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HOLOMORPHIC ISOMETRIC EMBEDDINGS OF THE PROJECTIVE LINE INTO QUADRICS

OSCAR MACIA, YASUYUKI NAGATOMO AND MASARO TAKAHASHI

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Abstract. We discuss holomorphic isometric embeddings of the projective line into quadrics using the generalisation of the theorem of do Carmo–Wallach in [14] to provide a description of their moduli spaces up to image and gauge equivalence. Moreover, we show rigidity of the real standard map from the projective line into quadrics.

CONTENTS

1. Introduction. Suppose f is an embedding of the complex projective line $\mathbf{C}P^1$ into the Grassmannian manifold $\mathrm{Gr}_n(\mathbf{R}^{n+2})$ of codimension-2 oriented linear subspaces of $\mathbf{R}^{n+2}$. Notice that by standard arguments (cf. [10, pp. 278–282]) the target space can be regarded as the complex quadric hypersurface in $\mathbf{C}P^{n+1}$. We say that f is a holomorphic isometric embedding if it is compatible with the obvious holomorphic and metric structures. It is convenient to regard two such embeddings as indistinguishable if related through composition with an holomorphic isometry of the target space; the maps are then said to be congruent or image equivalent. The natural question of classification comes up: are holomorphic isometric embeddings $\mathbf{C}P^1 \hookrightarrow \mathrm{Gr}_n(\mathbf{R}^{n+2})$ unique up to image equivalence? If not, what is the moduli space?

The purpose of this article is to obtain some results concerning the moduli space $\mathbf{M}_k$ of holomorphic isometric embeddings $f : \mathbf{C}P^1 \hookrightarrow \mathrm{Gr}_n(\mathbf{R}^{n+2})$ of degree k up to image equivalence using methods and results of [14]. To this aim we shall study first a more fundamental

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object: the moduli space $\mathcal{M}_k$ of full holomorphic isometric embeddings of degree k up to gauge equivalence. The meaning of fullness and gauge equivalence of maps will be made precise in §2, where a summary of the main results in [14] has been included for convenience of the reader.

Our first result serves as an illuminating application of the general theory. The assertion (Theorem 5.4) is that in an important special case (called *real standard*) $\mathcal{M}_k$ consists of a single point, i.e. there is a single gauge equivalence class. Next, we introduce our main result (Theorem 6.4) which shows that generally $\mathcal{M}_k$ is an open convex bounded body in certain real SU(2)-module which can be described explicitly. The precise statement is

MAIN THEOREM. If $f : \mathbf{C}P^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ is a full holomorphic isometric embedding of degree k, then $n \leq 2k$. Let $\mathcal{M}_k$ be the moduli space of full holomorphic isometric embeddings of degree k of the complex projective line into $Gr_{2k}(\mathbf{R}^{2k+2})$ by the gauge equivalence of maps. Then, $\mathcal{M}_k$ can be regarded as an open bounded convex body in $2\bigoplus_{r=1}^{k\geqq 2r} S_0^{k-2r}\mathbf{R}^3$.

Let $\overline{\mathcal{M}_k}$ be the closure of the moduli $\mathcal{M}_k$ by the inner product. Boundary points of $\overline{\mathcal{M}_k}$ describe those maps whose images are included in some totally geodesic submanifold $Gr_p(\mathbf{R}^{p+2})$ of $Gr_{2k}(\mathbf{R}^{2k+2})$, where $p < 2k$.

The totally geodesic submanifold $Gr_p(\mathbf{R}^{p+2})$ can be regarded as the common zero set of some sections of the universal quotient bundle $Q \to Gr_{2k}(\mathbf{R}^{2k+2})$, which belongs to $\mathbf{R}^{2k+2}$.

The theorem generalises to arbitrary degree k what had been proved for lower degrees $k = 1, 2$ in [14, Theorems 7.26–7.27].

The principal advantage of our approach is that we are able to compute the exact dimension of $\mathcal{M}_k$ (Equation 6.1). Moreover, we also discuss its compactification in the L^2-topology, $\overline{\mathcal{M}_k}$, and conclude that the boundary points describe maps whose images are included in some totally geodesic submanifold of $\mathrm{Gr}_n(\mathbf{R}^{n+2})$. Finally, we study the moduli up to image equivalence $\mathbf{M}_k$ and show that it can be described as an S^1-quotient of $\mathcal{M}_k$ (Theorem 7.1).

The study of harmonic maps from the complex projective line into complex quadrics has a long history and has been pursued by various authors in different ways, e.g. [4, 7, 12, 17]. Our particular standpoint is a generalisation by the second-named-author of the methods of Takahashi [15] and of do Carmo and Wallach [5], which can be summarised as follows: A well-known theorem by Takahashi [15, Theorem 3] proves that an isometric immersion of a Riemannian manifold in Euclidean space is an eigenvector for the Laplacian iff it is a minimal immersion in some Euclidean sphere. The energy density of the immersion is then related to the corresponding eigenvalue. A generalisation of this result via vector bundles can be found in [14, Theorem 3.5]. The statement is that a smooth map f of a Riemannian manifold into the Grassmannian $Gr_p(W)$, where W is a real or complex vector space with a scalar product, is harmonic iff W satisfies the *zero property for the Laplacian*: for arbitrary $t \in W \subset \Gamma(f^*Q)$, $\Delta t = -At$, where $Q \to Gr_p(W)$ is the universal quotient bundle, Δ is the Laplace operator acting on sections and A is the *mean curvature operator* (defined in [14, §2]) related to the energy density of f. This viewpoint leads to a description in which a harmonic map from a Riemannian manifold into a Grassmannian is induced by a triple composed by a vector bundle, a space of sections of this bundle and a Laplace operator.

A celebrated example of such induced map is Kodaira's embedding of an algebraic manifold into complex projective space [11], which in the aforesaid description is induced by a holomorphic line bundle and the space of holomorphic sections.

Takahashi's original result finds major application in do Carmo and Wallach [5] undertaking of the classification of minimal (isometric) immersions of spheres into spheres. Their result reveals [5, Theorems 1.4 and 1.5] that for the lower-dimensional cases minimal immersions of spheres into spheres are unique whenever they exist. Alternatively, starting at dimension three there are continuous families of image-inequivalent minimal immersions of spheres into standard spheres of dimension high enough. Each of these families is parametrised by a moduli space depending on no less than eighteen parameters, yielding a lower bound of the moduli dimension. A key role in do Carmo–Wallach theory is played by certain symmetric positive semi-definite linear operators [5, Proposition 1.3] interweaving minimal immersions: finding the space of the image-inequivalent operators amounts to describe the moduli space, an endeavour which is dealt successfully with representation theory.

From the generalised version of the theorem of Takahashi [14, Theorem 3.5], a generalisation of do Carmo–Wallach theory in terms of vector bundles is possible [14, Theorem 5.5]. We recall its principal features in Theorem 2.4 below. In essence, the theorem affirms that the harmonic induced map f of a Riemannian manifold into the Grassmannian $Gr_p(W)$ by the aforementioned harmonic triple is naturally equipped with a family of symmetric positive semi-definite operators determining the moduli space, as in the classical do Carmo–Wallach theory. Uniqueness of the associated symmetric operator reduces the moduli to a single point yielding rigidity of the induced map: this is the case of the *real standard* map of our Theorem 5.4.

An important illustration of such behaviour is Bando and Ohnita's result in [1] stating the rigidity of the minimal immersion of the complex projective line into complex projective spaces, originally proved employing twistor methods. Another instance is rigidity of holomorphic isometric embeddings between complex projective spaces, which is part of Calabi's diastasis result in [3]. Actually, this latter example is more amenable to the vector bundle approach favoured in the present work.

Closer to our viewpoint is Toth's analysis of polynomial minimal immersions between projective spaces [16] where the spaces of harmonic polynomials in complex space are used to define polynomial maps between spheres, and the Hopf fibration to get a map between complex projective spaces. Representation theory of unitary groups is then put in practice to determine a lower bound for the moduli dimension, as in the original do Carmo–Wallach paper.

It is remarkable that neither the problems undertaken in [5, 16] do require the vector bundle viewpoint of [14] since these cases correspond to straightforward situations: in the original do Carmo–Wallach construction the associated vector bundle is the trivial bundle; Toth's results follow from considering a complex line bundle with canonical connection. The authors would like however to emphasise that the present approach can be applied advantageously allowing to compute the exact dimension of the relevant moduli spaces. Also, notice that by considering complex quadrics as target instead as complex projective spaces (cf. [3]), positive-dimensional moduli spaces appear.

The article is organised as follows: In §2 we introduce the required preliminaries to the theory culminating in the statement of the generalisation of the theorem of do Carmo–Wallach (Theorem 2.4) as developed in [14]. §3 is devoted to show that holomorphic isometric embeddings $\mathbf{C}\mathrm{P}^1 \hookrightarrow \mathrm{Gr}_n(\mathbf{R}^{n+2})$ of degree k satisfy the hypothesis of this theorem. Next, §4 deals with the study of the space of Hermitian/symmetric operators and concludes with a detailed description of its various subspaces (Proposition 4.7) which encode the information about the moduli spaces. Applications of the theory first appear in §5. By showing that the space of symmetric operators yielding the moduli space restricts to a single point, we prove rigidity for the real standard map (Theorem 5.4). Finally, in §6 and §7 the moduli spaces up to gauge and image equivalence are introduced and described (Theorems 6.4 and 7.1).

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2. Preliminaries. In this section we give a short account of results in [14] needed to state a version of the generalisation of the theorem of do Carmo–Wallach (Theorem 2.4), whose implications will be applied later in this article.

Let W be a complex (resp. real/real oriented) N-dimensional vector space and $Gr_p(W)$ the complex (resp. real, resp. real oriented) Grassmannian of p-planes in W. Generically, $\underline{W}$ will stand for the total space of a trivial vector bundle $\underline{W} \to B$ with fibre W over some specified base manifold B. Denote by $\underline{W} \to Gr_p(W)$ the trivial bundle of fibre W over $Gr_p(W)$. Then, there is a natural bundle injection $i_S : S \to \underline{W}$ of the *tautological vector bundle* $S \to Gr_p(W)$ into the aforementioned trivial bundle. The *universal quotient bundle* $Q \to Gr_p(W)$ is defined by the exactness of the sequence $0 \to S \to \underline{W} \to Q \to 0$. Denote by π_Q the natural projection $\underline{W} \to Q$ and use it to regard W as a subspace of $\Gamma(Q)$, the space of sections of the universal quotient bundle.

By fixing a Hermitian (resp. symmetric) inner product on W the tautological and universal quotient bundles $S, Q \to Gr_p(W)$ inherit a fibre-metric, and can be given canonical connections and second fundamental forms in the sense of Kobayashi [9, p. 21].

Suppose $V \to M$ is a complex (resp. real/real oriented) vector bundle of rank q and consider an N-dimensional space of sections $W \subset \Gamma(V)$. By definition of $\underline{W} \to M$, there is a bundle homomorphism $ev : \underline{W} \to V$, called *evaluation*, defined by $(x, t) \mapsto t(x)$ for all $t \in W, x \in M$. The vector bundle $V \to M$ is said to be *globally generated by* W if the evaluation is surjective. Under this hypothesis, there is a map $f : M \to Gr_p(W)$, where $Gr_p(W)$ is a complex (resp. real/real oriented) Grassmannian and $p = N - q$, defined by

$$f(x) = \mathrm{Ker}\, ev_x = \{t \in W \mid t(x) = 0\} ,$$

where $ev_x \equiv ev(x, \cdot)$. The map f is said to be *induced by* the couple $(V \to M,\ W)$, or simply by W if the vector bundle $V \to M$ is specified (cf. [14]).

Notice that, by the definition of induced map, $V \to M$ can be *naturally identified* with $f^*Q \to M$. Therefore, given a smooth map $f : M \to Gr_p(W)$, it can be regarded as the induced map determined the by the couple $(f^*Q \to M,\ W)$. If the inclusion of $W \subset \Gamma(V)$ into $\Gamma(f^*Q)$ is injective, we say that the map f is *full* [14, Definition 5.2]. This definition of fullness coincides with the ones used in [5, 16] when the target space is the sphere or complex projective space.

Moreover, assume M to be Riemannian and $V \to M$ to be equipped with a fibre-metric and a connection. From these data a Laplace operator acting on sections can be defined.

The model special case is that in which M is a compact reductive homogeneous space G/K (where G is a compact Lie group and K is a closed subgroup of G), and $V \to M$ is a homogeneous complex (resp. real) vector bundle of rank q, i.e. $V \cong G \times_K V_0$ where V_0 is a q-dimensional complex (resp. real) K-module (cf. [14]). If additionally V_0 admits a K-invariant Hermitian (resp. symmetric) inner product, $V \to M$ inherits a G-invariant Hermitian (resp. symmetric) fibre-metric.

Because of reductivity, $V \to M$ is equipped with a canonical connection too, the one for which the horizontal subspace on the principal K-bundle $G \to M$ is given by the complement $\mathfrak{m}$ to $\mathfrak{k} = L(K)$ in $\mathfrak{g} = L(G)$.

Using the Levi–Civita connection and the canonical connection, $\Gamma(V)$ can be decomposed into eigenspaces of the Laplacian each being a finite-dimensional not necessarily irreducible G-module and equipped with a G-invariant L^2-inner product. Then, we say that the induced map by $(V \to M,\ W)$ is *standard* if a G-submodule $W \subseteq W_\mu$ globally generates the bundle, where W_μ is the eigenspace of the Laplacian with eigenvalue μ.

Evidently, the definition of standard map generalises the special homogeneous case. However, the homogeneous setting will be enough for the purposes of the present work.

The spaces of sections inducing standard maps have the following interesting property which will be useful later:

LEMMA 2.1 ([14, Lemma 5.17]). *Let W be a G-subspace of W_μ. If W globally generates $V \to G/K$, then V_0 can be regarded as a subspace of W.*

Denote by U_0 the orthogonal complement of V_0 in W. Then, the induced standard map $f_0 : M \to Gr_p(W)$ is expressed as

$$f_0([g]) = gU_0 \subset W\,,$$

for all $[g] \in G/K$, and is G-equivariant.

Notice that, besides its assumed fibre-metric and connection, $V \to M$ is endowed with a secondary couple of fibre-metric and connection inherited from the natural identification $\phi : V \cong f^*Q$, i.e. the fibre-metric and canonical connection on $Q \to Gr_p(W)$ pulled-back to $f^*Q \to M$. In general, these structures do not need to be gauge equivalent unless the splitting $W = U_0 \oplus^\perp V_0$ satisfies extra conditions:

LEMMA 2.2 ([14, Lemma 5.18]). *The pull-back connection is gauge equivalent to the canonical connection if and only if*

$$\mathfrak{m}V_0 \subset U_0 \,.$$

For any mapping $f : M \to Gr_p(W)$ the second fundamental forms H and K of the tautological and universal quotient bundles respectively can be assembled together in the *mean curvature operator of* f, a section of $\mathrm{End}(f^*Q)$ defined in [14], as

$$A = \sum_{i=1}^{n} \mathrm{H}_{df(e_i)} \mathrm{K}_{df(e_i)}$$

where $\{e_i\}_{1 \le i \le n}$ is an orthonormal basis. The mean curvature operator A satisfies the following important

LEMMA 2.3 ([14, Lemma 5.19]). *If a G-representation $W \subseteq W_\mu$ globally generates $V \to M$ and satisfies the condition $\mathfrak{m}V_0 \subset U_0$, then the standard map $f_0 : M \to Gr_p(W)$ is harmonic with constant energy density $e(f_0) = q\mu$ and the mean curvature operator is proportional to the identity $A = -\mu Id_V$.*

Let us introduce the two increasingly stronger equivalence relations [14, Definitions 5.3 and 5.4], up to which we shall later define moduli spaces of maps. Let f_1 and $f_2 : M \to Gr_p(W)$. Then f_1 is called *image equivalent* to f_2 if there exists an isometry ϕ of $Gr_p(W)$ such that $f_2 = \phi \circ f_1$. Furthermore, denote by $\tilde{\phi}$ the bundle isomorphism of $Q \to Gr_p(W)$ which covers the isometry ϕ of $Gr_p(W)$. Then, the pair (f_1, ϕ_1) is said to be *gauge equivalent* to (f_2, ϕ_2), where $\phi_i : V \to f_i^* Q (i = 1, 2)$ are bundle isomorphisms, if there exists an isometry ϕ of $Gr_p(W)$ such that $f_2 = \phi \circ f_1$ and $\phi_2 = \tilde{\phi} \circ \phi_1$.

Aside from the geometric background, some algebraic preliminaries regarding Hermitian/symmetric operators are needed.

Let G be a compact Lie group, W a complex G-module together with an invariant Hermitian product $(\ ,\ )_W$ and denote by $\mathrm{H}(W)$ the set of Hermitian endomorphisms of W. We equip $\mathrm{H}(W)$ with a G-invariant inner product $(A, B)_H = \mathrm{trace}\, AB$, for $A, B \in \mathrm{H}(W)$. Define a Hermitian operator $\mathrm{H}(u, v)$ for $u, v \in W$ as

$$\mathrm{H}(u, v) := \frac{1}{2} \left\{ u \otimes (\cdot, v)_W + v \otimes (\cdot, u)_W \right\} \,.$$

If U and V are subspaces of W, we define a real subspace $\mathrm{H}(U, V) \subset \mathrm{H}(W)$ spanned by $\mathrm{H}(u, v)$ where $u \in U$ and $v \in V$. In a similar fashion, $\mathrm{GH}(U, V)$ denotes the subspace of $\mathrm{H}(W)$ spanned by $g\mathrm{H}(u, v)$, where $g \in \mathrm{G}$.

If W is a real G-module together with an invariant inner product, then symmetric endomorphisms take the place of Hermitian ones and we get analogous definitions of $\mathrm{S}(W)$, $\mathrm{S}(u, v)$, $\mathrm{S}(U, V)$, $\mathrm{GS}(U, V)$.

Now we have all the needed ingredients to introduce a version of the generalisation of the theorem of do Carmo–Wallach for holomorphic maps.

THEOREM 2.4. *Let $M = G/K$ be a compact irreducible Hermitian symmetric space with decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{m}$ and fix a complex homogeneous line bundle $V = G \times_K V_0$ over M with invariant metric h and canonical connection ∇. Regard $V \to M$ as a real vector bundle with complex structure J. Finally, let $f : M \to Gr_n(\mathbf{R}^{n+2})$ be a full holomorphic map satisfying the following two conditions:*

(G) *The pull-back $f^*Q \to M$ of the universal quotient bundle $Q \to Gr_n(\mathbf{R}^{n+2})$ with the pull-back metric, connection and complex structure is gauge equivalent to $V \to M$ with h, ∇ and J.*

(EH) *The mean curvature operator $A \in \Gamma(\mathrm{End}\, V)$ of f is expressed as $-\mu\ Id_V$ with some positive real number μ, and so $e(f) = 2\mu$.*

Hence the space of holomorphic sections $W = H^0(V) \subset \Gamma(V)$ is also an eigenspace of the Laplacian with eigenvalue μ. Regard W as a real vector space with L^2-inner product $(\cdot,\cdot)_W$ induced from the L^2-Hermitian product. Then, there exists a positive semi-definite symmetric endomorphism $T \in \mathrm{S}(W)$ such that the pair (W, T) satisfies the following four conditions:

(I) *The vector space $\mathbf{R}^{n+2}$ is a subspace of W with the inclusion $\iota : \mathbf{R}^{n+2} \to W$ preserving the orientation, and $V \to M$ is globally generated by $\mathbf{R}^{n+2}$.*

(II) *As a subspace, $\mathbf{R}^{n+2} = \mathrm{Ker}\, T^{\perp}$ and the restriction of T is a positive definite symmetric endomorphism of $\mathbf{R}^{n+2}$.*

(III) *The endomorphism T satisfies the orthogonality conditions*

$$\left(T^2 - Id_W, \mathrm{GH}(V_0, V_0)\right)_H = 0, \qquad \left(T^2, \mathrm{GH}(\mathfrak{m}V_0, V_0)\right)_H = 0\,. \tag{2.1}$$

(IV) *The endomorphism T provides a holomorphic embedding of $Gr_n(\mathbf{R}^{n+2})$ into $Gr_{n'}(W)$, where $n' = n + \dim \mathrm{Ker}\, T$ and also provides a bundle isomorphism $\phi : V \to f^*Q$.*

Therefore, $f : M \to Gr_n(\mathbf{R}^{n+2})$ can be expressed as

$$f([g]) = \left(\iota^* T \iota\right)^{-1} \left(f_0([g]) \cap \mathrm{Ker}\, T^{\perp}\right), \tag{2.2}$$

where ι^* denotes the adjoint operator of ι under the induced inner product on $\mathbf{R}^{n+2}$ from $(\cdot,\cdot)_W$ on W and f_0 is the standard map by W. Moreover two such pairs (f_i, ϕ_i), $(i = 1, 2)$ are gauge equivalent if and only if $\iota_1^* T_1 \iota_1 = \iota_2^* T_2 \iota_2$, where (T_i, ι_i) correspond to f_i $(i = 1, 2)$ under the expression in (2.2), *respectively.*

Conversely, suppose that a vector space $\mathbf{R}^{n+2}$, the space of holomorphic sections $W \subset \Gamma(V)$ regarded as real vector space and a positive semi-definite symmetric endomorphism $T \in \mathrm{S}(W)$ satisfying conditions (I), (II) *and* (III) *are given. Then we have a unique holomorphic embedding of $Gr_n(\mathbf{R}^{n+2})$ into $Gr_{n'}(W)$ and the map $f : M \to Gr_n(\mathbf{R}^{n+2})$ defined by* (2.2) *is a full holomorphic map into $Gr_n(\mathbf{R}^{n+2})$ satisfying conditions* (G) *and* (EH) *with bundle isomorphism $V \cong f^*Q$.*

PROOF. This is obtained by a combination of Theorems 5.16 and 5.20 in [14], themselves refinements and following the same proof as that of Theorem 5.5. □

REMARK 1. Conditions (G) and (EH) in the theorem are named respectively gauge and Einstein–Hermitian conditions.

REMARK 2. To define a complex strucure of $Gr_n(\mathbf{R}^{n+2})$, we need to fix an orientation of $\mathbf{R}^{n+2}$. We denote by $\mathbf{R}^{n+2}_{+}$ the Euclidean space with the orientation. The Euclidean space with the reversed orientation is denoted by $\mathbf{R}^{n+2}_{-}$. Let $\tau : Gr_n(\mathbf{R}^{n+2}) \to Gr_n(\mathbf{R}^{n+2})$ be the map obtained by switching the orientation of n-dimensional subspaces of $\mathbf{R}^{n+2}$. Then $\tau : Gr_n(\mathbf{R}^{n+2}_{+}) \to Gr_n(\mathbf{R}^{n+2}_{-})$ is a *holomorphic* isometry. In the sequel, we do not distinguish a map $f : G/K \to Gr_n(\mathbf{R}^{n+2})$ from a map $\tau \circ f : G/K \to Gr_n(\mathbf{R}^{n+2})$.

3. Holomorphic isometric embeddings. The aim of this section is to introduce holomorphic isometric embeddings from $\mathbf{C}P^1$ into $\mathrm{Gr}_n(\mathbf{R}^{n+2})$ and to show that they satisfy the hypothesis of Theorem 2.4. Then the universal quotient bundle has a holomorphic bundle structure. Notice that the curvature two-form R of the canonical connection on the universal quotient bundle is the fundamental two-form ω_Q on $Gr_n(\mathbf{R}^{n+2})$ up to a constant multiple

$$R = -2\pi\sqrt{-1}\omega_Q \,.$$

Denote by ω_0 the fundamental two-form on $\mathbf{C}P^1$. When R_1 denotes the curvature two-form of the canonical connection on the hyperplane bundle $\mathcal{O}(1) \to \mathbf{C}P^1$ (cf. [8, p. 145]), we also have $R_1 = -2\pi\sqrt{-1}\omega_0$. In what follows, we will denote by $\mathcal{O}(k) \to \mathbf{C}P^1$ the k-th tensor power of the hyperplane bundle.

DEFINITION 1. Let $f : \mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ be a holomorphic embedding. Then f is called an isometric embedding of degree k if $f^*\omega_Q = k\omega_0$ (and so, k must be a positive integer).

In order to show that holomorphic isometric embeddings $\mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ satisfy the conditions of Theorem 2.4 we need the following two lemmas. Their proofs rely heavily on properties of the (unique) Einstein–Hermitian connection. For additional details we refer the interested reader to the excellent book by Kobayashi [9, Ch. IV].

LEMMA 3.1. *Let $f : \mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ be a holomorphic embedding. Then f is an isometric embedding of degree k if and only if the pull-back bundle $f^*Q \to \mathbf{C}P^1$ with the pull-back connection is gauge equivalent to $\mathcal{O}(k) \to \mathbf{C}P^1$ with the canonical connection.*

PROOF. If the degree of the isometric embedding f equals k, the pull-back of the universal quotient bundle is holomorphically isomorphic to the holomorphic line bundle of degree k on $\mathbf{C}P^1$ (by uniqueness of the holomorphic bundle structure), which by homogeneity admits a unique Einstein–Hermitian structure up to homotheties of the fibre-metric (cf. [9, Proposition IV.6.1]). Uniqueness of the Einstein–Hermitian connection yields the result.

Conversely, if the pull-back of the universal quotient bundle is holomorphically isomorphic as Einstein–Hermitian bundle to the holomorphic line bundle, the pull-back fibre-metric and the Einstein–Hermitian connection coincide up to homothety, and the statement in the lemma follows. □

LEMMA 3.2. *Let $f : \mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ be a holomorphic isometric embedding of degree k. Then, the mean curvature operator $A \in \Gamma(V)$ of f is the identity on V up to a negative real constant.*

PROOF. It is well-known that every holomorphic section t of $\mathcal{O}(k) \to \mathbf{C}P^1$ satisfies $\Delta t - K_{EH}t = 0$ (cf. [14, Lemma 4.2]), where the Laplacian is defined through a compatible connection, and K_{EH} is the mean curvature arising from the Hermitian structure in the sense of Kobayashi [9, p. 99]. Since the canonical connection is the Einstein–Hermitian connection, $K_{EH} = \mu Id$.

On the other hand, a generalisation of the theorem of Takahashi yields that $\Delta t + At = 0$ for $t \in \mathbf{R}^{n+2}$. Regarding $\mathbf{R}^{n+2}$ as a subspace of $H^0\left(\mathbf{C}P^1, \mathcal{O}(k)\right)$, then $\mathbf{R}^{n+2}$ globally generates $\mathcal{O}(k) \to \mathbf{C}P^1$. Therefore $K_{EH} = -A$, and the lemma follows. □

These two lemmas amount to say that the holomorphic embedding f is isometric if and only if it satisfies the gauge condition (G), and then the (EH) condition is automatically satisfied. Hence we can apply Theorem 2.4 to obtain the moduli space $\mathcal{M}_k$ of holomorphic isometric embeddings of degree k by the gauge equivalence of maps.

REMARK 3. Unlike the case of holomorphic isometric embeddings, for general harmonic maps and minimal immersions the (EH) condition is independent of the gauge condition (G). We shall discuss harmonic maps and minimal immersions satisfying gauge and (EH) conditions in the forthcoming paper [13].

4. Hermitian/Symmetric endomorphisms. In order to apply the generalised do Carmo–Wallach theory we need a deeper understanding of the space of symmetric endomorphisms of the space of holomorphic sections of the bundles of interest. Since in the present work the spaces of holomorphic sections are real SU(2)-modules, in this section we describe how the space of symmetric endomorphisms of a real irreducible SU(2)-module splits into irreducible components. To do so we need certain spectral formulae for decomposing tensor products of real SU(2)-modules. Being standard, proofs of the spectral formulae (Lemmas 4.4–4.6) are ommitted. The interested reader might consult [2].

Let W be a $\mathbf{C}$-vector space with a Hermitian inner product and write $W_{\mathbf{R}}$ for the underlying $\mathbf{R}$-vector space naturally equipped with the complex structure J. The Hermitian inner product induces a symmetric inner product on $W_{\mathbf{R}}$, simply by taking the real part.

If $\mathrm{H}(W)$ denotes the $\mathbf{R}$-vector space of Hermitian endomorphisms on W and $\mathrm{S}(W_{\mathbf{R}})$ the $\mathbf{R}$-vector space of all symmetric endomorphisms on $W_{\mathbf{R}}$, it follows from general considerations above that $\mathrm{H}(W) \subset \mathrm{S}(W_{\mathbf{R}})$, while $\mathbf{C}$-linearity of $A \in \mathrm{H}(W)$ is reflected in $\mathrm{S}(W_{\mathbf{R}})$ by commutation of A and J.

Suppose that W has a real (resp. quaternionic) structure denoted by σ compatible with the Hermitian inner product. Then $\mathrm{H}(W)$ has a regular action of σ such that $A \mapsto \sigma A \sigma$, where A is a Hermitian endomorphism. Hence, we can define the subspaces $\mathrm{H}_{\pm}(W)$ of $\mathrm{H}(W)$ as the set of invariant/anti-invariant Hermitian endomorphisms with respect to σ. The action of σ extends to $\mathrm{S}(W_{\mathbf{R}})$ in the obvious way.

LEMMA 4.1. *If $A \in \mathrm{H}_+(W)$, then real endomorphisms σA and $J\sigma A$ are symmetric endomorphisms on $W_{\mathbf{R}}$.*

PROOF. For simplicity, we assume that σ is a real structure. If σ is a quaternionic structure the proof goes along the same lines.

Let $A \in \mathrm{H}_+(W)$ so that $\sigma A = A\sigma$. Also, denote the Hermitian inner product on W by $(\ ,\ )$, with the convention in which it is $\mathbf{C}$-linear in the first argument, and let $\langle\ ,\ \rangle$ be the induced symmetric inner product on $W_{\mathbf{R}}$. Then, for $u, v \in W \cong W_{\mathbf{R}}$,

$$\begin{aligned}\langle \sigma A u, v\rangle &= \mathrm{Re}(\sigma A u, v) = \mathrm{Re}\overline{(Au, \sigma v)} = \mathrm{Re}(u, A\sigma v)\\ &= \mathrm{Re}(A\sigma v, u) = \mathrm{Re}(\sigma A v, u) = \langle \sigma A v, u\rangle\,.\end{aligned}$$

Therefore, $\sigma A \in \mathrm{S}(W_{\mathbf{R}})$. The proof for $J\sigma A$ is analogous. □

Notice that σA (resp. $J\sigma A$) above is not a Hermitian operator since σ is by definition conjugate-linear. We put

$$\sigma\mathrm{H}_+(W) := \{\sigma A \mid A \in \mathrm{H}_+(W)\} \subset \mathrm{S}(W_{\mathbf{R}})\,,$$

$$J\sigma\mathrm{H}_+(W) := \{J\sigma A \mid A \in \mathrm{H}_+(W)\} \subset \mathrm{S}(W_{\mathbf{R}})\,.$$

A characterisation of these subspaces is given as follows:

LEMMA 4.2. *Let B be a symmetric endomorphism of $W_{\mathbf{R}}$. Then,*

(1) *B belongs to $\sigma\mathrm{H}_+(W)$ if and only if $JB = -BJ$ and $\sigma B\sigma = B$;*
(2) *B belongs to $J\sigma\mathrm{H}_+(W)$ if and only if $JB = -BJ$ and $\sigma B\sigma = -B$.*

PROOF. For B in $\sigma\mathrm{H}_+(W)$ (resp. in $J\sigma\mathrm{H}_+(W)$), there exists $A \in \mathrm{H}_+(W)$ such that $B = \sigma A$ (resp. $J\sigma A$). Writing $BJ,\ \sigma B\sigma$ in terms of A, then commutation relations for $A,\ J,\ \sigma$ yield the implications.

Conversely, condition $JB = -BJ$ implies that B is not Hermitian. Hence, $A := \sigma B$ (resp. $A := J\sigma B$) is Hermitian, for commutation relations between J and σ lead to $AJ = JA$. Invariance under the regular action of σ on $\mathrm{H}(W)$ shows $A \in \mathrm{H}_+(W)$, therefore B belongs to $\sigma\mathrm{H}_+(W)$ (resp. $J\sigma\mathrm{H}_+(W)$). □

Subspaces $\sigma\mathrm{H}_+(W)$ and $J\sigma\mathrm{H}_+(W)$ are orthogonal with respect to the inherited inner product on $\mathrm{S}(W_{\mathbf{R}})$, Then, counting dimensions we have

COROLLARY 4.3. *We have a decomposition of $\mathrm{S}(W_{\mathbf{R}})$:*

$$\mathrm{S}(W_{\mathbf{R}}) = \mathrm{H}_+(W) \oplus \mathrm{H}_-(W) \oplus \sigma\mathrm{H}_+(W) \oplus J\sigma\mathrm{H}_+(W)\,.$$

REMARK 4. As a result, the orthogonal complement of $\mathrm{H}(W)$ in $\mathrm{S}(W_{\mathbf{R}})$ has the induced complex structure.

Let $S^k\mathbf{C}^2$ be the k-th symmetric power of the standard complex $\mathrm{SU}(2)$-module $\mathbf{C}^2$. Since $\mathbf{C}^2$ has an invariant quaternionic structure j, $S^{2k}\mathbf{C}^2$ inherits an invariant real structure $\sigma = j^{2k}$, while $S^{2k+1}\mathbf{C}^2$ is equipped with an induced invariant quaternionic structure j^{2k+1}. We shall denote the standard real $\mathrm{SO}(3)$-module by $\mathbf{R}^3$ and its l-th symmetric power by $S^l\mathbf{R}^3$.

The fundamental relation between real irreducible SU(2)- and SO(3)-modules is as follows.

LEMMA 4.4. *For $k \geqq 2$, $S^k\mathbf{R}^3$ admits the following decomposition:*

$$S^k\mathbf{R}^3 = S_0^k\mathbf{R}^3 \oplus S^{k-2}\mathbf{R}^3$$

where

$$S_0^k\mathbf{R}^3 = (S^{2k}\mathbf{C}^2)^{\mathbf{R}}$$

is the real irreducible SU(2)*-module defined as the σ-invariant real subspace of $S^{2k}\mathbf{C}^2$.*

Once we have identified the real irreducible SU(2)-modules we would like to have a spectral formula for the tensor product. To that end, it is enough to restrict to the real stable subspace of the real structure.

LEMMA 4.5. *For $k \geqq l$, we have*

$$S_0^k\mathbf{R}^3 \otimes S_0^l\mathbf{R}^3 = \bigoplus_{r=0}^{2l} S_0^{k+l-r}\mathbf{R}^3 . \tag{4.1}$$

Any complex irreducible SU(2)-module $S^n\mathbf{C}^2$ can be interpreted as a real module by considering its underlying $\mathbf{R}$-vector space $\mathbf{R}^{2n+2}$. For odd n, this is a real irreducible module. When n is even, this is reducible and we have further splittings into the stable subspaces for the action of the induced real structure.

It will be useful to have a spectral formula for the decomposition of tensor products of the underlying $\mathbf{R}$-vector spaces of a given complex SU(2)-module into real irreducible ones.

LEMMA 4.6. *When we regard $S^{2k}\mathbf{C}^2$ as a real* SU(2)*-module $\mathbf{R}^{4k+2}$, the second symmetric power $S^2\mathbf{R}^{4k+2}$ has the following irreducible decomposition:*

$$S^2\mathbf{R}^{4k+2} = 3\left(\bigoplus_{r=0}^{k} S_0^{2k-2r}\mathbf{R}^3\right) \oplus \left(\bigoplus_{r=0}^{k-1} S_0^{(2k-1)-2r}\mathbf{R}^3\right) . \tag{4.2}$$

When we regard $S^{2k+1}\mathbf{C}^2$ as a real SU(2)*-module $\mathbf{R}^{4k+4}$, the second symmetric power $S^2\mathbf{R}^{4k+4}$ has the following irreducible decomposition:*

$$S^2\mathbf{R}^{4k+4} = 3\left(\bigoplus_{r=0}^{k} S_0^{(2k+1)-2r}\mathbf{R}^3\right) \oplus \left(\bigoplus_{r=0}^{k-1} S_0^{2k-2r}\mathbf{R}^3\right) . \tag{4.3}$$

Applying Corollary 4.3 to the real SU(2)-modules discussed in the previous three lemmas yields

PROPOSITION 4.7.

$$\mathrm{H}_+(S^{2k}\mathbf{C}^2) = \bigoplus_{r=0}^{k} S_0^{2k-2r}\mathbf{R}^3,\ \mathrm{H}_-(S^{2k}\mathbf{C}^2) = \bigoplus_{r=0}^{k-1} S_0^{2k-1-2r}\mathbf{R}^3 ,$$

$$\mathrm{H}_+(S^{2k+1}\mathbf{C}^2) = \bigoplus_{r=0}^{k} S_0^{2k+1-2r}\mathbf{R}^3, \ \mathrm{H}_-(S^{2k+1}\mathbf{C}^2) = \bigoplus_{r=0}^{k} S_0^{2k-2r}\mathbf{R}^3 .$$

5. Rigidity of the real standard map. Let G be a compact Lie group. An irreducible G-module is said to be a *class-one representation of* (G, K), for K a closed subgroup of G, if it contains non-zero K-invariant elements.

Essential at this stage is to prove Proposition 5.3 (and its real invariant counterpart Proposition 5.5). This is a technical result that states in short that if each factor in the normal decomposition of a G-module W is inequivalent as a K-module to any other factor, there is a certain G-orbit in $\mathrm{H}(W)$ which contains all class-one representations of (G, K). Since in our case $\mathrm{H}(W)$ itself is composed of class-one representations only, the G-orbit mentioned earlier fills $\mathrm{H}(W)$.

The proposition has a practical reading: the Hermitian/symmetric operators parametrising the moduli spaces belong to the orthogonal complement in $\mathrm{H}(W)$ to the aforesaid G-orbit, but in the present situation this space is null. Therefore the induced map will be rigid. We use this information to study the real standard map, the outcome naming the section (Theorem 5.4).

A detailed description of the normal decomposition can be found in [5]. Let us sketch the central ideas: Consider the situation described in §2, i.e. $W \subset \Gamma(V)$ is a space of sections of the vector bundle $V \to M,\ M = G/K$ associated to the principal homogeneous bundle $G \to G/K$ with standard fibre the irreducible K-module $V_0 \subset W$. Furthermore, suppose $V \to M$ to be equipped with its canonical connection. Let $f : G/K \to Gr_p(W)$ be the corresponding induced map by $(V \to M,\ W)$. The space of sections W splits into V_0 and its orthogonal complement $N_0 = U_0$. Assume the condition of Lemma 2.1, i.e. $\mathfrak{m}V_0 \subset U_0$ such that the canonical connection and the pull-back connection coincide.

From now on, our considerations will be restricted at a point $o \in M$ for the sake of simplicity. The second fundamental form K at $o \in M$ is an element of $T_o^*M \otimes V_0^* \otimes U_0$ so that for all $X \in T_oM,\ v \in V_0,\ (\mathrm{K}_X(v))_o \in U_0$. The image of this mapping, also designated by B_1, is a well-defined subspace of N_0 and thus gives a further orthogonal decomposition of W as $V_0 \oplus \mathrm{Im}B_1 \oplus (V_0 \oplus \mathrm{Im}B_1)^\perp$. Call $N_1 = (V_0 \oplus \mathrm{Im}B_1)^\perp$ the *first normal subspace*. Applying the connection to the second fundamental form at the point $o \in M$ we have $\nabla\mathrm{K} \in S^2T_o^*M \otimes V_0^* \otimes U_0$ (where symmetrisation follows from Gauss–Codazzi equations and flatness of the connection on $\underline{W}$). If π_1 denotes the orthogonal projection $\pi_1 : W \to N_1$, then B_2 is defined as $\pi_1 \circ \nabla\mathrm{K} \in S^2T_o^*M \otimes V_0^* \otimes N_1$, and we have $W = V_0 \oplus \mathrm{Im}B_1 \oplus \mathrm{Im}B_2 \oplus N_2$ where N_2 is the *second normal subspace*. Recursively, $B_p = \pi_{p-1} \circ \nabla^{p-1}\mathrm{K} \in S^pT_o^*M \otimes V_0^* \otimes N_{p-1}$. This reiterative process leads to

$$W = V_0 \oplus \mathrm{Im}\, B_1 \oplus \mathrm{Im}\, B_2 \oplus \cdots \oplus \mathrm{Im}\, B_n \oplus N_n .$$

If $N_n = 0$ this is called the *normal decomposition of W with respect to V_0*.

Let us enunciate without proof two results regarding the normal decomposition which are needed in the sequel to establish Proposition 5.3.

PROPOSITION 5.1 ([14, Prop. 7.7]). *If W is an irreducible G-module, then for any K-module, $V_0 \subset W$ there exists a positive integer n such that $N_n = 0$, i.e.*

$$W = V_0 \oplus \operatorname{Im} B_1 \oplus \cdots \oplus \operatorname{Im} B_n \tag{5.1}$$

which is a normal decomposition of (W, V_0).

PROPOSITION 5.2 ([14, Prop. 7.8]). *Let W be a G-module and $V_0 \subset W$ a K-module. Suppose that (W, V_0) has a normal decomposition. Assume that each term in the decomposition* (5.1) *shares no common K-irreducible factor with any other term in the decomposition. Let T be a non-negative Hermitian endomorphism of W which satisfies $(Tgv_1, Tgv_2) = (v_1, v_2)$ for all $g \in G$, $v_1, v_2 \in V_0$. Then, if T is K-equivariant, $T = Id_W$.*

REMARK 5. See also Lemma 4.2 in [5].

Then, we can state the following

PROPOSITION 5.3. *Let $W = H^0(\mathbf{C}P^1, \mathcal{O}(k))$ and V_0 the K-module regarded as the standard fibre for $\mathcal{O}(k) \to \mathbf{C}P^1$. Then,* $\mathrm{GH}(V_0, V_0) = \mathrm{H}(W)$.

PROOF. By Borel–Weil theorem, W is identified with the SU(2)-module $S^k\mathbf{C}^2$ and, using Lemma 2.1, V_0 can be regarded as a subspace of W. The space W decomposes under the U(1)-action as

$$W = \mathbf{C}_{-k} \oplus \mathbf{C}_{-k+2} \oplus \cdots \oplus \mathbf{C}_k \,,$$

where $\mathbf{C}_l$ denotes the irreducible U(1)-module of weight l. Indeed, this is the normal decomposition by Proposition 5.1 where $V_0 = \mathbf{C}_{-k}$.

Let H be a class-one subrepresentation of (G, K) in $\mathrm{H}(W)$. Suppose that $H \not\subset \mathrm{GH}(V_0, V_0)$. Then, by a standard argument, we can assume that $H \perp \mathrm{GH}(V_0, V_0)$. Since H is a class-one representation, there exists a non-zero $C \in H$ such that $kCk^{-1} = C$ for all $k \in K$. It follows from the orthogonality assumption that

$$\begin{aligned} 0 =&(C, gH(v_1, v_2))_{\mathrm{H}(W)} = (C, H(gv_1, gv_2))_{\mathrm{H}(W)} \\ =&\frac{1}{2}\left\{(Cgv_1, gv_2)_W + (Cgv_2, gv_1)_W\right\}\,, \end{aligned}$$

for arbitrary $g \in G$ and $v_1, v_2 \in V_0 \subset W$. Polarisation gives

$$0 = (Cgv_1, gv_2), \quad g \in G,\ v_1, v_2 \in V_0\,.$$

If C is sufficiently small, then $Id + C > 0$ and so, we can define a positive Hermitian operator T satisfying $T^2 = Id + C$. Then we have

$$(Tgv_1, Tgv_2) = (v_1, v_2) \quad g \in G,\ v_1, v_2 \in V_0\,.$$

Since T is also K-equivariant, Proposition 5.2 yields that $T = Id$ and so, $C = 0$, which is a contradiction. Hence, every class-one subrepresentation of (G, K) in $\mathrm{H}(W)$ is included in $\mathrm{GH}(V_0, V_0)$. However, it follows from the Clesbsch–Gordan formulae that $\mathrm{H}(W)$ is composed by class-one representation of (G, K) only, therefore $\mathrm{GH}(V_0, V_0) = \mathrm{H}(W)$. □

REMARK 6. A more general version of our Proposition 5.3 can be found in [14, Proposition 7.9]. Our proof is essentially the same with the obvious particularisations.

We shall prove the following interesting result.

THEOREM 5.4. *Let $W = S^{2k}\mathbf{C}^2$ such that $W^{\mathbf{R}} = S_0^k\mathbf{R}^3 \cong \mathbf{R}^{2k+1}$. If $f : \mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_{2k-1}(\mathbf{R}^{2k+1})$ is a holomorphic isometric embedding of degree $2k$, then f is the standard map by $W^{\mathbf{R}}$ up to gauge equivalence.*

Before proving Theorem 5.4, let us clarify the construction of the mapping $f : \mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_{2k-1}(\mathbf{R}^{2k+1})$ from the vector bundle viewpoint.

If we regard the complex projective line as the symmetric space G/K where $G = \mathrm{SU}(2)$ and $K = \mathrm{U}(1)$, then by Borel–Weil theorem the space of sections $\Gamma(\mathcal{O}(2k))$ becomes a G-module such that $W = H^0(\mathbf{C}P^1; \mathcal{O}(2k)) \cong S^{2k}\mathbf{C}^2$. The decomposition of $S^{2k}\mathbf{C}^2$ into irreducible $\mathrm{U}(1)$-modules is as follows:

$$S^{2k}\mathbf{C}^2 = \bigoplus_{r=0}^{2k} \mathbf{C}_{2k-2r}\,. \tag{5.2}$$

The typical fibre of $\mathcal{O}(2k) \to \mathbf{C}P^1$ is regarded as a subspace $\mathbf{C}_{-2k}$ in the decomposition by Lemma 2.1.

Since W has an invariant real structure, we have an invariant real subspace denoted by $W^{\mathbf{R}} = (S^{2k}\mathbf{C}^2)^{\mathbf{R}} \cong S_0^k\mathbf{R}^3$ of real dimension $2k+1$. The real structure descends to the splitting (5.2) but now each irreducible $\mathrm{U}(1)$-module is not invariant under the real structure, but $\sigma(\mathbf{C}_{2k-2r}) = \mathbf{C}_{-2k+2r}$. Therefore for each $r = 0, \dots, k$ the space $(\mathbf{C}_{2k-2r} \oplus \mathbf{C}_{-2k+2r})$ is stable under the real structure and decomposes in two real isomorphic irreducible $\mathrm{U}(1)$-modules, denoted by $(\mathbf{C}_{2k-2r} \oplus \mathbf{C}_{-2k+2r})^{\mathbf{R}}$, such that (5.2) would be rewritten as

$$S_0^k\mathbf{R}^3 = \bigoplus_{r=0}^{2k} (\mathbf{C}_{2k-2r} \oplus \mathbf{C}_{-2k+2r})^{\mathbf{R}}\,. \tag{5.3}$$

This implies that $\mathcal{O}(2k) \to \mathbf{C}P^1$ is globally generated by $W^{\mathbf{R}}$. Thus, we can define a *real standard map* $f_0 : \mathbf{C}P^1 \to Gr_{2k-1}(\mathbf{R}^{2k+1})$ by $W^{\mathbf{R}}$, which turns out to be a holomorphic isometric embedding of degree $2k$ by Lemma 2.3. Using the inner product on $W^{\mathbf{R}}$ and the fibre-metric on $\mathcal{O}(2k) \to \mathbf{C}P^1$, it is possible to define the adjoint of the evaluation which at the identity of G/K determines a mapping $ev_{[e]}^* : \mathcal{O}(2k) \to \underline{W}^{\mathbf{R}}$ whose image is just $(\mathbf{C}_{2k} \oplus \mathbf{C}_{-2k})^{\mathbf{R}}$.

Within this framework we have a real version of Proposition 5.3, which is the core of the proof of Theorem 5.4:

PROPOSITION 5.5. *Let $W = H^0(\mathbf{C}P^1, \mathcal{O}(2k))$ and V_0 the K-module regarded as the standard fibre for $\mathcal{O}(2k) \to \mathbf{C}P^1$. Then,* $\mathrm{GS}(V_0, V_0) = \mathrm{S}(W^{\mathbf{R}})$.

PROOF. Equation (5.3) gives the normal decomposition of $W^{\mathbf{R}}$ where now $V_0 = (\mathbf{C}_{-2k} \oplus \mathbf{C}_{2k})^{\mathbf{R}}$. The space of symmetric endomorphisms of $W^{\mathbf{R}}$ can be identified by decomposing first the tensor product using Lemma 4.1, and identifying the symmetric components

$$\mathrm{S}(W^{\mathbf{R}}) = \bigoplus_{r=0}^{k} S_0^{4k-4r}\mathbf{R}^3 \subset \otimes^2 W^{\mathbf{R}} = \bigoplus_{r=0}^{2k} S_0^{4k-2r}\mathbf{R}^3 .$$

Notice that all these modules are class-one representations. Then, a similar argument as the one in the proof of Proposition 5.3 yields the desired result. □

We can now proceed to prove Theorem 5.4.

PROOF. Consider the real standard map by the holomorphic line bundle $\mathcal{O}(2k) \to \mathbf{C}\mathrm{P}^1$ and $W^{\mathbf{R}}$ as depicted above. Therefore by Proposition 5.5, $\mathrm{S}(W^{\mathbf{R}}) = \mathrm{GS}(V_0, V_0)$ and replacing $\mathbf{R}^{n+2}$ by $W^{\mathbf{R}}$ in Theorem 2.4 the real standard map admits no deformations as holomorphic isometric embedding of degree $2k$ into $Gr_{2k-1}(\mathbf{R}^{2k+1})$. □

REMARK 7. If the target space is replaced by a higher-dimensional Grassmannian including $Gr_{2k-1}(\mathbf{R}^{2k+1})$ as a totally geodesic submanifold the resulting moduli space could be non-trivial. This situation will be discussed in the next section.

6. Moduli space by gauge equivalence. We undertake now the task of giving an accurate description of the moduli space of holomorphic isometric embeddings $\mathbf{C}\mathrm{P}^1 \to Gr_p(W)$ up to gauge equivalence. Our strategy will be to capitalise on the representation-theoretic formulae of §4 to explicitly determine the subspaces of linear operators in $\mathrm{S}(W)$ which specify the moduli. Such subspaces are sharply characterised by condition (III) in Theorem 2.4. This is achieved after a sequence of stepping-stone results culminating in Lemma 6.2 and its Corollary, which allows to compute the moduli dimension.

As indicated by condition (IV) in Theorem 2.4, the gauge equivalence relation is to be taken into account to obtain the moduli space and to give a geometric meaning to its compactification in the natural L^2-topology. A qualitative description of these spaces is given in Theorem 6.4.

Let W be the space of holomorphic sections of $\mathcal{O}(k) \to \mathbf{C}\mathrm{P}^1$ which, by Borel–Weil theorem, is identified with the SU(2)-module $S^k\mathbf{C}^2$. Equation (5.2) gives a weight decomposition of W with respect to U(1). When $\mathcal{O}(k) \to \mathbf{C}P^1$ is regarded as the homogeneous line bundle $\mathrm{SU}(2) \times_{\mathrm{U}(1)} V_0 \to \mathbf{C}\mathrm{P}^1$, then V_0 is identified with the U(1)-irreducible subspace $\mathbf{C}_{-k}$ of W by Lemma 2.1.

In order to apply Theorem 2.4 we shall regard the universal quotient bundle as a real vector bundle of rank 2. Following the generalisation of do Carmo–Wallach theory, we must determine the subspaces $\mathrm{GS}(V_0, V_0)$ and $\mathrm{GS}(\mathfrak{m}V_0, V_0)$ of $\mathrm{S}(W)$.

From now on V_0 and W shall stand either for the complex modules or for their underlying $\mathbf{R}$-vector spaces whenever the meaning is clear, avoiding the heavier notation $(V_0)_{\mathbf{R}}$ or $W_{\mathbf{R}}$. In the remaining sections, we will adopt this convention.

Since $\mathrm{GH}(V_0, V_0)$ is a proper subspace of $\mathrm{GS}(V_0, V_0)$, we have that $\mathrm{H}(W) \subset \mathrm{GS}(V_0, V_0)$. We must determine the intersection between $\mathrm{GS}(V_0, V_0)$ and subspaces $\sigma\mathrm{H}_+(W) \oplus J\sigma\mathrm{H}_+(W)$

appearing in Corollary 4.3. The same is true for the intersection $\mathrm{GS}(\mathfrak{m}V_0, V_0)$ with $\sigma \mathrm{H}_+(W) \oplus J\sigma\mathrm{H}_+(W)$ as we shall consider immediately.

LEMMA 6.1. $\mathfrak{m}V_0 = \mathbf{C}_{-k-2}$.

PROOF. By the decomposition of $S^2\mathbf{C}^2$ into irreducible U(1)-modules $S^2\mathbf{C}^2 = \mathbf{C}_2 \oplus \mathbf{C}_0 \oplus \mathbf{C}_{-2}$ and using the real structure we have $(S^2\mathbf{C}^2)^{\mathbf{R}} \cong \mathfrak{su}(2)$, $(\mathbf{C}_0)^{\mathbf{R}} \cong \mathfrak{u}(1)$ therefore $(\mathbf{C}_2 \oplus \mathbf{C}_{-2})^{\mathbf{R}} \cong \mathfrak{m}$. Then,

$$\mathfrak{m} \otimes V_0 = (\mathbf{C}_2 \oplus \mathbf{C}_{-2}) \otimes \mathbf{C}_{-k} = \mathbf{C}_{-k+2} \oplus \mathbf{C}_{-k-2}\,.$$

The action of $\mathfrak{m}$ on V_0 is then obtained by projecting $\mathfrak{m} \otimes V_0$ back to $S^k\mathbf{C}^2$. Therefore

$$\mathfrak{m}V_0 = (\mathfrak{m} \otimes V_0) \cap S^k\mathbf{C}^2 = \mathbf{C}_{-k+2}\,.$$

$\square$

LEMMA 6.2. $\mathrm{GS}(\mathfrak{m}V_0, V_0) \cap \sigma\mathrm{H}_+(W) \oplus J\sigma\mathrm{H}_+(W)$ *is the highest weight representations of SU*(2) *appeared in Proposition* 4.7.

PROOF. Let u_{-k} and u_{-k+2} be unitary bases for the complex one-dimensional U(1)-modules $V_0 = \mathbf{C}_{-k}$ and $\mathfrak{m}V_0 = \mathbf{C}_{-k+2}$, respectively. Then, the space $\mathrm{H}(\mathfrak{m}V_0, V_0) \equiv \mathrm{H}(\mathbf{C}_{-k+2}, \mathbf{C}_{-k})$ is the real span of

$$2\mathrm{H}(u_{-k+2}, u_{-k}) = u_{-k+2} \otimes (\cdot, u_{-k})_W + u_{-k} \otimes (\cdot, u_{-k+2})_W$$

where $(\,,\,)_W$ denotes the Hermitian inner product on $S^k\mathbf{C}^2$. When $\mathbf{C}_{-k}$ and $\mathbf{C}_{-k+2}$ are regarded as their underlying two-dimensional $\mathbf{R}$-vector spaces $\mathbf{R}^2_k$ and $\mathbf{R}^2_{k-2}$, real bases are given respectively by $\{u_{-k}, Ju_{-k}\}$ and $\{u_{-k+2}, Ju_{-k+2}\}$ where J is the almost complex structure induced by the multiplication by the imaginary unit. Using these real bases the complex form $2\mathrm{H}(u_{-k+2}, u_{-k})$ can be rewritten as a real operator

$$\begin{aligned}2\mathrm{H}(u_{-k+2}, u_{-k})|\mathbf{R} = {}& u_{-k+2} \otimes \langle \cdot, u_{-k}\rangle_W + Ju_{-k+2} \otimes \langle \cdot, Ju_{-k}\rangle_W \\ & + u_{-k} \otimes \langle \cdot, u_{-k+2}\rangle_W + Ju_{-k} \otimes \langle \cdot, Ju_{-k+2}\rangle_W\end{aligned}$$

where $\langle\,,\,\rangle_W$ is the inner product on $W_{\mathbf{R}}$ induced from the Hermitian inner product on W. Write the basis for $\mathrm{S}(\mathfrak{m}V_0, V_0) \equiv \mathrm{S}(\mathbf{R}^2_{-k+2}, \mathbf{R}^2_{-k})$ as $\{\mathrm{S}(u_{-k+2}, u_{-k}),\ \mathrm{S}(Ju_{-k+2}, u_{-k}),\ \mathrm{S}(u_{-k+2}, Ju_{-k}),\ \mathrm{S}(Ju_{-k+2}, Ju_{-k})\}$, e.g. ,

$$2\mathrm{S}(u_{-k+2}, u_{-k}) = u_{-k+2} \otimes \langle \cdot, u_{-k}\rangle_W + u_{-k} \otimes \langle \cdot, u_{-k+2}\rangle_W\,, \quad \text{etc}\,.$$

Comparing both equations we have that

$$\mathrm{H}(u_{-k+2}, u_{-k})|\mathbf{R} = \mathrm{S}(u_{-k+2}, u_{-k}) + \mathrm{S}(Ju_{-k+2}, Ju_{-k})\,.$$

Analogously,

$$\mathrm{H}(u_{-k+2}, iu_{-k})|\mathbf{R} = \mathrm{S}(u_{-k+2}, Ju_{-k}) - \mathrm{S}(Ju_{-k+2}, u_{-k})\,.$$

Let us define a new elements $\{X, Y\}$

$$\begin{aligned} X &= \mathrm{S}(u_{-k+2}, u_{-k}) - \mathrm{S}(Ju_{-k+2}, Ju_{-k})\,, \\ Y &= \mathrm{S}(u_{-k+2}, Ju_{-k}) + \mathrm{S}(Ju_{-k+2}, u_{-k})\,. \end{aligned}$$

$X, Y \in \mathrm{S}(W_{\mathbf{R}})$ are orthogonal to the subspace of Hermitian matrices $\mathrm{H}(W) \subset \mathrm{S}(W_{\mathbf{R}})$, therefore they belong to $\sigma H_+(W) \oplus J\sigma H_+(W)$ according to Corollary 4.3.

Let us consider the contragredient action of the structure map σ on X, the case of Y being analogous. Firstly,

$$\sigma\left(u \otimes \langle\cdot, v\rangle_W\right)\sigma = \sigma u \otimes \langle\sigma\cdot, v\rangle_W = \sigma u \otimes \langle\cdot, \sigma v\rangle_W$$

and as such $\sigma \mathrm{S}(u, v)\sigma = \mathrm{S}(\sigma u, \sigma v)$.

Secondly, the U(1)-modules $\mathbf{C}_i$ are not σ-invariant but $\sigma(\mathbf{C}_{\pm i}) = \mathbf{C}_{\mp i}$, for all i that is, $\sigma u_{\pm i} = u_{\mp i}$. which, together with conjugate-linearity of the structure map yields $\sigma(\mathbf{R}^2_{\pm i}) = \mathbf{R}^2_{\mp i} : \{u_{\pm i}, Ju_{\pm i}\} \mapsto \{u_{\mp i}, -Ju_{\mp i}\}$. Hence we have

$$\begin{aligned} X^\sigma &= \sigma X \sigma = \mathrm{S}(\sigma u_{-k+2}, \sigma u_{-k}) - \mathrm{S}(\sigma J u_{-k+2}, \sigma J u_{-k}) \\ &= \mathrm{S}(u_{k-2}, u_k) - \mathrm{S}(Ju_{k-2}, Ju_k)\,. \end{aligned}$$

This is not an element of $\mathrm{S}(\mathfrak{m}V_0, V_0) \equiv \mathrm{S}(\mathbf{R}^2_{-k+2}, \mathbf{R}^2_{-k})$ but $X^\sigma \in \mathrm{S}(\mathbf{R}^2_{k-2}, \mathbf{R}^2_k)$. Note that we can find $g \in \mathrm{SU}(2)$ such that $\mathrm{S}(u_{k-2}, u_k) = \mathrm{S}(gu_{-k+2}, gu_{-k}) = g \cdot \mathrm{S}(u_{-k+2}, u_{-k}) \in \mathrm{GS}(\mathfrak{m}V_0, V_0)$ up to a sign. Let us add $Y^\sigma = \mathrm{S}(u_{k-2}, Ju_k) + \mathrm{S}(Ju_{k-2}, u_k)$ for the sake of completeness.

The preceding argument also shows that a subspace of $\mathrm{GS}(\mathfrak{m}V_0, V_0)$ is spanned by $\{S(u_{k-2}, u_k),\ S(u_{k-2}, Ju_k),\ S(Ju_{k-2}, u_k),\ S(Ju_{k-2}, Ju_k)\}$.

Moreover, using the characterisation given in Lemma 4.2 we have

$$X + X^\sigma \in \sigma\mathrm{H}_+(W), \qquad X - X^\sigma \in J\sigma\mathrm{H}_+(W)\,.$$

The same inclusions are also true for $Y \pm Y^\sigma$.

From the expression of the action of σ on $\mathrm{H}(u, v)$

$$\begin{aligned} \sigma \cdot \mathrm{H}(u, v) &= \sigma\left(u \otimes (\cdot, v)_W + v \otimes (\cdot, u)_W\right)\sigma = \sigma u \otimes (\sigma\cdot, v)_W + \sigma v \otimes (\sigma\cdot, u)_W \\ &= \sigma u \otimes \overline{(\cdot, \sigma v)_W} + \sigma v \otimes \overline{(\cdot, \sigma u)_W}\,, \end{aligned}$$

it is easy to write $X \pm X^\sigma$ back in terms of Hermitian operators as

$$X \pm X^\sigma = \sigma \cdot (\mathrm{H}(u_{k-2}, u_{-k}) \pm \mathrm{H}(u_{-k+2}, u_k))\,|\mathbf{R}\,.$$

The toral action of a U(1)-element of SU(2) on $u_{\pm k},\ u_{\pm(k-2)}$ yields

$$\exp(i\theta) \cdot u_{\pm k} = \exp(\pm ik\theta)u_{\pm k}, \qquad \exp(i\theta) \cdot u_{\pm(k-2)} = \exp(\pm i(k-2)\theta)u_{\pm(k-2)}$$

and as such, $X \pm X^\sigma$ (considered as the Hermitian operators above) have weight $\pm(2k-2)$. However, from Corollary 4.7 we know that the only component in the real decomposition of $\sigma\mathrm{H}_+(W)$ and $J\sigma\mathrm{H}_+(W)$ (both isomorphic to $\mathrm{H}_+(W)$) which can host such a vector is the top term $S^k_0\mathbf{R}^3$ on each space. Therefore

$$\mathrm{GS}(\mathfrak{m}V_0, V_0) \cap \sigma\mathrm{H}_+(W) = S^k_0\mathbf{R}^3 \qquad (\text{ resp. for } J\sigma\mathrm{H}_+(W)\ )\,.$$

And as a result

$$\mathrm{GS}(\mathfrak{m}V_0, V_0) = \mathrm{H}(W) \oplus S^k_0\mathbf{R}^3 \oplus S^k_0\mathbf{R}^3\,.$$

□

In other words, we obtain

COROLLARY 6.3. *The orthogonal complement to* $\mathrm{GS}(\mathfrak{m}V_0, V_0) \oplus \mathbf{R}\, Id$ *in* $\mathrm{S}(W)$ *is*

$$2 \bigoplus_{r=1}^{k \geq 2r} S_0^{k-2r} \mathbf{R}^3 .$$

This follows from applying the previous lemma to the explicit expressions for the components of $\mathrm{S}(W)$ as described in Proposition 4.7, and accounts for the space of symmetric operators T described by the second relation in (2.1), i.e. condition (III) in Theorem 2.4.

REMARK 8. The first condition in (2.1) is for all our purposes inessential. Let $\mathrm{GS}_0(V_0, V_0)$ be the orthogonal complement of the G-invariant, irreducible subrepresentation generated by the identity in $\mathrm{GS}(V_0, V_0)$. We denote by $\mathrm{S}_0(W)$ the set of tracefree symmetric operators on W with the induced inner product from $\mathrm{S}(W)$. Then,

$$\mathrm{GS}_0(V_0, V_0) \subset \mathrm{GS}(\mathfrak{m}V_0, V_0) ,$$

which stems from an analogous result to Lemma 6.2 applied to $\mathrm{GS}_0(V_0, V_0)$. The proof is equivalent, changing the weight $\pm(2k-2)$ by $\pm 2k$ in the crucial final step.

Condition (III) in Theorem 2.4 is fulfilled by the family of operators in Corollary 6.3 (see remark above) thus accounting for all holomorphic embeddings $f : \mathbf{C}P^1 \hookrightarrow Gr_p(\mathbf{R}^{p+2})$ up to possible degeneracies. Quantitative information about the moduli (i.e. its dimension) can therefore be derived from the Corollary:

$$\dim_{\mathbf{R}} \mathcal{M}_k = k(k-1) . \tag{6.1}$$

The following theorem summarises the qualitative information about the moduli space and gives a neat geometric interpretation to its compactification.

THEOREM 6.4. *If* $f : \mathbf{C}P^1 \hookrightarrow Gr_n(\mathbf{R}^{n+2})$ *is a full holomorphic isometric embedding of degree* k, *then* $n \leq 2k$.

Let $\mathcal{M}_k$ *be the moduli space of full holomorphic isometric embeddings of degree* k *of the complex projective line into* $Gr_{2k}(\mathbf{R}^{2k+2})$ *by the gauge equivalence of maps. Then,* $\mathcal{M}_k$ *can be regarded as an open bounded convex body in* $2 \bigoplus_{r=1}^{k \geqq 2r} S_0^{k-2r} \mathbf{R}^3$.

Let $\overline{\mathcal{M}_k}$ *be the closure of the moduli* $\mathcal{M}_k$ *by the inner product. Boundary points of* $\overline{\mathcal{M}_k}$ *describe those maps whose images are included in some totally geodesic submanifold* $Gr_p(\mathbf{R}^{p+2})$ *of* $Gr_{2k}(\mathbf{R}^{2k+2})$, *where* $p < 2k$.

The totally geodesic submanifold $Gr_p(\mathbf{R}^{p+2})$ *can be regarded as the common zero set of some sections of* $Q \to Gr_{2k}(\mathbf{R}^{2k+2})$, *which belongs to* $\mathbf{R}^{2k+2}$.

PROOF. The restriction $n \leq 2k$ follows from (I) in Theorem 2.4 and Borel–Weil theorem.

It is evident from (III) in Theorem 2.4 that $\mathrm{GS}(\mathfrak{m}V_0, V_0)^{\perp}$ is a parametrisation of the space of full holomorphic isometric embeddings $f : \mathbf{C}P^1 \hookrightarrow Gr_{2k}(\mathbf{R}^{2k+2})$ of degree k.

Positivity of T being guaranteed by fullness, we can apply the original do Carmo–Wallach argument [5, §5.1], to conclude that $\mathcal{M}_k$ is a bounded connected *open* convex body in $\mathrm{H}(W)$ with the topology induced by the L^2 scalar product.

Under the natural compactification in the L^2-topology, the boundary points correspond to operators T which are not positive definite, but positive semi-definite. It follows from (IV) in Theorem 2.4 that each of these operators defines in turn a full holomorphic isometric embedding $\mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_p(\mathbf{R}^{p+2})$, of degree k with $p = 2k - \dim \operatorname{Ker} T$, whose target embeds in $Gr_{2k}(\mathbf{R}^{2k+2})$ as a totally geodesic submanifold. The image Z of the embedding $Gr_p(\mathbf{R}^{p+2}) \hookrightarrow Gr_{2k}(\mathbf{R}^{2k+2})$ is determined by the common zero-set of sections in Ker T. □

7. Moduli space by image equivalence. The moduli space $\mathcal{M}_k$ has a natural complex structure induced by that on $Q \to Gr_{2k}(\mathbf{R}^{2k+2})$ which coincides with the one in Remark 4. Hence, $\mathcal{M}_k$ can be regarded as holomorphically included in the $\mathbf{C}$-vector space $\bigoplus_{r=1}^{k\geq 2r} S^{2k-4r}\mathbf{C}^2$. We can show that the centraliser of the holonomy group acts on $\mathcal{M}_k$ with weight $-k$. Hence we have

THEOREM 7.1. *Let $\mathbf{M}_k$ be the moduli space of holomorphic isometric embeddings of the complex projective line into $Gr_{2k}(\mathbf{R}^{2k+2})$ of degree k by the image equivalence of maps. Then we have $\mathbf{M}_k = \mathcal{M}_k/S^1$.*

PROOF. Assume two full holomorphic isometric embeddings $\mathbf{C}\mathrm{P}^1 \hookrightarrow Gr_{2k}(\mathbf{R}^{2k+2})$ of degree k to be image equivalent. They may represent distinct points in $\mathcal{M}_k$. By definition of image equivalence, there is an isometry ψ of $Gr_{2k}(\mathbf{R}^{2k+2})$ such that $f_2 = \psi \circ f_1$, then $f_2^*Q = f_1^*\tilde{\psi}Q$ as sets. Using the natural identifications ϕ_1, ϕ_2 of (IV) in Theorem 2.4 we introduce new bundle isomorphisms $\mathcal{O}(k) \to f_2^*Q$ defined by $\tilde{\psi} \circ \phi_1$ and ϕ_2. Hence, we have a gauge transformation $\phi_2^{-1}\tilde{\psi}\phi_1$ on the line bundle $\mathcal{O}(k) \to M$ preserving the metric and the connection. By connectedness of $\mathbf{C}\mathrm{P}^1$ such a gauge transformation is regarded as an element of the centraliser of the holonomy group of the connection in the structure group of V, i.e. $\mathrm{U}(1) \equiv S^1$ acting with weight $-k$ on the standard fibre $V_0 \cong \mathbf{C}_{-k}$. Modding out the S^1-action yields the true moduli space by image equivalence $\mathbf{M}_k$. □

REMARK 9. The moduli space $\mathcal{M}_k$ has a complex structure (see remark in §4) and a metric induced by the inner product both preserved by the S^1-action. Hence, it is a Kähler manifold together with an S^1-action preserving the Kähler structure. Therefore, $\mathcal{M}_k$ is naturally equipped with a moment map $\mu : \mathcal{M}_k \to \mathbf{R}$ expressed as $\mu = |T|^2$.

COROLLARY 7.2. *There exists a one-parameter family $\{f_t\}$, $t \in [0, 1]$, of* $\mathrm{SU}(2)$*-equivariant image-inequivalent holomorphic isometric embeddings of even degree of* $\mathbf{C}\mathrm{P}^1$ *into complex quadrics, where f_0 corresponds to the standard map and f_1 is the real standard map.*

PROOF. The moduli space by gauge equivalence $\mathcal{M}_k$ sits in $\bigoplus_{r=1}^{k\geq 2r} S^{2k-4r}\mathbf{C}^2$. For even k this last expression includes the trivial representation $\mathbf{C}$, which using the real structure can be described as $\mathbf{C} = \mathbf{R}\sigma \oplus \mathbf{R}J\sigma$. Let $C \in \mathbf{C} \subset \bigoplus_{r=1}^{k\geq 2r} S^{2k-4r}\mathbf{C}^2$. If it is small enough, then by Theorem 2.4, $Id + C$ determines a holomorphic isometric embedding into $Gr_{2k}(\mathbf{R}^{2k+2})$. The group SU(2) acts on each component of $Id + C$ trivially, so the associated holomorphic isometric embedding is SU(2)-equivariant. The S^1 action of the centraliser of the holonomy group acts on $\mathbf{C}$ with weight $-k$ (see proof of Theorem 7.1) therefore, taking quotient by the S^1-action, we obtain a half-open segment parametrising the described maps, which becomes a closed segment under the natural compactification in the L^2-topology. Let $C = t\sigma + sJ\sigma$. Then we can show that $Id + C$ is positive if and only if $t^2 + s^2 < 1$. Suppose that $t^2 + s^2 = 1$. Then $(t + sJ)\sigma$ is also an invariant real structure on $S^{2k}\mathbf{C}^2$. Hence we may consider only the case that $t = 1$ and $s = 0$. Since the kernel of $Id + \sigma$ is $JW^{\mathbf{R}}$, Theorem 2.4 implies that $Id + \sigma$ determines a totally geodesic submanifold $Gr_{2k-1}(\mathbf{R}^{2k+1})$ of $Gr_{4k}(\mathbf{R}^{4k+2})$ and a holomorphic isometric embedding into the submanifold $Gr_{2k-1}(\mathbf{R}^{2k+1})$ represented by $2\,Id_{W^{\mathbf{R}}}$. This map is nothing but the real standard map by $W^{\mathbf{R}}$, because constant multiples of the identity give the same subspace of $W^{\mathbf{R}}$. □

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DEPARTMENT OF GEOMETRY AND TOPOLOGY
FACULTY OF MATHEMATICAL SCIENCES
UNIVERSITY OF VALENCIA
C. DR MOLINER, 50, BURJASSOT
46100, VALENCIA
SPAIN

E-mail address: oscar.macia@uv.es

DEPARTMENT OF GENERAL EDUCATION
KURUME NATIONAL COLLEGE OF TECHNOLOGY
KURUME, FUKUOKA 830–8555
JAPAN

E-mail address: masaro@kurume-nct.ac.jp

DEPARTMENT OF MATHEMATICS
MEIJI UNIVERSITY
HIGASHI-MITA, TAMA-KU
KAWASAKI-SHI
KANAGAWA 214–8571
JAPAN

E-mail address: yasunaga@meiji.ac.jp