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GAUSS MAPS OF TORIC VARIETIES

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Abstract. We investigate Gauss maps of (not necessarily normal) projective toric varieties over an algebraically closed field of arbitrary characteristic. The main results are as follows: (1) The structure of the Gauss map of a toric variety is described in terms of combinatorics in any characteristic. (2) We give a developability criterion in the toric case. In particular, we show that any toric variety whose Gauss map is degenerate must be the join of some toric varieties in characteristic zero. (3) As applications, we provide two constructions of toric varieties whose Gauss maps have some given data (e.g., fibers, images) in positive characteristic.

1. Introduction. Let $X \subseteq \mathbb{P}^N$ be an n-dimensional projective variety over an algebraically closed field $\Bbbk$ of arbitrary characteristic. The *Gauss map* γ of X is defined as a rational map

$$\gamma : X \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)\,,$$

which sends each smooth point $x \in X$ to the embedded tangent space $\mathbb{T}_x X$ of X at x in $\mathbb{P}^N$. The Gauss map is a classical subject and has been studied by many authors. For example, it is well known that a general fiber of the Gauss map γ is (an open subset of) a linear subvariety of $\mathbb{P}^N$ in characteristic zero (P. Griffiths and J. Harris [13, (2.10)], F. L. Zak [22, I, 2.3. Theorem (c)]; S. L. Kleiman and R. Piene gave another proof in terms of the projective dual [16, pp. 108–109]). The linearity of general fibers of γ also holds in arbitrary characteristic if γ is separable [11, Theorem 1.1]. We denote by $\delta_\gamma(X)$ the dimension of a general fiber of γ, and call it the *Gauss defect* of X (see [7, 2.3.4]). The Gauss map γ is said to be *degenerate* if $\delta_\gamma(X) > 0$.

In this paper, we investigate the Gauss map of *toric* $X \subseteq \mathbb{P}^N$; more precisely, we consider a (not necessarily normal) toric variety $X \subseteq \mathbb{P}^N$ such that the action of the torus on X extends to the whole space $\mathbb{P}^N$. It is known that such X is projectively equivalent to a projective toric variety X_A associated to a finite subset A of a free abelian group M (see [12, Ch. 5, Proposition 1.5]). The construction of X_A is as follows.

Let M be a free abelian group of rank n and let $\Bbbk[M] = \bigoplus_{u \in M} \Bbbk z^u$ be the group ring of M over $\Bbbk$. We denote by T_M the algebraic torus $\operatorname{Spec} \Bbbk[M]$. For a finite subset $A = \{u_0, \dots, u_N\} \subseteq M$, we define the toric variety X_A to be the closure of the image of the morphism

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$$\varphi_A : T_M \to \mathbb{P}^N \ : \ t \mapsto [z^{u_0}(t) : \cdots : z^{u_N}(t)] \,. \tag{1.1}$$

We set $\langle A - A \rangle \subseteq M$ (resp. $\langle A - A \rangle_{\Bbbk} \subseteq M_{\Bbbk} := M \otimes_{\mathbb{Z}} \Bbbk$) to be the subgroup of M (reap. the $\Bbbk$-vector subspace of $M_{\Bbbk}$) generated by $A - A := \{u - u' \mid u, u' \in A\}$. The algebraic torus $T_{\langle A-A \rangle}$ acts on X_A, and $T_{\langle A-A \rangle}$ is contained in X_A as an open dense orbit. In this paper, we call such X_A a *projectively embedded toric variety*, or simply *toric variety*.

We denote by $\mathrm{Aff}(A)$ (resp. $\mathrm{Aff}_{\Bbbk}(A)$) the affine sublattice of M (resp. the $\Bbbk$-affine subspace of $M_{\Bbbk}$) spanned by A. In other words, $\mathrm{Aff}(A)$ (resp. $\mathrm{Aff}_{\Bbbk}(A)$) is the set of linear combinations $\sum_i a_i u_i \in M$ with $a_i \in \mathbb{Z}$ (resp. $a_i \in \Bbbk$), $\sum_i a_i = 1$, and $u_i \in A$. We say that A *spans* the affine lattice M (resp. the $\Bbbk$-affine space $M_{\Bbbk}$) if $\mathrm{Aff}(A) = M$ (resp. $\mathrm{Aff}_{\Bbbk}(A) = M_{\Bbbk}$).

The projective geometry of X_A has been investigated in view of the projective dual in many papers ([3], [4], [5], [6], [12], [18], etc.). On the other hand, the Gauss map of X_A has not been well studied yet. We note that, in the notion of the *m-th Gauss map* of an n-dimensional variety $X \subseteq \mathbb{P}^N$ for $n \leqslant m \leqslant N - 1$ by Zak, the (ordinary) Gauss map is nothing but the n-th Gauss map, and the dual variety appears as the image of the $(N - 1)$-th Gauss map ([22, I, 2.2. Remark]). For example, the *dual defect*, which is equal to the dimension of a general fiber of the $(N - 1)$-th Gauss map, is greater than or equal to the Gauss defect (see [7, §2.3.4, Proposition]).

In the following result, we describe the structure of Gauss maps of toric varieties in terms of combinatorics.

THEOREM 1.1 (= Theorem 3.1). *Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic, and let M be a free abelian group of rank n. For a finite subset $A = \{u_0, \dots, u_N\} \subseteq M$ which spans the affine lattice M, set*

$$B := \{u_{i_0} + u_{i_1} + \cdots + u_{i_n} \in M \mid u_{i_0}, u_{i_1}, \dots, u_{i_n} \text{ span the } \Bbbk\text{-affine space } M_{\Bbbk}\}$$

and let $\pi : M \to M' := M/(\langle B - B \rangle_{\mathbb{R}} \cap M)$ be the natural projection. Let $\gamma : X_A \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)$ be the Gauss map of the toric variety $X_A \subseteq \mathbb{P}^N$. Then the following hold.

(1) *The closure $\overline{\gamma(X_A)}$ of the image of γ, which is embedded in a projective space by the Plücker embedding of $\mathbb{G}(n, \mathbb{P}^N)$, is projectively equivalent to the toric variety X_B.*
(2) *The restriction of $\gamma : X_A \dashrightarrow \overline{\gamma(X_A)} \cong X_B$ on $T_M \subseteq X_A$ is the morphism*

$$T_M = \operatorname{Spec} \Bbbk[M] \twoheadrightarrow T_{\langle B-B \rangle} = \operatorname{Spec} \Bbbk[\langle B - B \rangle] \subseteq X_B$$

induced by the inclusion $\langle B - B \rangle \subseteq M$.
(3) *Let $F \subseteq T_M$ be an irreducible component of any fiber of $\gamma|_{T_M}$ with the reduced structure. Let $T_{M'} \hookrightarrow T_M$ be the subtorus induced by π. Then F is a translation of $T_{M'}$ by an element of T_M, and the closure $\overline{F} \subseteq X_A$ is projectively equivalent to the toric variety $X_{\pi(A)}$.*

In particular, we have $\delta_\gamma(X_A) = \operatorname{rk} M' = n - \operatorname{rk}\langle B - B \rangle$.

Note that there is no loss of generality in assuming that A spans the affine lattice M in the above theorem (see Remark 3.2). We also study when the Gauss map of X_A is degenerate (i.e., $\mathrm{rk}\langle B - B\rangle < n$), and give a developability criterion for covering families of X_A (see §4.1).

REMARK 1.2. Essentially, the construction of the set B in Theorem 1.1 was already introduced in the study of Nash blowups of toric varieties [1, Subsection 2.3]. Since the authors of [1] work on *affine* toric varieties, they take the sum of u_i which span $M_{\Bbbk}$ as the $\Bbbk$-*vector* space.

In the following example, we illustrate the notation in Theorem 1.1.

EXAMPLE 1.3. Let $M = \mathbb{Z}^2$ and

$$A = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\} \subseteq \mathbb{Z}^2,$$

where we write elements in $\mathbb{Z}^2$ by column vectors. We consider the Gauss map γ of the toric surface $X_A \subseteq \mathbb{P}^3$. When $\mathrm{char}\, \Bbbk \neq 2$,

$$B = \left\{ \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{Z}^2.$$

Hence $\langle B - B\rangle = M = \mathbb{Z}^2$, and γ is birational due to (2) in Theorem 1.1. On the other hand, when $\mathrm{char}\, \Bbbk = 2$,

$$B = \left\{ \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}, \ \langle B - B\rangle = \left\langle \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\rangle, \ \langle B - B\rangle_{\mathbb{R}} \cap M = \left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\rangle \subseteq \mathbb{Z}^2,$$

and $\pi(A) = \{0, 1, -1\} \subseteq \mathbb{Z}^2/(\langle B - B\rangle_{\mathbb{R}} \cap M) = \mathbb{Z}^1$ as in Figure 1. Thus (1) implies that $\overline{\gamma(X_A)} \cong X_B = \mathbb{P}^1$, and (2) implies that $\gamma|_{T_M} : T_M = (\Bbbk^\times)^2 \twoheadrightarrow T_{\langle B-B\rangle} = \Bbbk^\times$ is given by $(z_1, z_2) \mapsto z_1^2$. From (3), a general fiber of γ with the reduced structure is projectively equivalent to the smooth conic $X_{\pi(A)}$.

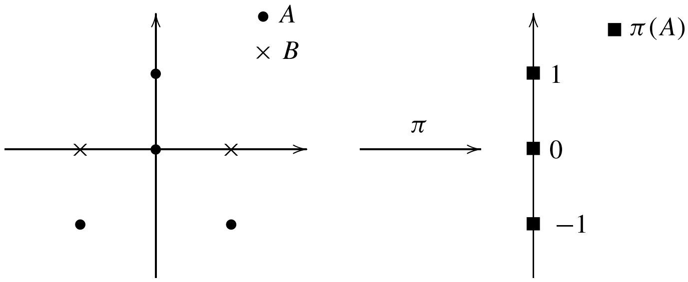

FIGURE 1. $\mathrm{char}\, \Bbbk = 2$.

In characteristic zero, it is well known that, if a projective variety is the join of some varieties, then its Gauss map is degenerate due to Terracini's lemma (see [7, 2.2.5], [22, Ch. II, 1.10. Proposition]). For toric varieties in characteristic zero, this is the only case when the Gauss map is degenerate; more precisely, we have:

COROLLARY 1.4 (= Corollary 4.13). *Let $X \subseteq \mathbb{P}^N$ be a projectively embedded toric variety in* $\operatorname{char} \Bbbk = 0$. *Then there exist disjoint torus invariant closed subvarieties $X_0, \dots, X_{\delta_\gamma(X)} \subseteq X$ such that X is the join of $X_0, \dots, X_{\delta_\gamma(X)}$.*

If the Gauss map of $X_A \subseteq \mathbb{P}^N$ is separable, A is written as a *Cayley sum* of certain finite subsets $A^0, \dots, A^{\delta_\gamma(X)}$ in any characteristic (see Theorem 4.8 for details). However, the statement of Corollary 1.4 does *not* hold in general in positive characteristic, even if the Gauss map is separable (see Example 4.15).

Next, let us focus on inseparable Gauss maps. A. H. Wallace [21, §7] showed that the Gauss map γ of a projective variety can be *inseparable* in positive characteristic. In this case, it is possible that a general fiber of γ is *not* a linear subvariety of $\mathbb{P}^N$; the fiber can be a union of points (H. Kaji [14, Example 4.1] [15], J. Rathmann [20, Example 2.13], A. Noma [19]), and can be a non-linear variety (S. Fukasawa [8, §7]). In fact, Fukasawa [9] showed that *any* projective variety appears as a general fiber of the Gauss map of some projective variety.

As we will see in Corollary 3.6, Theorem 1.1 provides several computations on the Gauss map γ of toric varieties (e.g., the rank, separable degree, inseparable degree). We also obtain the toric version of Fukasawa's result [9] as follows:

THEOREM 1.5 (Special case of Theorem 5.1). *Assume* $\operatorname{char} \Bbbk > 0$. *Let $Y \subseteq \mathbb{P}^{N'}$ and $Z \subseteq \mathbb{P}^{N''}$ be projectively embedded toric varieties. If $n := \dim(Y) + \dim(Z)$ is greater than or equal to N', then there exists an n-dimensional projectively embedded toric variety $X \subseteq \mathbb{P}^{n+N''}$ satisfying the following conditions:*

(i) *(The closure of) a general fiber of the Gauss map γ of X with the reduced structure is projectively equivalent to Y.*

(ii) *(The closure of) the image of γ is projectively equivalent to Z.*

By Theorem 1.5, any projectively embedded toric variety appears as a general fiber and the image of the Gauss map of a certain projectively embedded *toric* variety; moreover we can also control the *rank* of γ, and the *number* of the irreducible components of a general fiber of γ (see §5, for details).

This paper is organized as follows. In §2, we recall some basic properties of toric varieties. In §3, we describe the structure of the Gauss maps of toric varieties in a combinatorial way, and prove Theorem 1.1. In §4, we investigate when the Gauss maps are degenerate, and give a developability criterion. As a result, we show Corollary 1.4. In §5, we present two constructions of projectively embedded toric varieties, yielding Theorem 1.5.

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2. Preliminary on toric varieties. Two projective varieties $X_1 \subseteq \mathbb{P}^{N_1}$ and $X_2 \subseteq \mathbb{P}^{N_2}$ are said to be *projectively equivalent* if there exist linear embeddings $\jmath_i : \mathbb{P}^{N_i} \hookrightarrow \mathbb{P}^N$ (i.e., $\jmath_i^*(\mathcal{O}_{\mathbb{P}^N}(1)) = \mathcal{O}_{\mathbb{P}^{N_i}}(1)$) such that $\jmath_1(X_1) = \jmath_2(X_2)$. (Indeed, we can take $N = \max\{N_1, N_2\}$.)

The following two lemmas about toric varieties are well known, but we prove them for the convenience of the reader.

LEMMA 2.1. *Let $\pi : M \to M'$ be a surjective homomorphism between free abelian groups of finite ranks. Let $\iota : T_{M'} \hookrightarrow T_M$ be the embedding induced by π. For a finite set $A \subseteq M$ with $\mathrm{Aff}(A) = M$, the closure of $\iota(T_{M'})$ in X_A is projectively equivalent to $X_{\pi(A)}$. The translations $\{t \cdot \overline{\iota(T_{M'})}\}_{t \in T_M}$ of the closure $\overline{\iota(T_{M'})}$ under the action of T_M on X_A give a covering family of X_A, and each translation is also projectively equivalent to $X_{\pi(A)}$.*

PROOF. Let $A = \{u_0, \dots, u_N\}$ and $\pi(A) = \{u'_0, \dots, u'_{N'}\}$ for $N = \#A - 1$ and $N' = \#\pi(A) - 1$. We define a linear embedding $\jmath : \mathbb{P}^{N'} \to \mathbb{P}^N$ by

$$\jmath([X'_0 : \cdots : X'_{N'}]) = [X_0 : \cdots : X_N],$$

where for each i, we set $X_i := X'_j$ for j such that $\pi(u_i) = u'_j$. Then we have the following commutative diagram

$$\begin{tikzcd} T_{M'} \arrow[r, "\varphi_{\pi(A)}"] \arrow[d, hook, "\iota"'] \arrow[dr, phantom, "\circlearrowleft"] & \mathbb{P}^{N'} \arrow[d, hook, "\jmath"] \\ T_M \arrow[r, "\varphi_A"] & \mathbb{P}^N . \end{tikzcd}$$

Hence $\overline{\iota(T_{M'})}$ is projectively equivalent to $X_{\pi(A)}$. Since the action of T_M on X_A extends to $\mathbb{P}^N$ (see [12, Ch. 5, Proposition 1.5]), translations of $\overline{\iota(T_{M'})}$ are also projectively equivalent to $X_{\pi(A)}$. Since $\iota(T_{M'})$ is non-empty and contained in T_M, the translations give a covering family of X_A. □

Let $f : X \dashrightarrow Y$ be a rational map between varieties. For a smooth point $x \in X$, we denote by $d_x f : t_x X \to t_{f(x)} Y$ the tangent map between Zariski tangent spaces at x and $f(x)$. The *rank* of f, denoted by $\mathrm{rk}(f)$, is defined to be the rank of the $\Bbbk$-linear map $d_x f$ for general $x \in X$. Recall that f is said to be *separable* if the field extension $K(X)/K(f(X))$ is separable; this condition is equivalent to $\mathrm{rk}(f) = \dim(f(X))$.

LEMMA 2.2. *Let M be a free abelian group of finite rank. Let M'' be a subgroup of M and $g : T_M \twoheadrightarrow T_{M''}$ be the morphism induced by the inclusion $\beta : M'' \hookrightarrow M$.*

(a) *The inclusions $M'' \subseteq M''_{\mathbb{R}} \cap M \subseteq M$ induce a decomposition of g*

$$T_M \overset{g_1}{\to} T_{M''_{\mathbb{R}} \cap M} \overset{g_2}{\to} T_{M''},$$

where g_1 is a morphism with reduced and irreducible fibers, and g_2 is a finite morphism.

(b) *The rank of g is equal to the rank of the $\Bbbk$-linear map $\beta_{\Bbbk} : M''_{\Bbbk} \to M_{\Bbbk}$ obtained by tensoring $\Bbbk$ with $\beta : M'' \hookrightarrow M$. In particular, g is separable if and only if* $\operatorname{rk}(\beta_{\Bbbk}) = \operatorname{rk}(M'')$.

(c) *Assume $p = \operatorname{char}\Bbbk > 0$. Let a be the index $[M''_{\mathbb{R}} \cap M : M'']$ and write $a = p^s b$ with integers $s \geqslant 0, b \geqslant 1$ such that $p \nmid b$. Then the degree, separable degree, inseparable degree of the finite morphism g_2 are a, b, p^s respectively.*

PROOF. Set $n = \operatorname{rk} M, k = \operatorname{rk} M''$. By the elementary divisors theorem (see [17, III, Theorem 7.8], for example), there exists a basis $e_1, \dots, e_n$ of M such that

$$M'' = \mathbb{Z}a_1 e_1 \oplus \cdots \oplus \mathbb{Z}a_k e_k \subseteq \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_n = M$$

for some positive integers a_i. Set $e''_i := a_i e_i \in M''$. By the bases $e_1, \dots, e_n$ of M and $e''_1, \dots, e''_k$ of M'', we identify M and M'' with $\mathbb{Z}^n$ and $\mathbb{Z}^k$ respectively. Then $g : T_M = (\Bbbk^\times)^n \to T_{M''} = (\Bbbk^\times)^k$ is described as

$$(\Bbbk^\times)^n \to (\Bbbk^\times)^k \quad : \quad (z_1, \dots, z_n) \mapsto (z_1^{a_1}, \dots, z_k^{a_k}) . \tag{2.1}$$

(a) By (2.1), g is decomposed as

$$(\Bbbk^\times)^n \to (\Bbbk^\times)^k \to (\Bbbk^\times)^k \quad : \quad (z_1, \dots, z_n) \mapsto (z_1, \dots, z_k) \mapsto (z_1^{a_1}, \dots, z_k^{a_k}) .$$

Since $M''_{\mathbb{R}} \cap M = \mathbb{Z}e_1 \oplus \cdots \oplus \mathbb{Z}e_k \subseteq M$, the assertion of (a) follows.

(c) Write $a_i = p^{s_i} b_i$ by integers $s_i \geqslant 0, b_i \geqslant 1$ such that $p \nmid b_i$. Since g_2 is the morphism

$$(\Bbbk^\times)^k \to (\Bbbk^\times)^k \quad : \quad (z_1, \dots, z_k) \mapsto (z_1^{a_1}, \dots, z_k^{a_k}) = (z_1^{p^{s_1} b_1}, \dots, z_k^{p^{s_k} b_k}) ,$$

the degree, separable degree, inseparable degree of g_2 are $\prod_{i=1}^k a_i = a$, $\prod_{i=1}^k b_i = b$, $\prod_{i=1}^k p^{s_i} = p^s$ respectively.

(b) If $\operatorname{char}\Bbbk = 0$, this statement is clear from (2.1). Assume $p = \operatorname{char}\Bbbk > 0$ and use the notation a_i, s_i, b_i as above. Then the rank of g is equal to $\#\{ 1 \leqslant i \leqslant k \mid s_i = 0 \}$. On the other hand, $\beta_{\Bbbk} : M''_{\Bbbk} = \Bbbk^k \to M_{\Bbbk} = \Bbbk^n$ is the $\Bbbk$-linear map defined by $e''_i \mapsto a_i e_i = p^{s_i} b_i e_i \in M_{\Bbbk}$ for $1 \leqslant i \leqslant k$. Hence the rank of $\beta_{\Bbbk}$ is also equal to $\#\{ 1 \leqslant i \leqslant k \mid s_i = 0 \}$. Thus we have $\operatorname{rk}(g) = \operatorname{rk}(\beta_{\Bbbk})$. The last statement follows from $\dim g(T_M) = \dim T_{M''} = \operatorname{rk}(M'')$. □

3. Structure of Gauss maps. In this section, we prove Theorem 1.1 and describe several invariants (e.g., the rank) of Gauss maps of toric varieties by combinatorial data.

THEOREM 3.1 (= Theorem 1.1). *Let $\Bbbk$ be an algebraically closed field of arbitrary characteristic, and let M be a free abelian group of rank n. For a finite subset $A = \{u_0, \dots, u_N\} \subseteq M$ which spans the affine lattice M, set*

$$B := \{u_{i_0} + u_{i_1} + \cdots + u_{i_n} \in M \mid u_{i_0}, u_{i_1}, \dots, u_{i_n} \text{ span the } \Bbbk\text{-affine space } M_{\Bbbk}\}$$

and let $\pi : M \to M' := M/(\langle B - B \rangle_{\mathbb{R}} \cap M)$ be the natural projection. Let $\gamma : X_A \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)$ be the Gauss map of the toric variety $X_A \subseteq \mathbb{P}^N$. Then the following hold.

(1) *The closure $\overline{\gamma(X_A)}$ of the image of γ, which is embedded in a projective space by the Plücker embedding of $\mathbb{G}(n, \mathbb{P}^N)$, is projectively equivalent to the toric variety X_B.*

(2) *The restriction of* $\gamma : X_A \dashrightarrow \overline{\gamma(X_A)} \cong X_B$ *on* $T_M \subseteq X_A$ *is the morphism*

$$T_M = \operatorname{Spec} \Bbbk[M] \twoheadrightarrow T_{\langle B-B \rangle} = \operatorname{Spec} \Bbbk[\langle B - B \rangle] \subseteq X_B$$

induced by the inclusion $\langle B - B \rangle \subseteq M$.

(3) *Let* $F \subseteq T_M$ *be an irreducible component of any fiber of* $\gamma|_{T_M}$ *with the reduced structure. Let* $T_{M'} \hookrightarrow T_M$ *be the subtorus induced by* π. *Then* F *is a translation of* $T_{M'}$ *by an element of* T_M, *and the closure* $\overline{F} \subseteq X_A$ *is projectively equivalent to the toric variety* $X_{\pi(A)}$.

In particular, we have $\delta_\gamma(X_A) = \operatorname{rk} M' = n - \operatorname{rk}\langle B - B \rangle$.

In order to investigate a toric variety X_A for $A \subseteq M$, we may assume that A spans the affine lattice M due to the following remark.

REMARK 3.2. For a finite subset $A \subseteq M$, let $\theta : \mathbb{Z}^m \to \operatorname{Aff}(A)$ be an affine isomorphism for $m = \operatorname{rk}\langle A - A \rangle$. Then $\theta^{-1}(A)$ spans $\mathbb{Z}^m$ as an affine lattice and $X_{\theta^{-1}(A)}$ is naturally identified with X_A by [12, Chapter 5, Proposition 1.2]. Hence any projectively embedded toric variety X is projectively equivalent to X_A for some $A \subseteq M$ with $\operatorname{Aff}(A) = M$.

Let $A = \{u_0, u_1, \dots, u_N\} \subseteq M := \mathbb{Z}^n$ be a finite subset which spans the affine lattice M. We denote each u_i by a column vector as

$$u_i = \begin{bmatrix} u_{i,1} \\ \vdots \\ u_{i,n} \end{bmatrix}.$$

Then the morphism φ_A, defined by (1.1) in §1, is described as

$$\varphi_A : (\Bbbk^\times)^n \to \mathbb{P}^N \quad : \quad z = (z_1, \dots, z_n) \mapsto [z^{u_0} : z^{u_1} : \dots : z^{u_N}],$$

where $z^{u_i} := z_1^{u_{i,1}} z_2^{u_{i,2}} \cdots z_n^{u_{i,n}}$. By the assumption that A spans $\mathbb{Z}^n$ as an affine lattice, φ_A is an isomorphism onto an open subset of X_A.

Let us study the Gauss map $\gamma : X_A \dashrightarrow \mathbb{G}(n, \mathbb{P}^N)$ of $X_A \subseteq \mathbb{P}^N$.

LEMMA 3.3. *Let* A, φ_A *be as above, and let* $x \in (\Bbbk^\times)^n$. *Then* $\gamma(\varphi_A(x)) \in \mathbb{G}(n, \mathbb{P}^N)$ *is expressed by the* $\Bbbk$*-valued* $(n + 1) \times (N + 1)$ *matrix* $\Gamma(x)$; *more precisely,* $\gamma(\varphi_A(x))$ *corresponds to the* n*-plane (i.e.,* n*-dimensional linear subvariety of* $\mathbb{P}^N$*) spanned by the* $n+1$ *points which are given as the row vectors of* $\Gamma(x)$, *where*

$$\Gamma := \begin{bmatrix} z^{u_0} & z^{u_1} & \cdots & z^{u_N} \\ u_{0,1} \cdot z^{u_0} & u_{1,1} \cdot z^{u_1} & \cdots & u_{N,1} \cdot z^{u_N} \\ \vdots & \vdots & & \vdots \\ u_{0,n} \cdot z^{u_0} & u_{1,n} \cdot z^{u_1} & \cdots & u_{N,n} \cdot z^{u_N} \end{bmatrix}$$
$$= \left[z^{u_0} \cdot \begin{bmatrix} 1 \\ u_0 \end{bmatrix} \; z^{u_1} \cdot \begin{bmatrix} 1 \\ u_1 \end{bmatrix} \; \cdots \; z^{u_N} \cdot \begin{bmatrix} 1 \\ u_N \end{bmatrix} \right].$$

PROOF. Let $L_x \subseteq \mathbb{P}^N$ be the n-plane spanned by the $n+1$ points which are given as the row vectors of

$$\begin{bmatrix} z^{u_0} & \cdots & z^{u_N} \\ \partial(z^{u_0})/\partial z_1 & \cdots & \partial(z^{u_N})/\partial z_1 \\ \vdots & & \vdots \\ \partial(z^{u_0})/\partial z_n & \cdots & \partial(z^{u_N})/\partial z_n \end{bmatrix}(x)\,. \tag{3.1}$$

Then L_x coincides with the embedded tangent space $\mathbb{T}_x X$, because of the equality $t_{x'}\hat{L}_x = t_{x'}\hat{X}$ of Zariski tangent spaces in $t_{x'}\mathbb{A}^{N+1}$ with $x' \in \hat{x} \setminus \{0\}$, where $\hat{S} \subseteq \mathbb{A}^{N+1}$ means the affine cone of $S \subseteq \mathbb{P}^N$. On the other hand, (3.1) is calculated as

$$\begin{bmatrix} z^{u_0} & \cdots & z^{u_N} \\ u_{0,1} \cdot z^{u_0}/z_1 & \cdots & u_{N,1} \cdot z^{u_N}/z_1 \\ \vdots & & \vdots \\ u_{0,n} \cdot z^{u_0}/z_n & \cdots & u_{N,n} \cdot z^{u_N}/z_n \end{bmatrix}.$$

Since each row vector corresponds to the homogeneous coordinates of a point of $\mathbb{P}^N$, by multiplying z_i with the $(i+1)$-th row vector for $1 \leqslant i \leqslant n$, we have the matrix Γ and the assertion. □

We interpret Lemma 3.3 by the Plücker embedding. We regard $\mathbb{P}^N$ as $\mathbb{P}_*(V) = (V \setminus \{0\})/\Bbbk^\times$, the projectivization of $V := \Bbbk^{N+1}$. Let $\mathbb{G}(n, \mathbb{P}^N) \hookrightarrow \mathbb{P}_*(\bigwedge^{n+1} V)$ be the Plücker embedding and let $[p_{i_0,\dots,i_n}]_{(i_0,i_1,\dots,i_n)\in I}$ be the Plücker coordinates on $\mathbb{P}_*(\bigwedge^{n+1} V)$, where

$$I := \{(i_0, i_1, \dots, i_n) \in \mathbb{N}^{n+1} \mid 0 \leqslant i_0 < i_1 < \cdots < i_n \leqslant N\}\,.$$

LEMMA 3.4. *The composite morphism* $(\Bbbk^\times)^n \xrightarrow{\gamma\circ\varphi_A} \mathbb{G}(n, \mathbb{P}^N) \hookrightarrow \mathbb{P}_*(\bigwedge^{n+1} V)$ *maps* $z = (z_1, \dots, z_n) \in (\Bbbk^\times)^n$ *to*

$$\left[\mu_{i_0,i_1,\dots,i_n} \cdot z^{u_{i_0}+u_{i_1}+\cdots+u_{i_n}}\right]_{(i_0,i_1,\dots,i_n)\in I} \in \mathbb{P}_*\Big(\bigwedge^{n+1} V\Big),$$

where

$$\mu_{i_0,i_1,\dots,i_n} := \det \begin{bmatrix} 1 & 1 & \cdots & 1 \\ u_{i_0} & u_{i_1} & \cdots & u_{i_n} \end{bmatrix} \in \Bbbk\,.$$

PROOF. This directly follows from Lemma 3.3 and the definition of the Plücker embedding. □

Set

$$J := \{(i_0, i_1, \dots, i_n) \in I \mid \mu_{i_0,i_1,\dots,i_n} \neq 0\}\,.$$

By definition, $\mu_{i_0,i_1,\dots,i_n} \neq 0$ in $\Bbbk$ if and only if $u_{i_0}, u_{i_1}, \dots, u_{i_n}$ span $M_\Bbbk$ as an affine space. Hence the finite set B in Theorem 1.1 is described as

$$B = \{u_{i_0} + u_{i_1} + \cdots + u_{i_n} \in M \mid (i_0, i_1, \dots, i_n) \in J\}\,.$$

Write $B = \{b_0, b_1, \ldots, b_{\#B-1}\}$ for mutually distinct $b_j \in M$. We define a linear embedding

$$\begin{aligned} \mathbb{P}^{\#B-1} &\hookrightarrow \mathbb{P}_*\Big(\bigwedge^{n+1} V\Big) \\ [Y_0 : Y_1 : \cdots : Y_{\#B-1}] &\mapsto [p_{i_0,i_1,\ldots,i_n}]_{(i_0,i_1,\ldots,i_n)\in I} \end{aligned} \tag{3.2}$$

as follows. When $(i_0, i_1, \ldots, i_n) \in J$, there exists a unique $0 \leqslant j \leqslant \#B - 1$ such that $b_j = u_{i_0} + u_{i_1} + \cdots + u_{i_n}$. For this j, we set $p_{i_0,i_1,\ldots,i_n} = \mu_{i_0,i_1,\ldots,i_n} \cdot Y_j$. When $(i_0, i_1, \ldots, i_n) \notin J$, we set $p_{i_0,i_1,\ldots,i_n} = 0$.

By Lemma 3.4 and the definition of the embedding (3.2), we have the following commutative diagram:

$$\begin{tikzcd} (\Bbbk^\times)^n \arrow[r, hook, "\varphi_A"] \arrow[drr, "\varphi_B"'] & X_A \arrow[r, dashed, "\gamma"] & \mathbb{P}_*(\bigwedge^{n+1} V) \\ & & \mathbb{P}^{\#B-1} \arrow[u, hook] \end{tikzcd} \tag{3.3}$$

PROOF OF THEOREM 1.1. By taking a basis of M, we may assume that $M = \mathbb{Z}^n$ and use the notation as above. From the above diagram (3.3), the embedding $\mathbb{P}^{\#B-1} \hookrightarrow \mathbb{P}_*(\bigwedge^{n+1} V)$ gives an isomorphism between

$$\overline{\gamma(X_A)} = \overline{\gamma \circ \varphi_A((\Bbbk^\times)^n)} \text{ and } X_B = \overline{\varphi_B((\Bbbk^\times)^n)}\,.$$

Hence (1) in Theorem 1.1 holds.

The toric variety $X_B = \overline{\varphi_B((\Bbbk^\times)^n)}$ contains $T_{\langle B-B\rangle}$ as an open dense orbit and the restriction $\varphi_B|_{T_M}$ is nothing but $T_M = (\Bbbk^\times)^n \twoheadrightarrow T_{\langle B-B\rangle}$ induced by the inclusion $\langle B - B\rangle \hookrightarrow M$. Hence (2) in Theorem 1.1 holds by the diagram (3.3).

To show (3), we use the following claim.

CLAIM 3.5. *The morphism $\gamma|_{T_M} = \varphi_B$ is decomposed as*

$$T_M \xrightarrow{g_1} T_{\langle B-B\rangle_{\mathbb{R}}\cap M} \xrightarrow{g_2} T_{\langle B-B\rangle}$$

by $\langle B - B\rangle \subseteq \langle B - B\rangle_{\mathbb{R}} \cap M \subseteq M$, where g_1 is a morphism with reduced and irreducible fibers, and g_2 is a finite morphism.

PROOF OF CLAIM 3.5. By applying (a) in Lemma 2.2 to $\langle B - B\rangle \subseteq M$, we have the assertion. □

The short exact sequence

$$0 \to \langle B - B\rangle_{\mathbb{R}} \cap M \to M \xrightarrow{\pi} M/(\langle B - B\rangle_{\mathbb{R}} \cap M) \to 0$$

induces a short exact sequence of algebraic tori

$$1 \to T_{M/(\langle B-B\rangle_{\mathbb{R}}\cap M)} \to T_M \xrightarrow{g_1} T_{\langle B-B\rangle_{\mathbb{R}}\cap M} \to 1\,. \tag{3.4}$$

Hence $g_1^{-1}(1_{T_{\langle B-B\rangle_{\mathbb{R}}\cap M}}) = T_{M/(\langle B-B\rangle_{\mathbb{R}}\cap M)}$ holds for the identity element $1_{T_{\langle B-B\rangle_{\mathbb{R}}\cap M}}$ of the torus $T_{\langle B-B\rangle_{\mathbb{R}}\cap M}$. Applying Lemma 2.1 to the surjection $\pi : M \to M/(\langle B - B\rangle_{\mathbb{R}} \cap M)$, it holds that the closure

$$\overline{g_1^{-1}(1_{T_{\langle B-B\rangle_{\mathbb{R}}\cap M}})} \subseteq X_A$$

is projectively equivalent to $X_{\pi(A)}$.

Let F be an irreducible component of any fiber of $\gamma|_{T_M}$ with the reduced structure. From Claim 3.5, F is a fiber of g_1. By (3.4), F is the translation of $g_1^{-1}(1_{T_{\langle B-B\rangle_{\mathbb{R}}\cap M}}) = T_{M/(\langle B-B\rangle_{\mathbb{R}}\cap M)}$ by an element of T_M. Hence the closure $\overline{F}$ is projectively equivalent to $X_{\pi(A)}$ by Lemma 2.1. □

COROLLARY 3.6. *Let A, M, γ, B be as in Theorem 1.1. Let $\gamma|_{T_M} = g_2 \circ g_1$ be the decomposition of $\gamma|_{T_M}$ in Claim 3.5. Then the following hold.*

(1) *The rank of γ is equal to* $\dim(\mathrm{Aff}_{\Bbbk}(B))$. *In particular, γ is separable if and only if* $\dim(\mathrm{Aff}_{\Bbbk}(B)) = \mathrm{rk}(\mathrm{Aff}(B))$.
(2) *Assume* $p = \mathrm{char}\,\Bbbk > 0$. *Then we have*

$$\deg(g_2) = [\langle B-B\rangle_{\mathbb{R}} \cap M : \langle B-B\rangle]\,.$$

For the maximum integer $s \geqslant 0$ such that $p^s \mid \deg(g_2)$, the separable degree and the inseparable degree of g_2 are $\deg(g_2)/p^s$ and p^s respectively. Hence the number of the irreducible components of a general fiber of γ is equal to $\deg(g_2)/p^s$, which is coprime to p.

We note that $\dim(\mathrm{Aff}_{\Bbbk}(B)) = \dim\langle B-B\rangle_{\Bbbk}$ holds since $\mathrm{Aff}_{\Bbbk}(B)$ is a parallel translation of $\langle B-B\rangle_{\Bbbk}$ in $M_{\Bbbk}$.

PROOF OF COROLLARY 3.6. We apply Lemma 2.2 to $M'' = \langle B-B\rangle \subseteq M$. In this case, g in Lemma 2.2 is $\gamma|_{T_M}$. Since the image of $M''_{\Bbbk} \to M_{\Bbbk}$ is nothing but $\langle B-B\rangle_{\Bbbk} \subseteq M_{\Bbbk}$, it holds that $\mathrm{rk}(\gamma) = \dim\langle B-B\rangle_{\Bbbk} = \dim \mathrm{Aff}_{\Bbbk}(B)$ by (b) in Lemma 2.2. This implies (1). On the other hand, (2) follows from (c) in Lemma 2.2 directly. □

In the following examples, we denote the separable degree and the inseparable degree of a finite morphism f by $\deg_s(f)$ and $\deg_i(f)$ respectively.

EXAMPLE 3.7. Let A, B be as in Example 1.3 and assume $\mathrm{char}\,\Bbbk = 2$. Then $\mathrm{rk}(\gamma) = \dim\langle B-B\rangle_{\Bbbk} = 0$. From (2) of Corollary 3.6, we can calculate $\deg(g_2) = \deg_i(g_2) = 2$ and $\deg_s(g_2) = 1$.

EXAMPLE 3.8. Kaji's example [14, Example 4.1] can be interpreted as follows.

Let $A = \{0, 1, cp^m, cp^m + 1\} \subseteq M = \mathbb{Z}^1$, where c, m are positive integers and $p = \mathrm{char}\,\Bbbk > 0$. Assume that c and p are relatively prime. Then

$$B = \{1, cp^m + 1, 2cp^m + 1\}, \quad \langle B-B\rangle = \langle cp^m\rangle, \quad \langle B-B\rangle_{\mathbb{R}} \cap M = M$$

as in Figure 2. Therefore $\deg(\gamma) = cp^m$, $\deg_i(\gamma) = p^m$, $\deg_s(\gamma) = c$. In particular, a general fiber of γ with the reduced structure is equal to a set of c points.

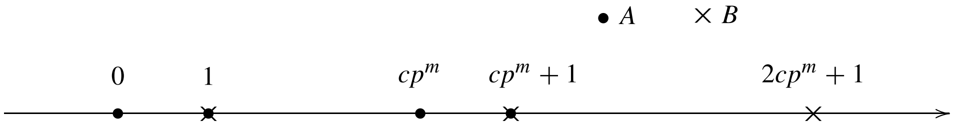

FIGURE 2. Kaji's example.

As in (2) of Corollary 3.6, the number of the irreducible components of a general fiber of γ is coprime to $p = \operatorname{char} \Bbbk$ in the toric case. However, the number can be a multiple of p in the non-toric case. The authors learned the following example from H. Kaji in a personal communication.

REMARK 3.9 (A variant of [14, Example 4.1]). We take $f : \mathbb{P}^1 \dashrightarrow \mathbb{P}^1$ to be a separable rational map whose degree is a multiple of p, and locally parameterize f by $t \mapsto [1 : f_1(t)]$. We set $\varphi : \mathbb{P}^1 \dashrightarrow \mathbb{P}^3$ to be the rational map which is locally parameterized by

$$t \mapsto [1 : t : f_1^p : t \cdot f_1^p]$$

and set $X \subseteq \mathbb{P}^3$ to be the projective curve $\overline{\operatorname{im}(\varphi)}$. Then a general fiber of $\gamma : X \dashrightarrow \mathbb{G}(1, \mathbb{P}^3)$ with the reduced structure is equal to a set of $\deg(f)$ points. In order to show this, we may check that γ is locally parameterized by $t \mapsto f_1^p(t)$, whose separable degree is equal to $\deg(f)$. We leave the details to the reader.

The following are examples of toric varieties $X_A \subseteq \mathbb{P}^N$ with codimension 1 or 2 such that the Gauss maps are birational. Later, these examples will be used in the proof of Theorem 5.4.

EXAMPLE 3.10. Assume $p = \operatorname{char} \Bbbk > 0$. Let $e_1, \ldots, e_n$ be the standard basis of $M := \mathbb{Z}^n$, and set

$$A = \{0, e_1, \ldots, e_n, a_1 e_1 + \cdots + a_n e_n\} \subseteq \mathbb{Z}^n$$

for $a_1, \ldots, a_n \in \mathbb{Z}$ such that

(i) $a_1, \ldots, a_n \not\equiv 0 \mod p$, and
(ii) $a_1 + \cdots + a_n \not\equiv 1 \mod p$.

Then the Gauss map of the toric hypersurface $X_A \subseteq \mathbb{P}^{n+1}$ is birational as follows.

By condition (i), $A \setminus \{e_i\}$ spans $M_\Bbbk = \Bbbk^n$ as an affine space for any $1 \leqslant i \leqslant n$. Hence it holds that

$$e_1 + \cdots + e_n + a_1 e_1 + \cdots + a_n e_n - e_i \in B\,. \tag{3.5}$$

By condition (ii), $A \setminus \{0\}$ spans $M_{\mathbb{R}} = \Bbbk^n$ as an affine space. Hence

$$e_1 + \cdots + e_n + a_1 e_1 + \cdots + a_n e_n \in B\,. \tag{3.6}$$

Considering the difference between (3.5) and (3.6), we have $e_i \in B - B$ for any i. Therefore $\langle B - B \rangle = M$. By (2) in Theorem 1.1, the Gauss map of X_A is birational.

REMARK 3.11. For example, the conditions (i) and (ii) are satisfied for $a_1 = \cdots = a_n = 1$ (resp. $a_1 = \cdots = a_n = -1$) when $n \not\equiv 1 \mod p$ (resp. $n \not\equiv -1 \mod p$). Hence there exists a toric hypersurface in $\mathbb{P}^{n+1}$ whose Gauss map is birational if $\operatorname{char} \Bbbk \neq 2$ or n is even.

If $\operatorname{char} \Bbbk = 2$ and n is odd, the Gauss map of any (not necessarily toric) hypersurface in $\mathbb{P}^{n+1}$ cannot be birational as we will see later in Remark 5.3.

EXAMPLE 3.12. Assume $n \geqslant 2$. Let

$$A = \{0, e_1, \dots, e_n, e_1 + e_2, e_2 + e_3 + \cdots + e_n\} \subseteq M := \mathbb{Z}^n .$$

For $S := \{0, e_1, \dots, e_n, e_1 + e_2\} \subseteq A$, each of

$$S \setminus \{e_1 + e_2\},\ S \setminus \{e_1\},\ S \setminus \{e_2\}$$

spans the affine space $M_\Bbbk$. Hence it holds that $e_1, e_2 \in B - B$. In addition, $\{0, e_1, \dots, e_n, e_2 + e_3 + \cdots + e_n\} \setminus \{e_i\}$ spans the affine space $M_\Bbbk$ for $2 \leqslant i \leqslant n$. Thus we have $e_i - e_2 \in B - B$ for $2 \leqslant i \leqslant n$. Hence $\langle B - B \rangle = M$ and the Gauss map of $X_A \subseteq \mathbb{P}^{n+2}$ is birational.

4. Degenerate Gauss maps.

4.1. Developability criterion. By Theorem 1.1, the Gauss map of any given toric variety and its general fibers can be explicitly determined from the computation of B and $\pi(A)$ as in Example 1.3. However, it is not so clear for what kind of A the Gauss map γ of $X_A \subseteq \mathbb{P}^N$ is degenerate, i.e., $\operatorname{rk}\langle B - B \rangle < n$. The following result gives conditions for which γ is degenerate.

PROPOSITION 4.1. *Let M, A, π be as in Theorem 1.1. Let $\tilde{\pi} : M \to \tilde{M}'$ be a surjective homomorphism of free abelian groups. Then $\tilde{\pi}$ factors through $\pi : M \to M'$, i.e., $\langle B - B \rangle \subseteq \ker \tilde{\pi}$ if and only if*

$$\sum_{j=0}^{\tilde{N}'} \dim \operatorname{Aff}_\Bbbk(A_j) = n - \tilde{N}' ,$$

where $\tilde{N}' := \#\tilde{\pi}(A) - 1$, $\tilde{\pi}(A) = \{\tilde{u}'_0, \dots, \tilde{u}'_{\tilde{N}'}\}$, and $A_j = \tilde{\pi}^{-1}(\tilde{u}'_j) \cap A$.

For a surjective homomorphism $\tilde{\pi} : M \to \tilde{M}'$ of free abelian groups, the toric variety X_A is covered by the translations of $\overline{T_{\tilde{M}'}} = X_{\tilde{\pi}(A)}$ by elements of T_M due to Lemma 2.1. The homomorphism $\tilde{\pi}$ factors through π if and only if $\overline{T_{\tilde{M}'}}$ (or equivalently, the translation of $\overline{T_{\tilde{M}'}}$ by any element of T_M) is contracted to one point by γ.

In general, a covering family $\{F_\alpha\}$ of a projective variety $X \subseteq \mathbb{P}^N$ by subvarieties $F_\alpha \subseteq X$ is said to be *developable* if F_α is contracted to one point by the Gauss map of X (i.e., $\mathbb{T}_x X$ is constant on general $x \in F_\alpha$) for general α. Proposition 4.1 is regarded as a toric version of the developability criterion (cf. [7, 2.2.4], [8], [11, §4]; see also §4.2 for the separable case).

Before the proof, we illustrate Proposition 4.1 by an example.

EXAMPLE 4.2. Let $A \subseteq \mathbb{Z}^2$ be as in Example 1.3. For the projection $\tilde{\pi} : \mathbb{Z}^2 \to \mathbb{Z}^1$ to the second factor, $\tilde{\pi}(A) = \{0, 1, -1\}$. Then $A_0 = \tilde{\pi}^{-1}(0) \cap A,\ A_1 = \tilde{\pi}^{-1}(1) \cap A,\ A_2 = \tilde{\pi}^{-1}(-1) \cap A$ are given by

$$A_0 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix} \right\},\ A_1 = \left\{ \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\},\ A_2 = \left\{ \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right\}.$$

Thus

$$\sum_{j=0}^{\tilde{N}'} \dim \mathrm{Aff}_{\Bbbk}(A_j) = \begin{cases} 1 & \text{when } \operatorname{char} \Bbbk \neq 2, \\ 0 & \text{when } \operatorname{char} \Bbbk = 2\,. \end{cases}$$

On the other hand, $n - \tilde{N}' = 2 - 2 = 0$ holds. Hence the equality in Proposition 4.1 holds if and only if $\operatorname{char} \Bbbk = 2$. Note that, in this example, the above $\tilde{\pi}$ can be identified with the natural projection $\pi : M \to M'$ in Theorem 1.1 when $\operatorname{char} \Bbbk = 2$.

To prove Proposition 4.1, we need the following lemma.

LEMMA 4.3. *In the setting of Proposition 4.1, the homomorphism $\tilde{\pi}$ factors through π if and only if*

$$\mathrm{Aff}_{\Bbbk}(A_j) = \mathrm{Aff}_{\Bbbk}(\{u_{i_0}, u_{i_1}, \dots, u_{i_n}\} \cap A_j) \quad (0 \leqslant j \leqslant \tilde{N}')$$

for any $u_{i_0}, \dots, u_{i_n} \in A$ which span the affine space $M_{\Bbbk}$.

PROOF. First, we show the "only if" part. Assume that $\tilde{\pi}$ factors through π. The inclusion "$\supseteq$" always holds. We show "$\subseteq$". Let $u \in A_j$. Since $u_{i_0}, \dots, u_{i_n}$ span the affine space $M_{\Bbbk}$, we can write $u = \sum_{k=0}^{n} c_{i_k} u_{i_k}$ with $\sum_{k=0}^{n} c_{i_k} = 1$ and $c_{i_k} \in \Bbbk$. For any k with $c_{i_k} \neq 0$, we have $u_{i_k} \in A_j$ as follows.

If $c_{i_k} \neq 0$, we find that $\{u\} \cup \{u_{i_{k'}}\}_{0 \leqslant k' \leqslant n, k' \neq k}$ span the affine space $M_{\Bbbk}$. Thus $u + \sum_{0 \leqslant k' \leqslant n, k' \neq k} u_{i_{k'}} \in B$, and then $u - u_{i_k} \in B - B$. Since $\tilde{\pi}$ factors through π, we have $\tilde{\pi}(u_{i_k}) = \tilde{\pi}(u) = \tilde{u}'_j$ by $u \in A_j = \tilde{\pi}^{-1}(\tilde{u}'_j) \cap A$; hence $u_{i_k} \in A_j$.

Thus we have $u = \sum_{u_{i_k} \in A_j} c_{i_k} u_{i_k}$ with $\sum_{u_{i_k} \in A_j} c_{i_k} = 1$, i.e., u is contained in $\mathrm{Aff}_{\Bbbk}(\{u_{i_0}, u_{i_1}, \dots, u_{i_n}\} \cap A_j)$. This implies the assertion.

Next, we show the "if" part. For any $b \in B$, we can write $b = u_{i_0} + \cdots + u_{i_n}$ with $u_{i_0}, \dots, u_{i_n} \in A$ which span the affine space $M_{\Bbbk}$. Since the n-dimensional affine space $M_{\Bbbk}$ is spanned by $n+1$ elements $u_{i_0}, \dots, u_{i_n} \in A$, we have

$$\#(\{u_{i_0}, \dots, u_{i_n}\} \cap A_j) = \dim \mathrm{Aff}_{\Bbbk}(\{u_{i_0}, u_{i_1}, \dots, u_{i_n}\} \cap A_j) + 1\,.$$

Hence $\#(\{u_{i_0}, \dots, u_{i_n}\} \cap A_j) = \dim \mathrm{Aff}_{\Bbbk}(A_j) + 1$ holds for each $0 \leqslant j \leqslant \tilde{N}'$ by assumption. In particular, it holds that

$$\tilde{\pi}(b) = \tilde{\pi}(u_{i_0} + \cdots + u_{i_n}) = \sum_{0 \leqslant j \leqslant \tilde{N}'} (\dim \mathrm{Aff}_{\Bbbk}(A_j) + 1) \cdot u'_j\,,$$

which does not depend on $b \in B$. Thus we have $B - B \subseteq \ker \tilde{\pi}$, i.e., $\tilde{\pi}$ factors through π. $\square$

REMARK 4.4. Assume that $\tilde{\pi}$ factors through π. From the "if" part in the above proof, $\#(\{u_{i_0}, \dots, u_{i_n}\} \cap A_j)$ does not depend on $u_{i_0}, \dots, u_{i_n} \in A$ which span the affine space $M_{\Bbbk}$. Thus each element of $B - B$ is written as a linear combination of elements of $\bigcup_{0 \leqslant j \leqslant \tilde{N}'}(A_j - A_j)$.

PROOF OF PROPOSITION 4.1. Let us take $u_{i_0}, \dots, u_{i_n} \in A$ which span the affine space $M_{\Bbbk}$. Then the latter condition of Lemma 4.3 holds if and only if

$$\dim \mathrm{Aff}_{\Bbbk}(A_j) = \#(\{u_{i_0}, u_{i_1}, \dots, u_{i_n}\} \cap A_j) - 1 \tag{4.1}$$

for any $0 \leqslant j \leqslant \tilde{N}'$ (we note that "$\geqslant$" always holds). On the other hand, since $A = A_0 \sqcup A_1 \sqcup \cdots \sqcup A_{\tilde{N}'}$, we have

$$\sum_{0 \leqslant j \leqslant \tilde{N}'} (\#(\{u_{i_0}, u_{i_1}, \dots, u_{i_n}\} \cap A_j) - 1) = \#\{u_{i_0}, u_{i_1}, \dots, u_{i_n}\} - (\tilde{N}' + 1) = n - \tilde{N}' .$$

Hence $\sum_{0 \leqslant j \leqslant \tilde{N}'} \dim \mathrm{Aff}_{\Bbbk}(A_j) = n - \tilde{N}'$ holds if and only if the equality (4.1) holds for any $0 \leqslant j \leqslant \tilde{N}'$. Therefore this proposition follows from Lemma 4.3. □

COROLLARY 4.5. *Let $A, \pi : M \to M'$ be as in Theorem 1.1. Then it holds that* $\mathrm{rk}(\gamma) \leqslant n - (\#\pi(A) - 1)$. *Moreover, if γ is separable, then we have $\#\pi(A) = \mathrm{rk}\, M' + 1$, which means $X_{\pi(A)}$ is a linear projective space of dimension* $\mathrm{rk}\, M'$.

PROOF. We apply Proposition 4.1 to the homomorphism π. Then it holds that $\sum_j \dim \langle A_j - A_j \rangle_{\Bbbk} = n - (\#\pi(A) - 1)$. From Corollary 3.6, we have $\mathrm{rk}(\gamma) = \dim \mathrm{Aff}_{\Bbbk}(B) = \dim \langle B - B \rangle_{\Bbbk}$. By Remark 4.4, $\langle B - B \rangle$ is contained in the space $\langle \{A_j - A_j\}_j \rangle$. Thus $\mathrm{rk}(\gamma) \leqslant n - (\#\pi(A) - 1)$ holds.

If γ is separable, $\mathrm{rk}(\gamma) = \mathrm{rk}\langle B - B \rangle = n - \mathrm{rk}(M')$ holds by Corollary 3.6. Hence $\mathrm{rk}(M') \geqslant \#\pi(A) - 1$ holds by the above inequality. The converse inequality "$\leqslant$" always holds since $\pi(A)$ spans the affine lattice M'. Since $\pi(A)$ spans the affine lattice M', the equality $\#\pi(A) = \mathrm{rk}\, M' + 1$ means $X_{\pi(A)}$ is a linear projective space of dimension $\mathrm{rk}\, M'$. □

REMARK 4.6. The equality "$\mathrm{rk}(\gamma) = n - (\#\pi(A) - 1)$" does *not* hold in general. For example, set $A = \{0, 1, p\} \subseteq M = \mathbb{Z}^1$ with $p = \mathrm{char}\, \Bbbk > 0$. Then we have

$$B = \{1, p + 1\}, \quad \langle B - B \rangle = \langle p \rangle, \quad \langle B - B \rangle_{\mathbb{R}} \cap M = M .$$

Thus $\pi : M = \mathbb{Z}^1 \to M/(\langle B - B \rangle_{\mathbb{R}} \cap M) = \{0\}$ is the zero map. Here we have $\mathrm{rk}(\gamma) = \dim \langle B - B \rangle_{\Bbbk} = 0$ and $n - (\#\pi(A) - 1) = 1 - (1 - 1) = 1$.

4.2. Separable Gauss maps, Cayley sums, and joins. In this subsection, we study the case when the Gauss map is separable, and prove Corollary 1.4 in the characteristic zero case.

DEFINITION 4.7. Let $l \leqslant n$ be non-negative integers. Let $e_1, \dots, e_l$ be the standard basis of $\mathbb{Z}^l$. For finite sets $A^0, \dots, A^l \subseteq \mathbb{Z}^{n-l}$, the *Cayley sum* $A^0 * \cdots * A^l$ of $A^0, \dots, A^l$ is defined to be

$$A^0 * \cdots * A^l := (A^0 \times \{0\}) \cup (A^1 \times \{e_1\}) \cup \cdots (A^l \times \{e_l\}) \subseteq \mathbb{Z}^{n-l} \times \mathbb{Z}^l .$$

Let A be the Cayley sum of $A^0, \ldots, A^l \subseteq \mathbb{Z}^{n-l}$, and assume that A spans the affine lattice $\mathbb{Z}^{n-l} \times \mathbb{Z}^l$. For the projection $\tilde{\pi} : \mathbb{Z}^{n-l} \times \mathbb{Z}^l \to \mathbb{Z}^l$ to the second factor, $X_{\tilde{\pi}(A)}$ is an l-plane since $\tilde{\pi}(A) = \{0, e_1, \ldots, e_l\}$. By Lemma 2.1, X_A is covered by l-planes, which are translations of $X_{\tilde{\pi}(A)} = \overline{T_{\mathbb{Z}^l}}$. For this $\tilde{\pi}$, $\tilde{N}'$ in Proposition 4.1 is equal to l. Thus, the subtorus $T_{\mathbb{Z}^l} \subseteq T_{\mathbb{Z}^{n-l} \times \mathbb{Z}^l}$ is contracted to one point by the Gauss map of X_A if and only if

$$\sum_{j=0}^{l} \dim \mathrm{Aff}_{\Bbbk}(A^j) = n - l . \tag{4.2}$$

In other words, (4.2) is the condition for the developability of the covering family obtained by translations of $\overline{T_{\mathbb{Z}^l}}$. In fact, any toric variety with separable Gauss map is described by a Cayley sum with the condition (4.2), as follows.

THEOREM 4.8. *Let A, M be as in Theorem 1.1. Assume that the Gauss map γ of $X_A \subseteq \mathbb{P}^N$ is separable, and set $l = \delta_\gamma(X)$. Then there exist finite subsets $A^0, \ldots, A^l \subseteq \mathbb{Z}^{n-l}$ with $\sum_{j=0}^{l} \dim \mathrm{Aff}_{\Bbbk}(A^j) = n - l$ such that A is identified with the Cayley sum of $A^0, \ldots, A^l \subseteq \mathbb{Z}^{n-l}$ under some affine isomorphism $M \simeq \mathbb{Z}^{n-l} \times \mathbb{Z}^l$.*

PROOF. Let $A = \{u_0, u_1, \ldots, u_N\}$ and $\pi : M \to M'$ be as in Theorem 1.1, where we may assume $u_0 = 0 \in M$. From Theorem 1.1 and Corollary 4.5, it follows that $\operatorname{rk} M' = l$ and $\#\pi(A) = l + 1$ since γ is separable by assumption. Set $\pi(A) = \{u'_0, \ldots, u'_l\}$. Without loss of generality, we may assume that $u'_0 = \pi(u_0) = 0 \in M'$. Then $u'_1, \ldots, u'_l$ form a basis of M' since $\pi(A)$ spans the affine lattice $M' \simeq \mathbb{Z}^l$. Set $A_j = \pi^{-1}(u'_j) \cap A$.

Fix a splitting $s : M' \to M$ of the short exact sequence $0 \to \ker \pi \to M \xrightarrow{\pi} M' \to 0$. Then the induced isomorphism

$$M \xrightarrow{\sim} \ker \pi \times M' \quad : \quad u \mapsto (u - s(\pi(u)), \pi(u))$$

gives an identification of $A \subseteq M$ with

$$\bigcup_{j=0}^{l} \left(A^j \times \{u'_j\} \right) \subseteq \ker \pi \times M' , \tag{4.3}$$

where $A^j := A_j - s(u'_j) \subseteq \ker \pi$ is the parallel translation of A_j by $s(u'_j)$. Since $u'_1, \ldots, u'_l$ form a basis of M', $u'_0 = 0$, and $\ker \pi \simeq \mathbb{Z}^{n-l}$, this theorem follows. □

REMARK 4.9. In [6], Di Rocco determined which smooth projective toric varieties over $\mathbb{C}$ have positive dual defects. Roughly, her result states that an n-dimensional smooth toric variety $X_A \subseteq \mathbb{P}^N$ has positive dual defect d if and only if there exist finite subsets $A^0, \ldots, A^{(n+d)/2} \subseteq \mathbb{Z}^{n-(n+d)/2}$ such that

- A is identified with the Cayley sum of $A^0, \ldots, A^{(n+d)/2}$ under some affine isomorphism $M \simeq \mathbb{Z}^{n-(n+d)/2} \times \mathbb{Z}^{(n+d)/2}$,

- all $\mathrm{Conv}(A^j)$'s have the same normal fan,
- $n + d \geqslant 4$.

In order to prove Corollary 1.4, we consider a relation between Cayley sums and joins. For projective varieties $X_1, \dots, X_m \subseteq \mathbb{P}^N$, we define the *join* of $X_1, \dots, X_m$ to be the closure of $\bigcup_{x_1 \in X_1, \dots, x_m \in X_m} \Lambda_{x_1,\dots,x_m} \subseteq \mathbb{P}^N$, where $\Lambda_{x_1,\dots,x_m}$ is the linear variety spanned by the points $x_1, \dots, x_m$.

LEMMA 4.10. *Let $A \subseteq \mathbb{Z}^{n-l} \times \mathbb{Z}^l$ be the Cayley sum of $A^0, \dots, A^l \subseteq \mathbb{Z}^{n-l}$ with $\mathrm{Aff}(A) = \mathbb{Z}^{n-l} \times \mathbb{Z}^l$. Then the following hold.*

(a) *$X_{A^0}, \dots, X_{A^l}$ are embedded into X_A as torus invariant subvarieties, and they are mutually disjoint.*

(b) *X_A is contained in the join of $X_{A^0}, \dots, X_{A^l} \subseteq \mathbb{P}^N$, and the codimension of X_A in the join is $l - n + \sum_{j=0}^{l} \mathrm{rk}\,\mathrm{Aff}(A^j)$.*

PROOF. (a) Write $A^j = \{u_0^j, \dots, u_{N_j}^j\} \subseteq \mathbb{Z}^{n-l}$ for $N_j = \#A^j - 1$. Set $N = \#A - 1 = \sum_{j=0}^{l}(N_j + 1) - 1$ and let $\{X_i^j\}_{0 \leqslant i \leqslant N_j, 0 \leqslant j \leqslant l}$ be the homogeneous coordinates on $\mathbb{P}^N$. By the definition of the Cayley sum A, it holds that

$$
\begin{aligned}
\varphi_A(z, w) &= [w_j z^{u_i^j}]_{0 \leqslant i \leqslant N_j, 0 \leqslant j \leqslant l} \\
&= [w_0 z^{u_0^0} : \cdots : w_0 z^{u_{N_0}^0} : w_1 z^{u_0^1} : \cdots : w_1 z^{u_{N_1}^1} : \cdots : w_l z^{u_0^l} : \cdots : w_l z^{u_{N_l}^l}]
\end{aligned}
\tag{4.4}
$$

in $\mathbb{P}^N$ for $(z, w) = (z_1, \dots, z_{n-l}, w_1, \dots, w_l) \in (\Bbbk^\times)^{n-l} \times (\Bbbk^\times)^l = T_{\mathbb{Z}^{n-l} \times \mathbb{Z}^l}$, where we set $w_0 = 1$. For fixed $0 \leqslant j \leqslant l$ and $z \in (\Bbbk^\times)^{n-l} = T_{\mathbb{Z}^{n-l}}$, $\varphi_A(z, w)$ converges to

$$
[0 : \cdots : 0 : z^{u_0^j} : \cdots : z^{u_{N_j}^j} : 0 : \cdots : 0] \in \mathbb{P}^N \tag{4.5}
$$

when $w_k / w_j \to 0$ for $0 \leqslant k \neq j \leqslant l$. Thus, the point (4.5) is contained in the closure $\overline{\varphi_A(T_{\mathbb{Z}^l \times \mathbb{Z}^{n-l}})} = X_A$. In other words, $\varphi_{A^j}(z) = [z^{u_0^j} : \cdots : z^{u_{N_j}^j}] \in \mathbb{P}^{N_j}$ is contained in X_A for any $z \in (\Bbbk^\times)^{n-l} = T_{\mathbb{Z}^{n-l}}$ by an embedding $\mathbb{P}^{N_j}$ into $\mathbb{P}^N$ as

$$
\mathbb{P}^{N_j} = (X_i^{j'} = 0)_{j' \neq j, 0 \leqslant i \leqslant N_{j'}} \subseteq \mathbb{P}^N . \tag{4.6}
$$

Since X_{A^j} is the closure of $\varphi_{A^j}(T_{\mathbb{Z}^{n-l}})$, $X_{A^j} \subseteq \mathbb{P}^{N_j}$ is contained in X_A.

The action of $T_{\mathbb{Z}^{n-l} \times \mathbb{Z}^l}$ on X_A is described as

$$
(z, w) \cdot [X_i^j]_{0 \leqslant i \leqslant N_j, 0 \leqslant j \leqslant l} = [w_j z^{u_i^j} X_i^j]_{0 \leqslant i \leqslant N_j, 0 \leqslant j \leqslant l}
$$

for $(z, w) = (z_1, \dots, z_{n-l}, w_1, \dots, w_l) \in T_{\mathbb{Z}^{n-l} \times \mathbb{Z}^l}$ and $[X_i^j]_{0 \leqslant i \leqslant N_j, 0 \leqslant j \leqslant l} \in X_A$, where $w_0 = 1$ as before. Therefore $X_{A^j} \subseteq X_A$ is a torus invariant subvariety. Since $\mathbb{P}^{N_0}, \dots, \mathbb{P}^{N_l} \subseteq \mathbb{P}^N$ are mutually disjoint by (4.6), so are $X_{A^0}, \dots, X_{A^l} \subseteq X_A$.

(b) For $(z, w) \in T_{\mathbb{Z}^{n-l} \times \mathbb{Z}^l}$, the image $\varphi_A(z, w) \in X_A \subseteq \mathbb{P}^N$ is described by (4.4), and $\varphi_{A^j}(z) \in X_{A^j} \subseteq \mathbb{P}^N$ is described by (4.5) for each $0 \leqslant j \leqslant l$. Hence $\varphi_A(z, w)$ is contained in the l-plane spanned by $\varphi_{A^0}(z), \varphi_{A^1}(z), \dots, \varphi_{A^l}(z)$. Thus X_A is contained in the join of

$X_{A^0}, \dots, X_{A^l}$. From (4.6), the dimension of the join of $X_{A^0}, \dots, X_{A^l}$ is $l + \sum_{j=0}^{l} \dim X_{A^j}$, which is equal to $l + \sum_{j=0}^{l} \operatorname{rk} \operatorname{Aff}(A^j)$. Since $\dim(X_A) = n$, the assertion about the codimension follows. □

EXAMPLE 4.11. Let $A \subseteq \mathbb{Z}^2 \times \mathbb{Z}^1$ be the Cayley sum of

$$A^0 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 0 \end{bmatrix} \right\}, \; A^1 = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 3 \end{bmatrix} \right\} \subseteq \mathbb{Z}^2 .$$

Then it holds that

$$l - n + \sum_{j=0}^{l} \operatorname{rk} \operatorname{Aff}(A^j) = 1 - 3 + (1 + 1) = 0 .$$

Hence X_A is the join of X_{A^0} and X_{A^1}. In fact,

$$X_A = \overline{\left\{ [1 : x : x^2 : w : wy : wy^2 : wy^3] \mid (x, y, w) \in (\Bbbk^\times)^3 = T_{\mathbb{Z}^2 \times \mathbb{Z}^1} \right\}} \subseteq \mathbb{P}^6 ,$$

and the conic $X_{A^0} \subseteq \mathbb{P}^2$ and the twisted cubic $X_{A^1} \subseteq \mathbb{P}^3$ are embedded into X_A as

$$X_{A^0} = \overline{\left\{ [1 : x : x^2 : 0 : 0 : 0 : 0] \mid x \in \Bbbk^\times \right\}} \subseteq X_A ,$$
$$X_{A^1} = \overline{\left\{ [0 : 0 : 0 : 1 : y : y^2 : y^3] \mid y \in \Bbbk^\times \right\}} \subseteq X_A .$$

REMARK 4.12. In Theorem 4.8, the codimension of X_A in the join of $X_{A^0}, \dots, X_{A^l}$ is $\sum_{j=0}^{l} (\operatorname{rk} \operatorname{Aff}(A^j) - \dim \operatorname{Aff}_\Bbbk(A^j))$ by Lemma 4.10.

Now we can prove Corollary 1.4 immediately.

COROLLARY 4.13 (= Corollary 1.4). *Let $X \subseteq \mathbb{P}^N$ be a projectively embedded toric variety in* $\operatorname{char} \Bbbk = 0$. *Then there exist disjoint torus invariant closed subvarieties $X_0, \dots, X_{\delta_\gamma(X)} \subseteq X$ such that X is the join of $X_0, \dots, X_{\delta_\gamma(X)}$.*

PROOF. We may assume that $X = X_A$ for some finite set $A \subseteq M$ with $\operatorname{Aff}(A) = M$. Then the assertion follows from Theorem 4.8 and Remark 4.12 since the equality $\operatorname{rk} \operatorname{Aff}(A^j) = \dim \operatorname{Aff}_\Bbbk(A^j)$ holds in $\operatorname{char} \Bbbk = 0$. □

COROLLARY 4.14. *Assume* $\operatorname{char} \Bbbk = 0$. *If a toric variety $X_A \subseteq \mathbb{P}^N$ is the join of some projective varieties, then X_A is the join of some toric varieties $X_{A^0}, X_{A^1}, \dots, X_{A^l}$ for some $l > 0$.*

PROOF. Since X_A is the join in $\operatorname{char} \Bbbk = 0$, the Gauss defect $\delta_\gamma(X_A)$ is positive (due to Terracini's lemma). Hence this corollary follows from Corollary 1.4 for $l = \delta_\gamma(X)$. □

The assumption $\operatorname{char} \Bbbk = 0$ is crucial in the above proof of Corollary 1.4. In positive characteristic, even if the Gauss map γ of X_A is separable (equivalently, a general fiber of γ is scheme-theoretically an open subset of a linear variety of $\mathbb{P}^N$), it is possible that γ is degenerate but X_A is *not* the join of any varieties, as follows.

EXAMPLE 4.15. Let $p = \operatorname{char} \Bbbk \geqslant 3$. Set

$$A^0 = \{0, 1, -1\},\ A^1 = \{0, p\} \subseteq \mathbb{Z}^1$$

and let $A \subseteq \mathbb{Z}^1 \times \mathbb{Z}^1$ be the Cayley sum of A^0, A^1. Then

$$\langle B - B \rangle = \mathbb{Z}^1 \times \{0\} \subseteq \mathbb{Z}^1 \times \mathbb{Z}^1$$

as in Figure 3. Hence $\pi : M \to M/(\langle B - B\rangle_{\mathbb{R}} \cap M)$ coincides with the projection $\mathbb{Z}^1 \times \mathbb{Z}^1 \to \mathbb{Z}^1$ to the second factor. In this setting, the following hold.

(1) The Gauss map γ of the surface $X_A \subseteq \mathbb{P}^4$ is separable. A general fiber of γ is a line; in particular, γ is degenerate.
(2) The conic X_{A^0} and the line X_{A^1} are embedded into X_A.
(3) X_A is of codimension one in the join of X_{A^0} and X_{A^1}. On the other hand, X_A itself is not the join of any varieties.

The reason is as follows. (1) The separability of γ follows from Corollary 3.6. A general fiber of γ is projectively equivalent to $X_{\pi(A)}$, which is a line. (2) The embedding of X_{A^j} is given as in Lemma 4.10. (3) It follows from Theorem 4.8 that X_A is contained in the join of X_{A^0} and X_{A^1}. Since the join is of dimension 3, the codimension of X_A in the join is equal to 1. By Remark 4.12, the codimension is also calculated from

$$\operatorname{rk} \operatorname{Aff}(A^0) - \dim \operatorname{Aff}_{\Bbbk}(A^0) = 1 - 1 = 0\,,$$
$$\operatorname{rk} \operatorname{Aff}(A^1) - \dim \operatorname{Aff}_{\Bbbk}(A^1) = 1 - 0 = 1\,.$$

On the other hand, X_A is not the join of any varieties; this is because, a projective surface $X \subseteq \mathbb{P}^N$ is the join of some varieties if and only if X is the cone of a curve with a vertex.

In Figure 3, the Cayley sum A of A^0 and A^1 corresponds to the disjoint union of black bullets $\bullet$ and white bullets $\circ$.

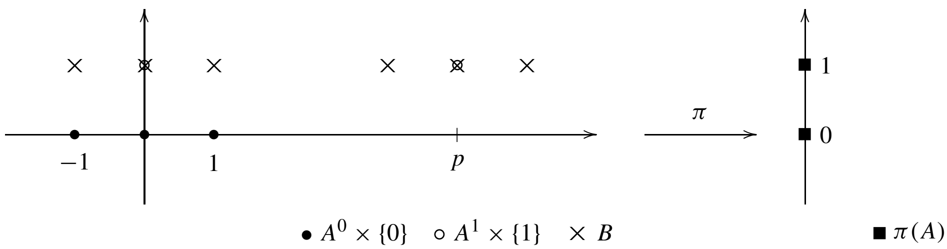

FIGURE 3. Cayley sum $A = A^0 * A^1$ $(p \geqslant 3)$.

5. Constructions in positive characteristic. This section presents two constructions of projectively embedded toric varieties in positive characteristic. We consider whether it is possible to find a toric variety whose Gauss map γ has given data about

(F) each irreducible component of a general fiber of γ;

(I) the image of γ;
(c) the number of the irreducible components of a general fiber of γ;
(r) the rank of γ.

The statement of Theorem 1.5 means that, a projectively embedded toric variety X is constructed for given (F) and (I). In fact, in the construction of Theorem 5.1, we can control (F), (I), and (c), but not (r) (indeed, $\mathrm{rk}(\gamma) = 0$ for X in our proof). On the other hand, in the construction of Theorem 5.4, we can control (F), (r), and (c). Hereafter we assume that $p = \mathrm{char}\,\Bbbk$ is positive.

THEOREM 5.1. *Assume* $p = \mathrm{char}\,\Bbbk > 0$. *Let* A' *and* A'' *be finite subsets of free abelian groups* M' *and* M'' *respectively such that* $\mathrm{Aff}(A') = M'$ *and* $\mathrm{Aff}(A'') = M''$. *Let* $c > 0$ *be an integer coprime to* p. *Assume* $n := \mathrm{rk}(M') + \mathrm{rk}(M'') \geqslant \#A' - 1$ *and* $\mathrm{rk}(M'') \geqslant 1$. *Then there exists a finite subset* $A \subseteq M := \mathbb{Z}^n$ *with* $\#A = n + \#A''$ *and* $\mathrm{Aff}(A) = M$ *such that the Gauss map* γ *of* $X_A \subseteq \mathbb{P}^{\#A-1}$ *satisfies the following conditions:*

(i) *(The closure of) each irreducible component of a general fiber of* γ *is projectively equivalent to* $X_{A'}$.
(ii) *(The closure of) the image* $\gamma(X_A)$ *is projectively equivalent to* $X_{A''}$.
(iii) *The number of the irreducible components of a general fiber of* γ *is equal to* c.

PROOF. We set $N' = \#A' - 1$ and $A' = \{u'_0, \dots, u'_{N'}\}$, and let $e_1, \dots, e_n$ be the standard basis of $M = \mathbb{Z}^n$. Without loss of generality, we may assume that $u'_0 = 0$. We define a group homomorphism π by

$$\pi : M \to M' \quad : \quad e_i \mapsto \begin{cases} u'_i & \text{for } 1 \leqslant i \leqslant N', \\ 0 & \text{for } N' + 1 \leqslant i \leqslant n. \end{cases}$$

We note that $n \geqslant N'$ holds by assumption. Since $\mathrm{Aff}(A') = M'$ and $u'_0 = 0 \in M'$, π is surjective. Hence $\ker \pi$ is a free abelian group whose rank is $\mathrm{rk}(M'')$. Since $\mathrm{rk}(M'') \geqslant 1$, we can take and fix an injective group homomorphism

$$M'' \hookrightarrow \ker \pi$$

whose cokernel is isomorphic to $\mathbb{Z}/\langle c \rangle$. Let $A'' = \{f_0, \dots, f_{N''}\} \subseteq M'' \subseteq \ker \pi$ for $N'' = \#A'' - 1$. Without loss of generality, we may assume that $f_0 = 0 \in M''$. Set

$$A = \{e_1, \dots, e_n, pf_0, \dots, pf_{N''}\} \subseteq M.$$

Since $e_1, \dots, e_n$ form a basis of M and $pf_0 = 0 \in M$, A spans the affine lattice M.

Let B be as in the statement of Theorem 1.1 for the above A. Choose $n + 1$ elements $u_{i_0}, u_{i_1}, \dots, u_{i_n} \in A$ which span the affine space $M_\Bbbk$. Since $pf_s = 0$ in $M_\Bbbk$ for $0 \leqslant s \leqslant N''$, at most one element of $\{pf_0, pf_1, \dots, pf_{N''}\}$ is contained in $\{u_{i_0}, u_{i_1}, \dots, u_{i_n}\}$. Hence $\{u_{i_0}, u_{i_1}, \dots, u_{i_n}\} = \{pf_s, e_1, \dots, e_n\}$ holds for some $0 \leqslant s \leqslant N''$. Thus we have

$$B = \{pf_0 + e_1 + \dots + e_n, \dots, pf_{N''} + e_1 + \dots + e_n\},$$

that is, B is the parallel translation of $p \cdot A''$ by $e_1 + \cdots + e_n$. Hence we have $X_B = X_{p \cdot A''} = X_{A''}$. Since the closure of the image $\gamma(X_A)$ is projectively equivalent to X_B by (1) in Theorem 1.1, the condition (ii) in this theorem holds.

Since $A'' = \{f_0, \dots, f_{N''}\}$ spans the affine lattice M'', it holds that

$$\langle B - B \rangle = p \cdot \langle A'' - A'' \rangle = p \cdot M'' \subseteq M'' \subseteq \ker \pi \,.$$

Therefore $\langle B - B \rangle_{\mathbb{R}} \cap M = \ker \pi$ and the natural projection $M \to M/(\langle B - B \rangle_{\mathbb{R}} \cap M)$ coincides with $\pi : M \to M'$. Since $\pi(A) = A'$ by the definition of π and A, the condition (i) in this theorem follows from (3) in Theorem 1.1.

Since $\ker \pi / M'' \simeq \mathbb{Z}/\langle c \rangle$, the order of the finite group

$$(\langle B - B \rangle_{\mathbb{R}} \cap \mathbb{Z}^n)/\langle B - B \rangle = \ker \pi / (p \cdot M'')$$

is $p^{\mathrm{rk}(M'')} c$. Hence (iii) in this theorem follows from (2) in Corollary 3.6. □

Note that, in the above construction, $\mathrm{rk}(\gamma) = \dim \langle B - B \rangle_{\Bbbk} = 0$.

EXAMPLE 5.2. In this example, we illustrate Theorem 5.1 for

$$A' = \left\{ \begin{bmatrix} 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\} \subseteq \mathbb{Z}^2, \ A'' = \{0, 1, 2, 3\} \subseteq \mathbb{Z}^1$$

and an integer $c > 0$ coprime to $p = \operatorname{char} \Bbbk$. In this case, $X_{A'} = \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$ is a smooth quadric surface and $X_{A''} \subseteq \mathbb{P}^3$ is a twisted cubic curve. Since $n = 2 + 1 \geqslant \# A' - 1 = 4 - 1$, we can apply Theorem 5.1.

We use the notations in the proof of Theorem 5.1. Since $\pi : M = \mathbb{Z}^3 \to M' = \mathbb{Z}^2$ is defined by

$$\pi(e_1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad \pi(e_2) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad \pi(e_3) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

for the standard basis e_1, e_2, e_3 of $\mathbb{Z}^3$, $\ker \pi$ is generated by $e_1 + e_2 - e_3$. Hence an injection $M'' = \mathbb{Z}^1 \hookrightarrow \ker \pi$ with cokernel $\mathbb{Z}/\langle c \rangle$ is given by mapping $1 \in \mathbb{Z}^1$ to $c(e_1 + e_2 - e_3)$. Thus A in the proof of Theorem 5.1 is

$$A = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, p \cdot 0, p \cdot f, p \cdot 2f, p \cdot 3f \right\} \text{ for } f = \begin{bmatrix} c \\ c \\ -c \end{bmatrix}.$$

We can see directly that (i) - (iii) in Theorem 5.1 hold for this A as follows: In this case, X_A is the image of $\varphi_A : (\Bbbk^\times)^3 \hookrightarrow \mathbb{P}^6$ defined by

(5.1) $$(x, y, z) \mapsto [x : y : z : 1 : (xyz^{-1})^{pc} : (xyz^{-1})^{2pc} : (xyz^{-1})^{3pc}] \,.$$

We embed $\mathbb{P}^3$ into $\mathbb{G}(3, \mathbb{P}^6)$ by mapping $[X : Y : Z : W] \in \mathbb{P}^3$ to the 3-plane in $\mathbb{P}^6$ spanned by the 4 points which are given as the row vectors of

$$\begin{bmatrix} 0 & 0 & 0 & X & Y & Z & W \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Then the image of $\varphi_A(x, y, z)$ by the Gauss map γ of X_A is

$$[1 : (xyz^{-1})^{pc} : (xyz^{-1})^{2pc} : (xyz^{-1})^{3pc}] \in \mathbb{P}^3 \subseteq \mathbb{G}(3, \mathbb{P}^6) \tag{5.2}$$

from Lemma 3.3. Hence the closure $\overline{\gamma(X_A)}$ is the twisted cubic curve $X_{A''} \subseteq \mathbb{P}^3$. Thus (ii) holds.

From (5.1) and (5.2), the fiber of $\gamma|_{T_M}$ over $[1 : 1 : 1 : 1] \in \overline{\gamma(X_A)} = X_{A''} \subseteq \mathbb{P}^3$ is

$$\left\{[x : y : z : 1 : 1 : 1 : 1] \in X_A \subseteq \mathbb{P}^6 \mid (x, y, z) \in (\mathbb{k}^\times)^3, (xyz^{-1})^{pc} = 1\right\}.$$

As a set, this is the disjoint union of

$$\left\{[s : t : \zeta^k st : 1 : 1 : 1 : 1] \in X_A \subseteq \mathbb{P}^6 \mid (s, t) \in (\mathbb{k}^\times)^2\right\} \tag{5.3}$$

for $0 \leqslant k \leqslant c - 1$, where $\zeta \in \mathbb{k}^\times$ is a primitive c-th root of unity. Since the closure of each (5.3) is projectively equivalent to $X_{A'} = \mathbb{P}^1 \times \mathbb{P}^1 \subseteq \mathbb{P}^3$, (i) and (iii) are satisfied.

Next, we consider how to construct X for a given integer $r > 0$ such that the rank of the Gauss map of X is equal to r. From the following remark, we need to assume $r \neq 1$ if $\operatorname{char} \mathbb{k} = 2$.

REMARK 5.3. In characteristic 2, it is known that the rank of the Gauss map of any projective variety $X \subseteq \mathbb{P}^N$ *cannot* be equal to 1. In addition, if X is a hypersurface, then the rank of the Gauss map is *even*. The reason is as follows.

Let $X \subseteq \mathbb{P}^N$ be a projective variety in $\operatorname{char} \mathbb{k} = 2$, and let $x \in X$ be a general point. As in [10, §2], choosing homogeneous coordinates on $\mathbb{P}^N$, we may assume that X is locally parameterized at $x = [1 : 0 : \cdots : 0]$ by $[1 : z_1 : \cdots : z_n : f_{n+1} : \cdots : f_N]$, where $z_1, \ldots, z_n$ form a regular system of parameters of $\mathcal{O}_{X,x}$, and $f_{n+1}, \ldots, f_N \in \mathcal{O}_{X,x}$. Then $\operatorname{rk} d_x\gamma$ is equal to the rank of the $n \times (n(N - n))$ matrix

$$\begin{bmatrix} H(f_{n+1}) & H(f_{n+2}) & \cdots & H(f_N) \end{bmatrix}, \tag{5.4}$$

where $H(f) := [\partial^2 f/\partial z_i \partial z_j]_{1 \leqslant i,j \leqslant n}$ is the Hessian matrix of a function f. Assume that $\operatorname{rk}(\gamma)$ $(= \operatorname{rk} d_x\gamma)$ is nonzero. Then the matrix (5.4) is nonzero; in particular, one of the Hessian matrix $H(f_k)$ is nonzero. Since $\operatorname{char} \mathbb{k} = 2$, we have $\partial^2 f_k/\partial z_i \partial z_i = 0$, i.e., the diagonal entries of $H(f_k)$ are zero. Hence some $\partial^2 f_k/\partial z_i \partial z_j$ with $i \neq j$ must be nonzero. Therefore $H(f_k)$ has 2×2 submatrix

$$\begin{bmatrix} \partial^2 f_k/\partial z_i \partial z_i & \partial^2 f_k/\partial z_j \partial z_i \\ \partial^2 f_k/\partial z_i \partial z_j & \partial^2 f_k/\partial z_j \partial z_j \end{bmatrix} = \begin{bmatrix} 0 & \partial^2 f_k/\partial z_i \partial z_j \\ \partial^2 f_k/\partial z_i \partial z_j & 0 \end{bmatrix},$$

whose determinant is nonzero. This implies that $\mathrm{rk}(\gamma) \geqslant 2$.

Now assume that $X \subseteq \mathbb{P}^N$ is a hypersurface. Then X is locally parametrized by $[1 : z_1 : \cdots : z_n : f_{n+1}]$, and hence $\mathrm{rk}\, d_x\gamma = \mathrm{rk}\, H(f_{n+1})$. Since $\mathrm{char}\,\Bbbk = 2$, the symmetric matrix $H(f_{n+1})$ is skew-symmetric, whose rank is even (for example, see [2, §5, n°1, Corollaire 3]).

THEOREM 5.4. *Assume $p = \mathrm{char}\,\Bbbk > 0$. Let A' be a finite subset of a free abelian group M' with $\mathrm{Aff}(A') = M'$. Let $r, c > 0$ be positive integers such that $(p, r) \neq (2, 1)$ and c is coprime to p. Assume that positive integers n, N satisfy*

$$n \geqslant \max\{(\#A' - 1) + r, \mathrm{rk}(M') + r + 1\}$$

and

$$N \geqslant \begin{cases} 2n - \mathrm{rk}(M') - r + 1 & \text{if } p \geqslant 3, \text{ or } p = 2, r : \text{even}, \\ 2n - \mathrm{rk}(M') - r + 2 & \text{if } p = 2, r : \text{odd}. \end{cases} \tag{5.5}$$

Then there exists a finite subset $A \subseteq M := \mathbb{Z}^n$ with $\mathrm{Aff}(A) = M$ and $\#A = N + 1$ such that the Gauss map γ of $X_A \subseteq \mathbb{P}^N$ satisfies the following conditions:

(i) *(The closure of) each irreducible component of a general fiber of γ is projectively equivalent to $X_{A'}$.*
(ii) *The rank of γ is equal to r.*
(iii) *The number of the irreducible components of a general fiber of γ is equal to c.*

PROOF. We set $n' = \mathrm{rk}(M')$, $N' = \#A' - 1$, and $A' = \{u'_0, \dots, u'_{N'}\}$. Let $e_1, \dots, e_n$ be the standard basis of $M = \mathbb{Z}^n$. Without loss of generality, we may assume that $u'_0 = 0$. As in the proof of Theorem 5.1, we define a surjective group homomorphism $\pi : \mathbb{Z}^n \to M'$ by $\pi(e_i) = u'_i$ for $1 \leqslant i \leqslant N'$ and $\pi(e_i) = 0$ for $N' + 1 \leqslant i \leqslant n$. Since $\ker\pi \simeq \mathbb{Z}^{n-n'}$ and $e_{N'+1}, \dots, e_{N'+r} \in \ker\pi$ (note that $N' + r = (\#A' - 1) + r \leqslant n$ holds by assumption), there exist

$$f_1, \dots, f_{n-n'-r} \in \ker\pi$$

such that $e_{N'+1}, \dots, e_{N'+r}, f_1, \dots, f_{n-n'-r}$ form a basis of $\ker\pi$. By assumption, $n - n' - r = n - \mathrm{rk}(M') - r \geqslant 1$ holds.

First, we consider the case when N is equal to the right hand side of (5.5). Set

$$A = C \cup D \subseteq M,$$

where

$$C = \{e_1, \dots, e_n, 0, cpf_1, pf_2, \dots, pf_{n-n'-r}\},$$

$$D = \begin{cases} \{e_{N'+1} + \cdots + e_{N'+r}\} & \text{for } r \not\equiv 1 \bmod p, \\ \{-e_{N'+1} - \cdots - e_{N'+r}\} & \text{for } r \equiv 1, r \not\equiv -1 \bmod p, \\ \{e_{N'+1} + e_{N'+2}, e_{N'+2} + \cdots + e_{N'+r}\} & \text{for } p = 2, r : \text{odd}, r \geqslant 3. \end{cases}$$

By a similar argument as in the proof of Theorem 5.1 and by Remark 3.11, Example 3.12, we have

$$\langle B - B\rangle = \bigoplus_{i=N'+1}^{N'+r} \mathbb{Z}e_i \oplus \mathbb{Z}cpf_1 \oplus \bigoplus_{j=2}^{n-n'-r} \mathbb{Z}pf_j \,. \tag{5.6}$$

Hence $\langle B - B\rangle_{\mathbb{R}} \cap M = \ker \pi$. Since $A' = \pi(A)$, (i) and (iii) in this corollary follows as in Theorem 5.1.

Since $e_{N'+1}, \dots, e_{N'+r}, f_1, \dots, f_{n-n'-r}$ form a basis of $\ker \pi$, we have $\langle B - B\rangle_{\Bbbk} = \bigoplus_{i=N'+1}^{N'+r} \Bbbk e_j$. Thus the rank of γ is $\dim \langle B - B\rangle_{\Bbbk} = r$ by Corollary 3.6.

Next, we consider any integer N satisfying the inequality (5.5). We take a finite subset E of the right hand side of (5.6) such that $N = \#(C \cup D \cup E) - 1$ for the above C and D. Set $A := C \cup D \cup E \subseteq M$. Since E is contained in the right hand side of (5.6), the subgroup $\langle B - B\rangle$ for this A is the same as (5.6). Hence the assertion follows. □

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