{N})$. Because of our property of $m(\mathfrak{N})$, $M(\mathfrak{N})$ is faithful. Since $\mathfrak{N}$ contains all the minimal invariant subgroups of

$\mathfrak{G}$, $M(\mathfrak{G})$ itself is then so too, and II is settled.

(To modify II in a similar manner as I′ is rather meaningless. For, if an invariant subgroup $\mathfrak{A}$ is contained in the kernel of a d.i.c. $V(\mathfrak{G})$ of r.r., then $V(\mathfrak{G})$ is essential a d.i.c. of the r.r. of the factor group $\mathfrak{G}/\mathfrak{A}$, as we may see readily again by the theorem of induced representations for instance.)

A similar study concerning representations with a certain number of direct components[15] is immediate, and in fact can easily be reduced to the above.

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Notes.

1) Finite groups with faithful irreducible and directly indecomposable modular representations, forthcoming in Proc. Imp. Acad. Tokyo.

2) K. Shoda, Über direkt zerlegbare Gruppen, Journ. Fac. Sci. Tokyo Imp. Univ. I, Vol. II—3 (1930); See also Vol. II—7 (1931). (*Added in proof* (August 14, 1948): Cf. also L. Weisner, Amer. J. Math. 61).

3) The following counter part is trivial: If $\mathfrak{G}$ possesses a faithful modular i.r. then any i.r. containing it is faithful.

4) Groups are all supposed to be finite.

5) Representations are considered, for the sake of simplicity, in algebraically closed fields.

6) R. Brauer, On modular and $\mathfrak{p}$-adic representations of algebras, Proc. Nat. Acad. Sci. 25, No. 5 (1939).

7) That is, if (*) and the same condition for the maximal abelian invariant p-subgroup are satisfied.

8) Take, for instance, $V(\mathfrak{G})$ in a form such that $V(\mathfrak{K})$ is actually decomposed in the said manner. There exists a (non-modular) $\mathfrak{p}$-adic representation which reduces itself to this $V(\mathfrak{G})$ when considered mod. $\mathfrak{p}$ and whose restriction on $\mathfrak{K}$ is decomposed in the manner corresponding to the decomposition of $V(\mathfrak{K})$ not only mod. $\mathfrak{p}$ but by itself (Cf. Brauer, l.c. 6)).

9) R. Brauer-C. Nesbitt, On the modular characters of groups, Ann. Math. 42 (1941), § 28.

10) Cf. Nakayama, l.c. 1); non-abelian factors are taken care of automatically.

11) Cf. Nakayama, l.c. 1).

12) Brauer-Nesbitt, l.c. 9); M. Osima, On primary decomposable group rings, Proc. Phys-Math. Soc. Japan 24 (1942).

13) Nakayama, Some studies on regular representations, induced representations and modular representations, Ann. Math. 39 (1938), § 3; Brauer-Nesbitt, l.c. 9), § 26.

14) The present consideration remains valid in the case of an arbitrary subgroup, not necessarily invariant, and we have: *Let $V(\mathfrak{G})$ be a d.i.c. of modular r.r. of $\mathfrak{G}$, and $v(\mathfrak{H})$ a d.i.c. of its restriction on a subgroup $\mathfrak{H}$. If a non-modular i.r. $m(\mathfrak{H})$ is contained in $v(\mathfrak{H})$, then some irreducible constituent of the representation of $\mathfrak{G}$ induced by $m(\mathfrak{H})$ is contained in $V(\mathfrak{G})$.* More accurate statement can be made in terms of so-called decomposition numbers.

15) M. Tazawa, Über die isomorphe Darstellung der endlichen Gruppe, Tohoku Math. J. 47 (1940); Nakayama, l.c. 1).