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EXPONENTIALLY WEIGHTED POLYNOMIAL APPROXIMATION FOR ABSOLUTELY CONTINUOUS FUNCTIONS

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Abstract. We discuss a polynomial approximation on $\mathbb{R}$ with a weight w in $\mathcal{F}(C^2+)$ (see Section 2). The de la Vallée Poussin mean $v_n(f)$ of an absolutely continuous function f is not only a good approximation polynomial of f, but also its derivatives give an approximation for the derivative f'. More precisely, for $1 \le p \le \infty$, we have $\lim_{n\to\infty} \|(f - v_n(f))w\|_{L^p(\mathbb{R})} = 0$ and $\lim_{n\to\infty} \|(f' - v_n(f)')w\|_{L^p(\mathbb{R})} = 0$ whenever $f''w \in L^p(\mathbb{R})$.

1. Introduction.

Let $\mathbb{R} = (-\infty, \infty)$. We consider an exponential weight

$$w(x) = \exp(-Q(x))$$

on $\mathbb{R}$, where Q is an even and nonnegative function on $\mathbb{R}$. Throughout this paper we always assume that w belongs to a relevant class $\mathcal{F}(C^2+)$. A function $T = T_w$ defined by

$$T(x) := \frac{xQ'(x)}{Q(x)}, \ x \neq 0 \tag{1.1}$$

is very important. We call w a Freud-type weight if T is bounded, and otherwise, w is called an Erdös-type weight.

For $x > 0$, the Mhaskar-Rakhmanov-Saff number (MRS number) $a_x = a_x(w)$ of $w = \exp(-Q)$ is defined by a positive root of the equation

$$x = \frac{2}{\pi}\int_0^1 \frac{a_x u Q'(a_x u)}{(1-u^2)^{1/2}} du\,. \tag{1.2}$$

When $w = \exp(-Q) \in \mathcal{F}(C^2+)$, Q' is positive and increasing on $(0, \infty)$, so that

$$\lim_{x\to\infty} a_x = \infty \ \text{ and } \ \lim_{x\to\infty} \frac{a_x}{x} = 0 \tag{1.3}$$

hold. Note that those convergences are all monotonically.

For $1 \le p \le \infty$, we denote by $L^p(\mathbb{R})$ the usual L^p space on $\mathbb{R}$. For $fw \in L^p(\mathbb{R})$ and $n \in \mathbb{N}$, the degree of weighted polynomial approximation $E_{p,n}(w; f)$ is defined by

$$E_{p,n}(w; f) := \inf_{P\in\mathcal{P}_n} \|(f - P)w\|_{L^p(\mathbb{R})}\,, \tag{1.4}$$

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where $\mathcal{P}_n$ is the class of all polynomials of degree not more than n. It is known that when $w \in \mathcal{F}(C^2+)$, then

$$\lim_{n\to\infty} E_{p,n}(w; f) = 0 \tag{1.5}$$

for every $fw \in L^p(\mathbb{R})$ (if $p = \infty$ we further assume that $fw \in C_0(\mathbb{R})$, i.e., it is continuous and vanishes at infty) (e.g., [4, Theorems 1.4 and 1.6]). Moreover, we can find a constant $C \geq 1$ such that

$$E_{p,n}(w; f) \leq C\frac{a_n}{n}E_{p,n-1}(w; f') \tag{1.6}$$

holds whenever f is absolutely continuous and $f'w \in L^p(\mathbb{R})$(see [6, Theorem 1]).

In [1, Corollary 14], we proved that if $w \in \mathcal{F}(C^2+)$ satisfies

$$T(a_n) \leq C_1\left(\frac{n}{a_n}\right)^{2/3} \tag{1.7}$$

with some constant $C_1 \geq 1$, then there exits a constant $C \geq 1$ such that

$$\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \leq CT^{1/4}(a_n)E_{p,n}(w; f) \tag{1.8}$$

holds for every $fw \in L^p(\mathbb{R})$, here $v_n(f)$ is the de la Vallée Poussin mean of f. Note that if

$$\frac{|Q'(x)|}{Q^\lambda(x)} \leq C \quad (|x| \geq 1)$$

holds for some $0 < \lambda < 2$ and $C \geq 1$, then (1.7) holds (see [7, Remark 16]).

In the present paper, we discuss a derivative version of (1.8). The following theorem is established.

THEOREM 1.1. *Let $w \in \mathcal{F}(C^2+)$ satisfy (1.7). Then there exists a constant $C \geq 1$ such that for every $1 \leq p \leq \infty$, $n \in \mathbb{N}$ and every absolutely continuous function f,*

$$\|(f' - v_n(f)')w\|_{L^p(\mathbb{R})} \leq CT^{3/4}(a_n)E_{p,n-1}(w; f') \tag{1.9}$$

holds whenever $f'w \in L^p(\mathbb{R})$.

By (1.3), (1.6) and (1.7), we see

$$T^{1/4}(a_n)E_{p,n}(w; f) \leq C\left(\frac{a_n}{n}\right)^{5/6}E_{p,n-1}(w; f') \;\to\; 0 \;(n \to \infty)$$

whenever $f'w \in L^p(\mathbb{R})$, and similarly

$$\lim_{n\to\infty} T^{3/4}(a_n)E_{p,n-1}(w; f') \leq \lim_{n\to\infty} C\left(\frac{a_n}{n}\right)^{1/2}E_{p,n-2}(w; f'') = 0$$

whenever $f''w \in L^p(\mathbb{R})$. Hence Theorem 1.1 means that if $f''w \in L^p(\mathbb{R})$, then the de la Vallée Poussin mean of f is not only a good approximation polynomial for f, but also its derivatives give an approximation for the derivative f'. Note also that if w is Freud-type, then (1.7) is always true and

$$\|(f' - v_n(f)')w\|_{L^p(\mathbb{R})} \leq CE_{p,n-1}(w; f') \tag{1.10}$$

holds. Theorem 1.1 is derived from the following estimate.

THEOREM 1.2. *Let $w \in \mathcal{F}(C^2+)$ be Erdös-type and assume* (1.7). *Then there exists a constant $C \geq 1$ such that for every absolutely continuous function f with $f'w \in L^p(\mathbb{R})$, if $\|(f-P)w\|_{L^p(\mathbb{R})} \leq \varepsilon$ holds for $n \in \mathbb{N}$, $P \in \mathcal{P}_{2n}$ and $\varepsilon > 0$, then*

$$\|(f'-P')w\|_{L^p(\mathbb{R})} \leq C\left(T^{3/4}(a_n)E_{p,n}(w;f') + \frac{n}{a_n}T^{1/2}(a_n)\varepsilon\right) \tag{1.11}$$

also holds.

H. N. Mhaskar discussed the above assertion for Freud-type weight ([5, Chapter 4]). Our proof is also done along his methods, however analyses for Erdös-type weights are necessary. The Favard type inequality (1.6) plays an important role in our argument, which was shown in our previous paper [6].

This paper is organized as follows. In Section 2, we give the definition of class $\mathcal{F}(C^2+)$ and recall some notations. Basic facts for MRS numbers are also provided. In Section 3, we prepare some propositions, which we use in the proof of our theorem. The main result (1.11) of this paper is shown in Section 4. Theorem 1.1 is its corollary. An estimate of higher order derivative is also discussed here.

Throughout this paper, C denote a positive constant which is independent of p, n, f and P. The value C is not necessary the same at each occurrence; it may vary within a line. For nonnegative functions f and g on a subset $I \subset \mathbb{R}$, we write $f(x) \sim g(x)$ for I if $C^{-1}g(x) \leq f(x) \leq Cg(x)$ holds for all $x \in I$. Similarly, for two positive sequences $\{c_n\}_{n=1}^\infty$ and $\{d_n\}_{n=1}^\infty$, we write $c_n \sim d_n$ if $C^{-1}c_n \leq d_n \leq Cc_n$ holds for all n.

2. Preliminary Results. We say that an exponential weight $w = \exp(-Q)$ belongs to class $\mathcal{F}(C^2+)$, when $Q : \mathbb{R} \to [0,\infty)$ is a continuous and even function and satisfies the following conditions:

(a) $Q'(x)$ is continuous in $\mathbb{R}$ and $Q(0) = 0$.
(b) $Q''(x)$ exists and is positive in $\mathbb{R} \setminus \{0\}$.
(c) $\lim_{x\to\infty} Q(x) = \infty$.
(d) The function

$$T(x) := \frac{xQ'(x)}{Q(x)}, \quad x \neq 0 \tag{2.1}$$

is quasi-increasing in $(0,\infty)$(i.e., there exists $C > 1$ such that $T(x) \leq CT(y)$ whenever $0 < x < y$), and there exists $\Lambda \in \mathbb{R}$ such that

$$T(x) \geq \Lambda > 1, \quad x \in \mathbb{R} \setminus \{0\}.$$

(e) There exists $C > 1$ such that

$$\frac{Q''(x)}{|Q'(x)|} \leq C\frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R}$$

and also there exist a compact subinterval $J(\ni 0)$ of $\mathbb{R}$, and $C > 1$ such that

$$C\frac{Q''(x)}{|Q'(x)|} \geq \frac{|Q'(x)|}{Q(x)}, \quad \text{a.e. } x \in \mathbb{R} \setminus J.$$

A typical example of Freud-type weight is $w(x) = \exp(-|x|^{\alpha})$ with $\alpha > 1$. For $u \geq 0$, $\alpha > 0$ with $\alpha + u > 1$ and $l \in \mathbb{N}$, we set

$$Q(x) := |x|^{u}(\exp_l(|x|^{\alpha}) - \exp_l(0)),$$

where

$$\exp_l(x) := \exp(\exp(\exp(\cdots(\exp x))))(l-\text{times}).$$

Then $w(x) := \exp(-Q(x))$ is an Erdös-type weight, which satisfies (1.7).

Let $\{a_t\}$ be MRS numbers for $w \in \mathcal{F}(C^2+)$. We use the following estimate frequently (cf. [3, Lemma 3.5 (3.27), (3.28)]).

LEMMA 2.1. *For $t > 0$,*

$$a_{2t} \sim a_t \tag{2.2}$$

and

$$T(a_{2t}) \sim T(a_t). \tag{2.3}$$

Next we recall the de la Vallée Poussin mean $v_n(f)$ of a function f with $fw \in L^p(\mathbb{R})$. Let $\{p_n\}$ be orthogonal polynomials for a weight $w \in \mathcal{F}(C^2+)$, that is, p_n is the polynomial of degree n such that

$$\int_{\mathbb{R}} p_n(x)p_m(x)w^2(x)dx = \delta_{mn}.$$

We set

$$s_n(f)(x) := \sum_{k=0}^{n-1} b_k(f)p_k(x) \text{ where } b_k(f) := \int_{\mathbb{R}} f(t)p_k(t)w^2(t)dt \tag{2.4}$$

for $n \in \mathbb{N}$ (the partial sum of Fourier series). Then $v_n(f)$ is given by

$$v_n(f)(x) := \frac{1}{n}\sum_{j=n+1}^{2n} s_j(f)(x). \tag{2.5}$$

We see the following lemma by a direct computation.

LEMMA 2.2. *For every $P \in \mathcal{P}_n$ and $f \in L^p(\mathbb{R})$, we have*

$$v_n(P)(x) = P(x) \tag{2.6}$$

and

$$\int_{\mathbb{R}} (f(t) - v_n(f)(t))P(t)w^2(t)dt = 0. \tag{2.7}$$

3. Propositions. In this section we prove some propositions which need in next section. Let $w \in \mathcal{F}(C^2+)$ and $1 \leq p \leq \infty$. We use q as the conjugate exponent to p, that is, $1/p + 1/q = 1$ holds, and if $p = \infty$ then $q = 1$ and $p = 1$ then $q = \infty$.

The first proposition may be known for $1 \leq p < \infty$ (cf. [5, Appendix A.1.1]), however we give a proof for $1 \leq p \leq \infty$, for the sake of completeness.

PROPOSITION 3.1. *Let $fw \in L^p(\mathbb{R})$, then*

$$E_{p,0}(w; f) = \sup_{\substack{\|gw\|_{L^q(\mathbb{R})} \le 1 \\ \int_{\mathbb{R}} g(t)w^2(t)dt = 0}} \int_{\mathbb{R}} f(t)g(t)w^2(t)dt \,. \tag{3.1}$$

PROOF. Let $fw \in L^p(\mathbb{R})$. If f is a constant function, (3.1) holds clearly, so we may assume that f is not constant. Then for any $a \in \mathbb{R}$ and $gw \in L^q(\mathbb{R})$ with $\|gw\|_{L^q(\mathbb{R})} \le 1$ and $\int_{\mathbb{R}} g(t)w^2(t)dt = 0$, we have

$$\begin{aligned}\left|\int_{\mathbb{R}} f(t)g(t)w^2(t)dt\right| &= \left|\int_{\mathbb{R}} (f(t)-a)g(t)w^2(t)dt\right| \\ &\le \|(f-a)w\|_{L^p(\mathbb{R})}\, \|gw\|_{L^q(\mathbb{R})} \le \|(f-a)w\|_{L^p(\mathbb{R})}\,,\end{aligned}$$

which implies

$$E_{p,0}(w; f) \ge \sup_{\substack{\|gw\|_{L^q(\mathbb{R})} \le 1 \\ \int_{\mathbb{R}} g(t)w^2(t)dt = 0}} \int_{\mathbb{R}} f(t)g(t)w^2(t)dt \,. \tag{3.2}$$

For the opposite direction, we first consider the case of $1 < p < \infty$. For $c \in \mathbb{R}$, we put

$$I(c) : = \int_{\mathbb{R}} \operatorname{sign}(f(t)-c)|f(t)-c|^{\frac{p}{q}} w^{\frac{p}{q}+1}(t)dt\,,$$

where $\operatorname{sign}(A) = 1$ if $A > 0$, $\operatorname{sign}(A) = -1$ if $A < 0$ and $\operatorname{sign}(A) = 0$ if $A = 0$. Since $I(c)$ is continuous for $c \in \mathbb{R}$ and $\lim_{c\to\infty} I(c) = -\infty$, $\lim_{c\to-\infty} I(c) = \infty$, we can find a constant c_0 such that $I(c_0) = 0$. Then the function g defined by

$$g(t) := \frac{\operatorname{sign}(f(t)-c_0)|f(t)-c_0|^{\frac{p}{q}} w^{\frac{p}{q}-1}(t)}{\|(f-c_0)w\|_{L^p(\mathbb{R})}^{\frac{p}{q}}}$$

satisfies $\|gw\|_{L^q(\mathbb{R})} = 1$, $\int_{\mathbb{R}} g(t)w^2(t)dt = I(c_0)/\|(f-c_0)w\|_{L^p(\mathbb{R})}^{\frac{p}{q}} = 0$ and

$$\int_{\mathbb{R}} f(t)g(t)w^2(t)dt = \int_{\mathbb{R}} (f(t)-c_0)g(t)w^2(t)dt = \|(f-c_0)w\|_{L^p(\mathbb{R})}.$$

Hence

$$E_{p,0}(w; f) \le \sup_{\substack{\|gw\|_{L^q(\mathbb{R})} \le 1 \\ \int_{\mathbb{R}} g(t)w^2(t)dt = 0}} \int_{\mathbb{R}} f(t)g(t)w^2(t)dt \,. \tag{3.3}$$

Next we discuss the case $p = 1$. For $c \in \mathbb{R}$, we put

$$I(c) := \int_{\mathbb{R}} (\operatorname{sign}(f(t)-c))w(t)dt$$

and let

$$c_1 := \sup\{c; I(c) > 0\}, \quad c_2 := \inf\{c; I(c) < 0\}\,.$$

Note that $-\infty < c_1 \le c_2 < \infty$. If $c_1 < c_2$, we put $c_0 := (c_1 + c_2)/2$. Then $c_1 < c_0$ implies $I(c_0) \le 0$ and $c_0 < c_2$ implies $I(c_0) \ge 0$, that is, $I(c_0) = 0$. In this case, if we put $g(t) := (\operatorname{sign}(f(t) - c_0))/w(t)$, then $\|gw\|_{L^\infty(\mathbb{R})} = 1$, $\int_{\mathbb{R}} g(t)w^2(t)dt = I(c_0) = 0$ and $\int_{\mathbb{R}} f(t)g(t)w^2(t)dt = \|(f - c_0)w\|_{L^1(\mathbb{R})}$ hold, which shows

$$E_{1,0}(w; f) \le \sup_{\substack{\|gw\|_{L^\infty(\mathbb{R})} \le 1 \\ \int_{\mathbb{R}} g(t)w^2(t)dt = 0}} \int_{\mathbb{R}} f(t)g(t)w^2(t)dt\,. \tag{3.4}$$

On the other hand, if $c_1 = c_2 = c_0$, we put

$$\lim_{c \to c_0 - 0} I(c) = A, \quad \lim_{c \to c_0 + 0} I(c) = B$$

and

$$\begin{aligned} E_+ &:= \{x \in \mathbb{R};\ f(x) - c_0 > 0\}\,, \\ E_- &:= \{x \in \mathbb{R};\ f(x) - c_0 < 0\}\,, \\ E_0 &:= \{x \in \mathbb{R};\ f(x) = c_0\}\,. \end{aligned}$$

We may suppose $B < A$. Since

$$\begin{aligned} A &= \lim_{c \to c_0 - 0} \left(\int_{\{f(t) > c\}} w(t)dt - \int_{\{f(t) < c\}} w(t)dt \right) \\ &= \lim_{c \to c_0 - 0} \left(\int_{\{c_0 > f(t) > c\}} w(t)dt + \int_{E_+ \cup E_0} w(t)dt - \int_{\{f(t) < c\}} w(t)dt \right) \\ &= \int_{E_+ \cup E_0} w(t)dt - \int_{E_-} w(t)dt \end{aligned}$$

and $B = \int_{E_+} w(t)dt - \int_{E_0 \cup E_-} w(t)dt$, we see $A - B = 2\int_{E_0} w(t)dt > 0$. We consider the case of $A \le \int_{E_0} w(t)dt$ (the case of $-B \le \int_{E_0} w(t)dt$ is similar). We put

$$g(t) := \frac{1}{w(t)} \times \begin{cases} 1 & t \in E_+ \\ 1 - A/(\int_{E_0} w(t)dt) & t \in E_0 \\ -1 & t \in E_- \end{cases}.$$

Then $\|gw\|_{L^\infty(\mathbb{R})} = 1$, $\int_{\mathbb{R}} g(t)w^2(t)dt = 0$ and

$$\int_{\mathbb{R}} f(t)g(t)w^2(t)dt = \int_{\mathbb{R}} (f(t) - c_0)g(t)w^2(t)dt = \|(f - c_0)w\|_{L^1(\mathbb{R})}$$

hold, and hence (3.4) follows.

Finally, we show the case of $p = \infty$. For $fw \in L^\infty(\mathbb{R})$, we put

$$I(c) := \operatorname{ess\,inf}(f - c)w + \operatorname{ess\,sup}(f - c)w\,.$$

As in the previous case, there exists a constant c_0 such that $I(c_0) = 0$. For any small $\varepsilon > 0$, let

$$F_+ := \left\{x \in \mathbb{R};\ (f(x) - c_0)w(x) > \|(f - c_0)w\|_{L^\infty(\mathbb{R})} - \varepsilon\right\}\,,$$

$$F_- := \left\{x \in \mathbb{R};\, (f(x) - c_0)w(x) < -\|(f - c_0)w\|_{L^\infty(\mathbb{R})} + \varepsilon\right\} .$$

Then F_+ and F_- are disjoint and

$$A := \int_{\mathbb{R}} \chi_{F_+}(t)w^2(t)dt > 0, \quad B := \int_{\mathbb{R}} \chi_{F_-}(t)w^2(t)dt > 0$$

hold, where χ_F is the characteristic function of a subset F of $\mathbb{R}$. We define

$$C := \frac{1}{2A}\int_{\mathbb{R}} \chi_{F_+}(t)w(t)dt + \frac{1}{2B}\int_{\mathbb{R}} \chi_{F_-}(t)w(t)dt$$

and

$$g(t) := \frac{1}{C}\left(\frac{1}{2A}\chi_{F+}(t) - \frac{1}{2B}\chi_{F_-}(t)\right) .$$

Then by direct computation, we see $\|gw\|_{L^1(\mathbb{R})} = 1$, $\int_{\mathbb{R}} g(t)w^2(t)dt = 0$ and

$$\begin{aligned}
\int_{\mathbb{R}} f(t)g(t)w^2(t)dt &= \int_{\mathbb{R}} (f(t) - c_0)g(t)w^2(t)dt \\
&= \frac{1}{C}\int_{F_+} \frac{(f(t) - c_0)}{2A}w^2(t)dt + \frac{1}{C}\int_{F_-} \frac{-(f(t) - c_0)}{2B}w^2(t)dt \\
&\geq \frac{1}{C}(\|(f - c_0)w\|_{L^\infty(\mathbb{R})} - \varepsilon)\left(\frac{1}{2A}\int_{F_+} w(t)dt + \frac{1}{2B}\int_{F_-} w(t)dt\right) \\
&= \|(f - c_0)w\|_{L^\infty(\mathbb{R})} - \varepsilon .
\end{aligned}$$

Letting $\varepsilon \to 0$, we have

$$E_{\infty,0}(w; f) \leq \sup_{\substack{\|gw\|_{L^1(\mathbb{R})} \leq 1 \\ \int_{\mathbb{R}} g(t)w^2(t)dt = 0}} \int_{\mathbb{R}} f(t)g(t)w^2(t)dt . \tag{3.5}$$

The proof of the opposite direction is completed, and from (3.2) to (3.5) we have (3.1). □

For $hw \in L^p(\mathbb{R})$ and $x \in \mathbb{R}$, we set

$$I(h)(x) := \frac{1}{w^2(x)}\int_x^\infty h(t)w^2(t)dt .$$

PROPOSITION 3.2. *Suppose that* $w \in \mathcal{F}(C^2(+)$ *is an Erdös-type weight. If* $hw \in L^p(\mathbb{R})$ *satisfies*

$$\int_{\mathbb{R}} h(t)w^2(t)dt = 0 , \tag{3.6}$$

then

$$\|I'(h)w\|_{L^p(\mathbb{R})} \leq C\|hw\|_{L^p(\mathbb{R})} . \tag{3.7}$$

Moreover, if g *is absolutely continuous and* $wg' \in L^q(\mathbb{R})$*, then*

$$\int_{\mathbb{R}} g(x)h(x)w^2(x)dx = \int_{\mathbb{R}} g'(t)I(h)(t)w^2(t)dt . \tag{3.8}$$

PROOF. Since

$$I'(h)(t)w(t) = \frac{2Q'(t)}{w(t)}\int_t^\infty h(u)w^2(u)du - h(t)w(t)$$
$$= 2Q'(t)I(h)(t)w(t) - h(t)w(t)\,,$$

(3.7) follows from

$$\|Q'I(h)w\|_{L^p(\mathbb{R})} \le C\|hw\|_{L^p(\mathbb{R})}\,. \tag{3.9}$$

We first prove (3.9) for $p = \infty$. If $t \ge 0$, then

$$\int_t^\infty w(x)dx \le \frac{1}{Q'(t)}\int_t^\infty Q'(x)\exp(-Q(x))dx = \frac{w(t)}{Q'(t)}\,,$$

because Q' is increasing. Hence

$$|Q'(t)I(h)(t)w(t)| = \left|\frac{Q'(t)}{w(t)}\int_t^\infty h(x)w^2(x)dx\right|$$
$$\le \|hw\|_{L^\infty(\mathbb{R})}\left|\frac{Q'(t)}{w(t)}\int_t^\infty w(x)dx\right| \le \|hw\|_{L^\infty(\mathbb{R})}\,.$$

For $t < 0$, since

$$\int_{-\infty}^t w(x)dx = \int_{|t|}^\infty w(x)dx \le \frac{w(t)}{|Q'(t)|}\,, \tag{3.10}$$

the assumption (3.6) implies

$$|Q'(t)I(h)(t)w(t)|$$
$$= \left|\frac{Q'(t)}{w(t)}\int_t^\infty h(x)w^2(x)dx\right| = \left|\frac{Q'(t)}{w(t)}\int_{-\infty}^t h(x)w^2(x)dx\right|$$
$$\le \|hw\|_{L^\infty(\mathbb{R})}\left|\frac{Q'(t)}{w(t)}\int_{|t|}^\infty w(x)dx\right| \le \|hw\|_{L^\infty(\mathbb{R})}\,,$$

that is, $\|Q'I(h)w\|_{L^\infty(\mathbb{R})} \le \|hw\|_{L^\infty(\mathbb{R})}$ holds true.

Next we discuss the case $p = 1$. Let $H(x) := |h(x)| + |h(-x)|$. Then by (3.6) again, we have

$$\|Q'I(h)w\|_{L^1(\mathbb{R})} = \int_{\mathbb{R}}\left|\frac{Q'(t)}{w(t)}\int_t^\infty h(x)w^2(x)dx\right|dt$$
$$= \int_0^\infty\left|\frac{Q'(t)}{w(t)}\int_t^\infty h(x)w^2(x)dx\right|dt + \int_{-\infty}^0\left|\frac{Q'(t)}{w(t)}\int_t^\infty h(x)w^2(x)dx\right|dt$$
$$= \int_0^\infty\left|\frac{Q'(t)}{w(t)}\int_t^\infty h(x)w^2(x)dx\right|dt + \int_{-\infty}^0\left|\frac{Q'(t)}{w(t)}\int_{-\infty}^t h(x)w^2(x)dx\right|dt$$
$$= \int_0^\infty\left|\frac{Q'(t)}{w(t)}\int_t^\infty h(x)w^2(x)dx\right|dt + \int_0^\infty\left|\frac{Q'(s)}{w(s)}\int_{-\infty}^{-s} h(x)w^2(x)dx\right|ds$$
$$\le \int_0^\infty \frac{Q'(t)}{w(t)}\left(\int_t^\infty H(x)w^2(x)dx\right)dt = \int_0^\infty H(x)w^2(x)\left(\int_0^x \frac{Q'(t)}{w(t)}dt\right)dx$$

$$= \int_0^\infty H(x)w^2(x)\left(\frac{1}{w(x)} - 1\right)dx \le \int_0^\infty H(x)w^2(x)\frac{1}{w(x)}dx$$
$$= \int_{\mathbb{R}} |h(x)|w(x)dx = \|hw\|_{L^1(\mathbb{R})}\,.$$

Therefore, for all h satisfying (3.6), we have (3.9) with $C = 1$ for $p = \infty$ and $p = 1$.

Now we define a linear operator

$$F(f) := \frac{\displaystyle\int_{\mathbb{R}} f(x)w^2(x)dx}{\displaystyle\int_{\mathbb{R}} w^2(x)dx}$$

for $fw \in L^p(\mathbb{R})$. Then $|F(f)| \le C\|fw\|_{L^p(\mathbb{R})}$ for $p=1$ and ∞, so that $\|(f-F(f))w\|_{L^p(\mathbb{R})} \le C\|fw\|_{L^p(\mathbb{R})}$ for $p = 1$ and ∞. Moreover

$$\int_{\mathbb{R}} (f(t) - F(f))w^2(t)dt = \int_{\mathbb{R}} f(t)w^2(t)dt - F(f)\int_{\mathbb{R}} w^2(t)dt = 0\,,$$

that is, $f - F(f)$ satisfies (3.6). For $fw \in L^p(\mathbb{R})$, we consider the operator

$$U(f)(x) := Q'(x)I(f - F(f))(x)\,.$$

Then by (3.9) for $p = 1$ and ∞, we have

$$\|U(f)w\|_{L^p(\mathbb{R})} \le \|(f - F(f))w\|_{L^p(\mathbb{R})} \le C\|fw\|_{L^p(\mathbb{R})}\,.$$

The Riesz-Thorin interpolation theorem shows that the above inequalities holds for all $1 \le p \le \infty$. If h satisfies (3.6), then $F(h) = 0$, so we have $\|Q'I(h)w\|_{L^p(\mathbb{R})} \le C\|hw\|_{L^p(\mathbb{R})}$ for all $1 \le p \le \infty$. This completes the proof of (3.9) and hence (3.7) follows.

Finally we discuss (3.8). Let $g'w \in L^q(\mathbb{R})$. Note that $gw \in L^q(\mathbb{R})$ (cf. [6, Theorem 6]). By (3.6), we may assume that $g(0) = 0$ and

$$\begin{aligned}
&\int_{-\infty}^0 g(x)h(x)w^2(x)dx \\
&= -\int_{-\infty}^0 h(x)w^2(x)\left(\int_x^0 g'(t)dt\right)dx \\
&= -\int_{-\infty}^0 g'(t)\left(\int_{-\infty}^t h(x)w^2(x)dx\right)dt \\
&= \int_{-\infty}^0 g'(t)\left(\int_t^\infty h(x)w^2(x)dx\right)dt \\
&= \int_{-\infty}^0 g'(t)I(h)(t)w^2(t)dt\,.
\end{aligned}$$

Similarly, we have

$$\int_0^\infty g(x)h(x)w^2(x)dx = \int_0^\infty g'(t)I(h)(t)w^2(t)dt\,.$$

Here we use the Fubini theorem, so we will check the integrability condition. Let $t \le -1$. For $1 < p < \infty$, as in (3.10), we see

$$|qQ'(t)|\int_{-\infty}^{t} w^q(x)dx \le w^q(t)$$

and since w is Erdös-type, $|Q'(t)| \ge C|t|^{2q/p}$ holds (see [6, (2.4)]). Hence by the Hölder inequality,

$$\begin{aligned}
&\int_{-\infty}^{-1} |g'(t)| \left(\int_{-\infty}^{t} |h(x)|w^2(x)dx\right) dt \\
&\le \int_{-\infty}^{-1} |g'(t)| \left(\int_{-\infty}^{t} (|h(x)|w(x))^p dx\right)^{1/p} \left(\int_{-\infty}^{t} w^q(x)dx\right)^{1/q} dt \\
&\le \|hw\|_{L^p(\mathbb{R})} \int_{-\infty}^{-1} |g'(t)| \left(\frac{w^q(t)}{|Q'(t)|}\right)^{1/q} dt \\
&\le C\|hw\|_{L^p(\mathbb{R})}\|g'w\|_{L^q(\mathbb{R})} \left(\int_{-\infty}^{-1} \frac{1}{t^2} dt\right)^{1/p} < \infty .
\end{aligned}$$

If $p = \infty$, then by (3.10) and $|Q'(t)| \ge |Q'(-1)|$ for $t \le -1$,

$$\begin{aligned}
&\int_{-\infty}^{-1} |g'(t)| \left(\int_{-\infty}^{t} |h(x)|w^2(x)dx\right) dt \\
&\le \int_{-\infty}^{-1} \left(|g'(t)|\|hw\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{t} w(x)dx\right) dt \\
&\le \|hw\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{-1} \frac{w(t)}{|Q'(t)|}|g'(t)|dt \\
&\le \frac{1}{|Q'(-1)|}\|hw\|_{L^\infty(\mathbb{R})} \int_{-\infty}^{-1} |g'(t)|w(t)dt \\
&\le C\|hw\|_{L^\infty(\mathbb{R})}\|g'w\|_{L^1(\mathbb{R})} < \infty .
\end{aligned}$$

When $p = 1$, then

$$\begin{aligned}
&\int_{-\infty}^{-1} |g'(t)| \left(\int_{-\infty}^{t} |h(x)|w^2(x)dx\right) dt \\
&= \int_{-\infty}^{-1} |h(x)|w^2(x) \left(\int_{x}^{-1} |g'(t)|dt\right) dx \\
&= \int_{-\infty}^{-1} |h(x)|w(x) \left(\int_{x}^{-1} |g'(t)|w(t)\frac{w(x)}{w(t)}dt\right) dx \\
&\le C\|hw\|_{L^1(\mathbb{R})}\|g'w\|_{L^\infty(\mathbb{R})} < \infty .
\end{aligned}$$

Here we use an estimate

$$\int_x^{-1} \frac{w(x)}{w(t)} dt = w(x) \int_x^{-1} \frac{Q'(t)}{w(t)} \frac{1}{Q'(t)} dt \le \frac{w(x)}{|Q'(-1)|} \int_x^{-1} \frac{Q'(t)}{w(t)} dt < C .$$

Clearly

$$\int_{-1}^{0} |g'(t)| \left(\int_{-\infty}^{t} |h(x)| w^2(x) dx \right) dt < \infty ,$$

so we have

$$\int_{-\infty}^{0} |g'(t)| \left(\int_{-\infty}^{t} |h(x)| w^2(x) dx \right) dt < \infty$$

for all $1 \le p \le \infty$. Similarly

$$\int_{0}^{\infty} |g'(t)| \left(\int_{t}^{\infty} |h(x)| w^2(x) dx \right) dt < \infty$$

holds and hence the proof of (3.8) is completed. □

Proposition 3.3. Suppose that $T(a_n) \le C_1 (n/a_n)^{2/3}$ for some $C_1 \ge 1$ and let $1 \le p \le \infty$. Then for every absolutely continuous g with $g'w \in L^p(\mathbb{R})$ and for any $n \in \mathbb{N}$, there exists a polynomial $V_n \in \mathcal{P}_{2n}$ such that $V_n' = v_n(g')$ and

$$\|(g - V_n)w\|_{L^p(\mathbb{R})} \le C \frac{a_n}{n} T^{1/4}(a_n) E_{p,n}(w; g') \tag{3.11}$$

holds, where $v_n(g')$ is the de la Vallée Poussin mean of g'.

PROOF. Without loss of generality, we may assume the $g(0) = 0$. Let

$$G(x) := \int_0^x (g'(t) - v_n(g')(t)) dt$$

and take a constant a such that $\|(G - a)w\|_{L^p(\mathbb{R})} \le 2E_{p,0}(w; G)$. Then

$$V_n(x) := a + \int_0^x v_n(g')(t) dt = a + g(x) - G(x) \in \mathcal{P}_{2n}$$

and $\|(g - V_n)w\|_{L^p(\mathbb{R})} = \|(G - a)w\|_{L^p(\mathbb{R})} \le 2E_{p,0}(w; G)$. By Proposition 3.1, we can find h which satisfies (3.6), $\|hw\|_{L^q(\mathbb{R})} = 1$ and

$$E_{p,0}(w; G) \le 2 \int_{\mathbb{R}} G(x) h(x) w^2(x) dx .$$

Note that $I(h)w \in L^q(\mathbb{R})$ by (3.9). Hence by changing the order of integration and taking $P \in \mathcal{P}_n$ such that $\|(I(h) - P)w\|_{L^q(\mathbb{R})} \le 2E_{q,n}(w; I(h))$, we see

$$\begin{aligned}
\|(g - V_n)w\|_{L^p(\mathbb{R})} &\le 2E_{p,0}(w; G) \le 4 \int_{\mathbb{R}} G(x) h(x) w^2(x) dx \\
&= 4 \int_{\mathbb{R}} \left(\int_0^x g'(t) - v_n(g')(t) dt \right) h(x) w^2(x) dx \\
&= 4 \int_{\mathbb{R}} (g'(t) - v_n(g')(t)) \left(\int_t^{\infty} h(x) w^2(x) dx \right) dt
\end{aligned}$$

$$
\begin{aligned}
&= 4\int_{\mathbb{R}} (g'(t) - v_n(g')(t)) I(h)(t) w^2(t)dt \\
&= 4\int_{\mathbb{R}} (g'(t) - v_n(g')(t))(I(h)(t) - P(t)) w^2(t)dt \\
&\le 8\|(g' - v_n(g'))w\|_{L^p(\mathbb{R})} E_{q,n}(w; I(h)) .
\end{aligned}
$$

Here we use (2.7). By [6, Theorem 1] and (3.7), we have

$$E_{q,n}(w; I(h)) \le C\frac{a_n}{n}\|I'(h)w\|_{L^q(\mathbb{R})} \le C\frac{a_n}{n}\|hw\|_{L^q(\mathbb{R})} = C\frac{a_n}{n}$$

and by (1.8), we see

$$\|(g' - v_n(g')w\|_{L^p(\mathbb{R})} \le CT^{1/4}(a_n)E_{p,n}(w; g') . \tag{3.12}$$

The desired result (3.11) follows from the combination of the above inequalities. □

4. Proof of Theorems.

PROOF OF THEOREM 1.2. By Proposition 3.3, we have

$$\|(f - V_n)w\|_{L^p(\mathbb{R})} \le C\frac{a_n}{n}T^{1/4}(a_n)E_{p,n}(w; f') \tag{4.1}$$

and by [3, Theorem 1.15 and Corollary 1.16] and Lemma 2.1, we see

$$\|(V_n' - P')w\|_{L^p(\mathbb{R})} \le C\frac{n}{a_n}T^{1/2}(a_n)\|(V_n - P)w\|_{L^p(\mathbb{R})} , \tag{4.2}$$

where $V_n' = v_n(f')$. Also by (1.8), we have

$$\|(f' - v_n(f'))w\|_{L^p(\mathbb{R})} \le CT^{1/4}(a_n)E_{p,n}(w; f') . \tag{4.3}$$

Hence

$$
\begin{aligned}
&\|(f' - P')w\|_{L^p(\mathbb{R})} \le \|(f' - v_n(f'))w\|_{L^p(\mathbb{R})} + \|(V_n' - P')w\|_{L^p(\mathbb{R})} \\
&\le CT^{1/4}(a_n)E_{p,n}(w; f') + C\frac{n}{a_n}T^{1/2}(a_n)\|(V_n - P)w\|_{L^p(\mathbb{R})} \\
&\le CT^{1/4}(a_n)E_{p,n}(w; f') \\
&\qquad + C\frac{n}{a_n}T^{1/2}(a_n)\left(\|(f - V_n)w\|_{L^p(\mathbb{R})} + \|(f - P)w\|_{L^p(\mathbb{R})}\right) \\
&\le C\left(T^{1/4}(a_n)E_{p,n}(w; f') + T^{3/4}(a_n)E_{p,n}(w; f') + \frac{n}{a_n}T^{1/2}(a_n)\varepsilon\right) \\
&\le C\left(T^{3/4}(a_n)E_{p,n}(w; f') + \frac{n}{a_n}T^{1/2}(a_n)\varepsilon\right) ,
\end{aligned}
$$

which shows (1.11). □

PROOF OF THEOREM 1.1. Let $P = v_n(f)$ in Theorem 1.2. Then by (1.8) we can take $\varepsilon = CT^{1/4}(a_n)E_{p,n}(w; f)$, hence (1.11) and (1.6) gives us

$$\begin{aligned}
&\|(f' - v_n(f)')w\|_{L^p(\mathbb{R})} \\
&\le C\left(T^{3/4}(a_n)E_{p,n}(w; f') + C\frac{n}{a_n}T^{3/4}(a_n)E_{p,n}(w; f)\right) \\
&\le CT^{3/4}(a_n)E_{p,n-1}(w; f') .
\end{aligned}$$

□

COROLLARY 4.1. *Let $w \in \mathcal{F}(C^2+)$ and p be the same as in Theorem 1.2. Let $j \ge 1$. Then there exists a constant $C \ge 1$ such that if an absolutely continuous function $f^{(j-1)}$ satisfies $f^{(j)}w \in L^p(\mathbb{R})$ and if $\|(f - P)w\|_{L^p(\mathbb{R})} \le \varepsilon$ holds for some $n > j$, $P \in \mathcal{P}_{2n}$ and $\varepsilon > 0$, then*

$$\begin{aligned}
&\|(f^{(i)} - P^{(i)})w\|_{L^p(\mathbb{R})} \\
&\le C\left(T^{(2i+1)/4}(a_n)E_{p,n-i+1}(w; f^{(i)}) + \left(\frac{n}{a_n}\right)^i T^{i/2}(a_n)\varepsilon\right)
\end{aligned}$$

holds for all $i = 1, 2, \dots, j$.

PROOF. Since $f'w \in L^p(\mathbb{R})$ implies $fw \in L^p(\mathbb{R})$, it follows from $f^{(j)}w \in L^p(\mathbb{R})$ that $f^{(i)}w \in L^p(\mathbb{R})$ for all $0 \le i \le j-1$. Hence by (1.11), we see

$$\|(f' - P')w\|_{L^p(\mathbb{R})} \le C(T^{3/4}(a_n)E_{p,n}(w; f') + \frac{n}{a_n}T(a_n)^{1/2}\varepsilon) =: \varepsilon_1 .$$

Applying Theorem 1.2 to f', P' and ε_1 and using (1.6) again, we have

$$\begin{aligned}
&\|(f'' - P'')w\|_{L^p(\mathbb{R})} \\
&\le C\left(T^{3/4}(a_n)E_{p,n}(w; f'') + \frac{n}{a_n}T^{1/2}(a_n)\varepsilon_1\right) \\
&\le C\left(T^{5/4}(a_n)E_{p,n-1}(w; f'') + \left(\frac{n}{a_n}\right)^2 T(a_n)\varepsilon\right) .
\end{aligned}$$

This shows the case $i = 2$. Repeating this process, we have Corollary 4.1. □

REMARK 4.2. Let $w \in \mathcal{F}(C^2+)$ satisfy (1.7) and let $j > 1$. If $f^{(j-1)}$ is absolutely continuous and it satisfies $f^{(j)}w \in L^p(\mathbb{R})$, then

$$\|(f^{(i)} - (v_n(f))^{(i)})w\|_{L^p(\mathbb{R})} \le CT^{(2i+1)/4}(a_n)E_{p,n-i}(w; f^{(i)}) \tag{4.4}$$

holds for $n > j$ and $i = 1, 2, \dots, j$.

In fact, by (1.6) and (1.8), we obtain

$$\begin{aligned}
&\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \le CT^{1/4}(a_n)E_{p,n}(w : f) \\
&\le CT^{1/4}(a_n)\frac{a_n}{n}E_{p,n-1}(w; f') \le CT^{1/4}(a_n)\frac{a_n a_{n-1}}{n(n-1)}E_{p,n-2}(w; f'')
\end{aligned}$$

$$\leq \cdots \leq CT^{1/4}(a_n)\frac{a_n a_{n-1}\cdots a_{n-i}}{n(n-1)\cdots(n-i)}E_{p,n-i}(w;\, f^{(i)})$$

$$\leq CT^{1/4}(a_n)\left(\frac{a_n}{n}\right)^i E_{p,n-i}(w;\, f^{(i)})\,.$$

This estimate and Corollary 4.1 show (4.4)

REMARK 4.3. Let $w \in \mathcal{F}(C^2+)$ satisfy (1.7). Then

$$\|(f' - v_n(f)')\frac{w}{T^{1/2}}\|_{L^p(\mathbb{R})} \leq CE_{p,n-1}(T^{1/4}w;\, f') \tag{4.5}$$

holds, whenever $f'T^{1/4}w \in L^p(\mathbb{R})$.

In fact, in the proof of Theorems 1.1 and 1.2, if we use

$$\|(f - v_n(f))w\|_{L^p(\mathbb{R})} \leq CE_{p,n}(T^{1/4}w;\, f)$$

(see [1, Corollary 14]) and

$$\|(V_n' - P')\frac{w}{T^{1/2}}\|_{L^p(\mathbb{R})} \leq C\frac{n}{a_n}\|(V_n - P)w\|_{L^p(\mathbb{R})}$$

(see [7, Theorem 6.1]) instead of (1.8) and (4.2), respectively, then the similar proof gives us (4.5).

REMARK 4.4. For the case $p = \infty$, under some additional assumption on w, the inequality

$$\|(f^{(i)} - (v_n(f))^{(i)})w\|_{L^p(\mathbb{R})} \leq CE_{p,n-i}(T^{(2i+1)/4}w;\, f^{(i)}) \tag{4.6}$$

follows from a result of Jung and Sakai [2], however we do not know whether this is true or not for general $1 \leq p < \infty$.

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