Tohoku Math. J.
69 (2017), 239–260

NON-HYPERBOLIC UNBOUNDED REINHARDT DOMAINS: NON-COMPACT AUTOMORPHISM GROUP, CARTAN'S LINEARITY THEOREM AND EXPLICIT BERGMAN KERNEL

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(Received May 18, 2015, revised August 3, 2015)

Abstract. In the study of the holomorphic automorphism groups, many researches have been carried out inside the category of bounded or hyperbolic domains. On the contrary to these cases, for unbounded non-hyperbolic cases, only a few results are known about the structure of the holomorphic automorphism groups. Main result of the present paper gives a class of unbounded non-hyperbolic Reinhardt domains with non-compact automorphism groups, Cartan's linearity theorem and explicit Bergman kernels. Moreover, a reformulation of Cartan's linearity theorem for finite volume Reinhardt domains is also given.

1. Introduction. In several complex variables, complex domains with properties *boundedness* or *hyperbolicity* are fundamental research objects. On the other hand, one may not expect some useful function-theoretic properties for complex domains without the two properties. Our primary motivation is to investigate some useful properties for unbounded non-hyperbolic domains.

1.1. Background. One of the most important problem in several complex variables is to classify all complex domains in $\mathbb{C}^n$. If $n = 1$, then the Riemann mapping theorem tells us that every simply connected proper subdomain of $\mathbb{C}$ is biholomorphically equivalent to the unit disk. By showing the inequivalence of the unit ball and the polydisk in $\mathbb{C}^2$, Poincaré found that the Riemann mapping theorem does not hold even simply connected domains in $\mathbb{C}^2$. Thus the purely topological condition "simply connectedness" is not enough for the holomorphic equivalence problem in $\mathbb{C}^n$. Because of this background, in several complex variables, it is important to investigate biholomorphic invariant objects of complex domains.

The purpose of this paper is to study two objects (the holomorphic automorphism group and the Bergman kernel) for a certain class of non-hyperbolic unbounded Reinhardt domains. The holomorphic automorphism group is a biholomorphic invariant object. Moreover, from the Bergman kernel, one can construct an invariant metric which is so-called the Bergman metric. In the theory of the automorphism groups, one important problem is to understand *how the information of the holomorphic automorphism group characterizes complex domains*. The next theorem is a notable result concerning this problem (cf. [24] and [29]):

2010 *Mathematics Subject Classification.* Primary 32M05; Secondary 32A25.

Key words and phrases. Bergman kernel, holomorphic automorphism group, Reinhardt domain.

The research of the author was supported by an NRF grant 2011-0030044 (SRC-GAIA) of the Ministry of Education, The Republic of Korea.

THEOREM 1.1 (Wong, Rosay). *Let D be a smoothly bounded strongly pseudoconvex domain in $\mathbb{C}^n$. If the holomorphic automorphism group* $\mathrm{Aut}(D)$ *is noncompact, then D is biholomorphic to the unit ball in $\mathbb{C}^n$.*

This monumental work have attracted many mathematicians. In fact, after Wong-Rosay's theorem, there are many characterization theorems for various important domains in several complex variables. For instance, for the Thullen domain, the following result is known (cf. [2] and [3]):

THEOREM 1.2. *If $D \subset \mathbb{C}^2$ is a bounded domain with real analytic boundary, and if the automorphism group of D is noncompact, then D is biholomorphically equivalent to the Thullen domain*

$$\{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^{2m} < 1\}$$

for some positive integer m.

We note that all non-hyperbolic pseudoconvex Reinhardt domains in $\mathbb{C}^2$ with noncompact automorphism group are explicitly described in [19]. For further information of these results and related topics, see survey papers [13], [15], [20] and [30].

Both the automorphism group and the Bergman kernel are usually hard to describe explicitly for a given domain. In view of this background, the unit ball and Thullen domain are important research objects in several complex variables since both domains have the following remarkable properties:

(i) the Bergman kernel with explicit description,
(ii) the noncompact automorphism group with explicit description.

One intuitively expects that domains with *small* automorphism groups (i.e. compact automorphism groups) are not suitable to obtain characterization results like Theorems 1.1 and 1.2. Thus, to find a complex domain with non-compact automorphism group is an important research direction.

For bounded and hyperbolic Reinhardt cases, there are many researches about the structure of the automorphism groups (cf. [9], [12], [25], [26], [27] and references therein). On the other hand, for non-hyperbolic unbounded Reinhardt cases, there is no guarantee of some useful properties which are known only for bounded (or hyperbolic) cases. Thus, essentially new ideas will be needed for the study of non-hyperbolic unbounded Reinhardt domains.

Now, let us recall the following theorem due to Cartan.

THEOREM 1.3 (Cartan's uniqueness theorem). *Let D be a bounded* (*or hyperbolic*) *domain in $\mathbb{C}^n$ and $f : D \to D$ be holomorphic such that $f(p) = p$ for some $p \in D$ and the Jacobian matrix of f at p is the identity matrix (i.e.* $\mathrm{Jac}(f, p) = \mathrm{id}$*). Then f is the identity mapping of D.*

As a consequence of this theorem, one can prove the next theorem, which is also due to Cartan:

THEOREM 1.4 (Cartan's linearity theorem). *Let D be a circular domain (i.e. D is invariant under $z \mapsto e^{i\theta} z$ for any $\theta \in \mathbb{R}$) in $\mathbb{C}^n$ and suppose D contains the origin 0 of $\mathbb{C}^n$. If f is an automorphism with $f(0) = 0$, then f is linear.*

Since Theorem 1.3 is known only for bounded (or hyperbolic) cases, it is nontrivial to ask whether or not Theorem 1.4 remains true for a given non-hyperbolic circular domain. Main result of this paper gives a family of non-hyperbolic unbounded Reinhardt domains with (i) and (ii). Moreover our domains have the following property:

(iii) all origin-preserving automorphisms are linear.

In other words, Theorem 1.4 remains true even though our Reinhardt domains are unbounded and non-hyperbolic. We note that our approach for Theorem 1.4 is based on the theory of the Bergman kernel. The readers will see the three properties (i), (ii), (iii) are closely related each other. Even for bounded cases, there are only few Reinhardt domains with (i), (ii). Thus our domains will be good models of unbounded non-hyperbolic Reinhardt domains such as the unit ball and the Thullen domain.

1.2. Outline of this paper. In §2, we first review basic definitions and examples of the Bergman kernel. Then we next recall an important formula, which is so-called the Forelli-Rudin construction. This formula plays a substantial role in the study of the Bergman kernel of our Reinhardt domains. In §3, we first prove Theorem 1.4 for our Reinhardt domain. Here we use the information of the Bergman kernel obtained in §2. Using Theorem 1.4 for our domains, we obtain an explicit description of the automorphism group. In §4, we reformulate Theorem 1.4 for finite volume Reinhardt domains (possibly unbounded non-hyperbolic).

Acknowledgement. The author thanks the referee for a careful reading of our manuscript and for giving useful comments.

2. Bergman kernel.

2.1. Preliminaries. Let D be a domain in $\mathbb{C}^n$ and ψ a positive continuous function on D. We will denote by $L^2(D, \psi)$ the set of square integrable functions with respect to a function ψ. The weighted Bergman space $A^2(D, \psi)$ is defined by $A^2(D, \psi) = L^2(D, \psi) \cap \mathcal{O}(D)$. The weighted Bergman space is a Hilbert space with the following inner product:

$$\langle f, g \rangle_{D,\psi} = \int_D f(z)\overline{g(z)}\psi(z)dV(z) .$$

The reproducing kernel $K_{D,\psi}$ is called the weighted Bergman kernel. In particular, if $\psi \equiv 1$, then we denote $A^2(D, \psi)$(resp. $K_{D,\psi}$) briefly $A^2(D)$ (resp. K_D). The space $A^2(D)$ (resp. K_D) is called the Bergman space (resp. the Bergman kernel) of D. Let $\{e_k\}_{k\in\mathbb{Z}_{\geq 0}}$ be a complete orthonormal basis of $A^2(D, \psi)$. Then the weighted Bergman kernel is given by

$$K_{D,\psi}(z, w) = \sum_{k\in\mathbb{Z}_{\geq 0}} e_k(z)\overline{e_k(w)} . \tag{1}$$

The unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$ is a typical example for which we can obtain the Bergman kernel in an explicit form.

EXAMPLE 2.1 (the unit ball). For the unit ball in $\mathbb{C}^n$ the following set S forms a complete orthonormal basis of the Bergman space:

$$S = \left\{ \left(\frac{(n+|k|)!}{k!\pi^n} \right)^{\frac{1}{2}} z^k \right\}_{k \in \mathbb{Z}^n_{\geq 0}}, \quad z \in \mathbb{B}_n, k \in \mathbb{Z}^n_{\geq 0}.$$

Here we follow the multi-index convention $|k| = k_1 + \cdots + k_n$ and $z^k = z_1^{k_1} \cdots z_n^{k_n}$. Using (1), one can find that the Bergman kernel of $\mathbb{B}_n$ has an explicit form as follows:

$$K_{\mathbb{B}_n}(z, w) = \frac{n!}{\pi^n (1 - \langle z, w \rangle)^{n+1}}, \quad z, w \in \mathbb{B}_n.$$

EXAMPLE 2.2 (generalized complex ellipsoids). Let us next consider generalized complex ellipsoids $\mathscr{E}_{p,m}$:

$$\mathscr{E}_{p,m} = \{\zeta = (\zeta_{(1)}, \dots, \zeta_{(\ell)}) \in \mathbb{C}^{|m|}; \|\zeta_{(1)}\|^{2p_1} + \cdots + \|\zeta_{(\ell)}\|^{2p_\ell} < 1\}, \quad \zeta_{(k)} \in \mathbb{C}^{m_k}.$$

Since $\mathscr{E}_{p,m}$ is a Reinhardt domain, the set of monomials $\zeta_{(1)}^{i_1} \cdots \zeta_{(\ell)}^{i_\ell}$ forms a complete orthogonal basis in $A^2(\mathscr{E}_{p,m})$ where $i_j \in \mathbb{Z}^{m_j}_{\geq 0}$. This fact, together with (1) gives us

$$K_{\mathscr{E}_{p,m}}(\zeta, \zeta') = \sum_i \frac{(\zeta_{(1)}\overline{\zeta'}_{(1)})^{i_1} \cdots (\zeta_{(\ell)}\overline{\zeta'}_{(\ell)})^{i_\ell}}{\|\zeta_{(1)}^{i_1} \cdots \zeta_{(\ell)}^{i_\ell}\|^2}.$$

It is known that $\|\zeta_{(1)}^{i_1} \cdots \zeta_{(\ell)}^{i_\ell}\|^{-2}$ has the following form [7]:

$$\|\zeta_{(1)}^{i_1} \cdots \zeta_{(\ell)}^{i_\ell}\|^{-2} = \frac{\Gamma(1 + \sum_{j=1}^{\ell}(|i_j| + m_j)/p_j) \prod_{j=1}^{\ell} p_j \Gamma(|i_j| + m_j)}{\pi^{|m|} \prod_{j=1}^{\ell} \Gamma((|i_j| + m_j)/p_j) i_j!}. \tag{2}$$

Thus the Bergman kernel $K_{\mathscr{E}_{p,m}}$ has the following form:

$$\begin{aligned} K_{\mathscr{E}_{p,m}}(\zeta, \zeta') &= \frac{\prod_{j=1}^{\ell} p_j}{\pi^{|m|}} \sum_i \frac{\Gamma(1 + \sum_{j=1}^{\ell}(|i_j| + m_j)/p_j) \prod_{j=1}^{\ell} \Gamma(|i_j| + m_j)}{\prod_{j=1}^{\ell} \Gamma((|i_j| + m_j)/p_j) i_j!} \\ &\quad \times (\zeta_{(1)}\overline{\zeta'_{(1)}})^{i_1} \cdots (\zeta_{(\ell)}\overline{\zeta'_{(\ell)}})^{i_\ell} \\ &= \sum_k \frac{\Gamma(1 + \sum_{j=1}^{\ell}(k_j + m_j)/p_j) \prod_{j=1}^{\ell}(k_j + 1)_{m_j}}{\pi^{|m|} \prod_{j=1}^{\ell} \Gamma((k_j + m_j)/p_j + 1)} \prod_{j=1}^{\ell} \langle \zeta_{(j)}, \zeta'_{(j)} \rangle^{k_j}. \end{aligned} \tag{3}$$

Here $(x)_m$ is the Pochhammer symbol and the second equality follows from the following identities:

$$\prod_{j=1}^{\ell} \frac{\Gamma(|i_j| + m_j)}{\Gamma((|i_j| + m_j)/p_j)} = \prod_{j=1}^{\ell} \frac{\Gamma(|i_j| + m_j + 1)}{p_j \Gamma((|i_j| + m_j)/p_j + 1)}.$$

$$\frac{(x_1 + \cdots + x_m)^k}{k!} = \sum_{|i|=k} \frac{x^i}{i!}.$$

In this case, we can construct a complete orthonormal basis like the unit ball case. However, unlike the unit ball case, the Bergman kernel cannot be expressed in terms elementary functions in general. Indeed, J.-D. Park [23] proved that the Bergman kernel of

$$D_{(p_1,p_2)} = \{(z_1, z_2) \in \mathbb{C}^2; |z_1|^{2p_1} + |z_2|^{2p_2} < 1\}$$

is represented by means of elementary functions if and only if $p = (1, p_2), (p_1, 1), (2, 2)$. We note that G. Francsics and N. Hanges [7] proved that the Bergman kernel $K_{\mathscr{E}_{p,m}}$ is expressed in terms of the hypergeometric functions.

REMARK 2.3. Besides the above examples, there are some domains with explicit Bergman kernels. The followings are such examples:

- symmetrized n-disc (see [6]),
- homogeneous Siegel domains of type II (see [10]),
- bounded symmetric domains (see [11]),
- the minimal ball (see [22]).

We now give an example of weighted Bergman kernel, which is so-called the Fock-Bargmann kernel.

EXAMPLE 2.4 (Fock-Bargmann kernel). Let us first recall that a weighted Bergman space $A^2(D, \psi)$ is called the Fock-Bargmann space if $D = \mathbb{C}^n$ and $\psi = e^{-s\|z\|^2}$ with $s > 0$. The reproducing kernel $K_{n,s}$ of the Fock-Bargmann space is called the Fock-Bargmann kernel. One can easily see that the set of monomials $\{{z_1}^{k_1} \cdots {z_n}^{k_n}\}_{k_1,\dots,k_n \geq 0}$ forms a complete orthogonal basis. After a straightforward computation, one find that

$$\begin{aligned}\|{z_1}^{k_1} \cdots {z_n}^{k_n}\|^2_{D,\psi} &= \int_{\mathbb{C}^n} |{z_1}^{k_1} \cdots {z_n}^{k_n}|^2 e^{-s\|z\|^2} dV(z)\\ &= \frac{\pi^n k!}{s^{|k|+n}}.\end{aligned}$$

It implies that the following set forms a complete orthonormal basis of the Fock-Bargmann space:

$$S = \left\{ \left(\frac{\pi^n k!}{s^{|k|+n}} \right)^{-1/2} z^k \right\}_{k \in \mathbb{Z}^n_{\geq 0}}.$$

This gives us an explicit form of the Fock-Bargmann kernel:

$$\begin{aligned}K_{n,s}(z, w) &= \sum_{k \in \mathbb{Z}^n_{\geq 0}} \left(\frac{\pi^n k!}{s^{|k|+n}} \right)^{-1} z^k \overline{w}^k\\ &= \frac{s^n}{\pi^n} \prod_{i=1}^{n} \sum_{k_i=0}^{\infty} \frac{s^{k_i}}{k_i!} z_i^{k_i} \overline{w}_i^{k_i}\\ &= \frac{s^n}{\pi^n} e^{s\langle z, w\rangle}.\end{aligned}$$

2.2. Forelli-Rudin construction. Let Ω_1, Ω_2 be domains such that $\Omega_1 \subset \Omega_2$ and φ a positive continuous function on Ω_1. Suppose that the Bergman kernel K_{Ω_2} and the weighted Bergman kernel K_{Ω_1,φ^k} exist for any $k \geq 1$. We begin with a general question.

QUESTION 2.5. What can we say about the relation between the two different kernels K_{Ω_2} and K_{Ω_1,φ^k}?

In general, we cannot expect a good relation between them. One important class of domains concerning this question is the Hartogs domain, which is defined as follows:

$$D_m^\varphi := \{(z,\zeta) \in D \times \mathbb{C}^m : \|\zeta\|^2 < \varphi(z)\}.$$

The following formula tells us how the Bergman kernel of D_m^φ and the weighted Bergman kernels $K_{D,\varphi^{k+m}}$ are related (cf. [21]):

$$K_{D_m^\varphi}((z,\zeta),(z',\zeta')) = \sum_{k=0}^{\infty} \frac{(k+1)_m}{\pi^m} K_{D,\varphi^{k+m}}(z,z')\langle \zeta, \zeta' \rangle^k. \tag{4}$$

This formula (4) is called the Forelli-Rudin construction.

In our previous paper [32], we obtained a generalization of the Forelli-Rudin construction. Let P be a real valued continuous function on $\mathbb{C}^m$ and $\alpha = (\alpha_1, \dots, \alpha_m) \in \mathbb{R}_+^m$. A function P satisfying the following condition is called quasi-homogeneous with weight α:

$$\lambda P(x_1, \dots, x_m) = P(\lambda^{\alpha_1} x_1, \dots, \lambda^{\alpha_m} x_m)$$

for any $x = (x_1, \dots, x_m) \in \mathbb{R}^m$. Consider the domain defined by

$$D_{P,m}^\varphi := \{(z,\zeta) \in D \times \mathbb{C}^m ; P(|\zeta_1|^2, \dots, |\zeta_m|^2) < \varphi(z)\}$$

with quasi-homogeneity on P. Especially, if $P(x_1, \dots, x_m) = x_1 + \cdots + x_m$, then $D_{P,m}^\varphi$ is the Hartogs domain. Let us define

$$D_P^m(r) := \{\zeta \in \mathbb{C}^m ; P(|\zeta_1|^2, \dots, |\zeta_m|^2) < r\},$$

where $r > 0$. If $r = 1$, then we simply denote it by D_P^m. In the following, we always assume the next conditions on P.

- P is quasi-homogeneous with weight α,
- $D_P^m(r)$ is bounded and complete.

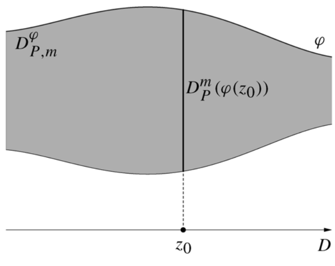


FIGURE 1.

The next theorem give a version of the Forelli-Rudin construction for $D^{\varphi}_{P,m}$.

THEOREM 2.6. *The Bergman kernel $K_{D^{\varphi}_{P,m}}$ of $D^{\varphi}_{P,m}$ has the following series representation:*

$$K_{D^{\varphi}_{P,m}}((z,\zeta),(z',\zeta')) = \sum_{k\in\mathbb{Z}^m_{\geq 0}} a_k K_{D,\varphi^{|\alpha(1+k)|}}(z,z')(\zeta\overline{\zeta'})^k\,,$$

where $a_k := \|\zeta_1^{k_1}\cdots\zeta_m^{k_m}\|^{-2}_{L^2_a(D^m_P)}$, $|\alpha(1+k)| := \sum_{r=1}^m \alpha_r(1+k_r)$.

In [32], this formula is applied to establish deflation type identities which is initiated by Boas-Fu-Straube [5]. Let us now consider the case $D = \mathbb{C}^n$, $\varphi(z) = e^{-\|z\|^2}$. The next theorem asserts that the Bergman kernel of the domain is expressed in terms of the Fock-Bargmann kernel and the Bergman kernel of D^m_P

THEOREM 2.7. *Let us assume that $D = \mathbb{C}^n$, $\varphi(z) = e^{-\|z\|^2}$. Then the Bergman kernel of the domain $D^{\varphi}_{P,m}$ is given by*

$$K_{D^{\varphi}_{P,m}}((z,\zeta),(z',\zeta')) = |\alpha|^{-n}K_{n,|\alpha|}(z,z')D^n_\alpha K_{D^m_P}(F_\alpha(t,\zeta),\zeta')\Big|_{t=\langle z,z'\rangle}\,,$$

where $D_\alpha = |\alpha| + \partial_t$ and $F_\alpha(t,\zeta) = (e^{\alpha_1 t}\zeta_1,\ldots,e^{\alpha_m t}\zeta_m)$.

PROOF. By Theorem 2.6 and Example 2.4, we have

$$\begin{aligned}K_{D^{\varphi}_{P,m}}((z,\zeta),(z',\zeta')) &= \sum_{k\in\mathbb{Z}^m_{\geq 0}} \frac{a_k}{\pi^n}|\alpha(1+k)|^n e^{|\alpha(1+k)|\langle z,z'\rangle}(\zeta\overline{\zeta'})^k \\ &= \frac{e^{|\alpha|\langle z,z'\rangle}}{\pi^n}\sum_{k\in\mathbb{Z}^m_{\geq 0}} a_k|\alpha(1+k)|^n e^{|\alpha k|\langle z,z'\rangle}(\zeta\overline{\zeta'})^k\,.\end{aligned}$$

By Example 2.4, we find that

$$|\alpha|^{-n}K_{n,|\alpha|}(z,z') = \frac{e^{|\alpha|\langle z,z'\rangle}}{\pi^n}\,. \tag{5}$$

By the definition of D_α, we easily see that

$$D^n_\alpha e^{|\alpha k|t} = (|\alpha| + |\alpha k|)^n e^{|\alpha k|t} = |\alpha(1+k)|^n e^{|\alpha k|t}. \tag{6}$$

Combining these two relations (5) and (6), we obtain

$$K_{D^{\varphi}_{P,m}}((z,\zeta),(z',\zeta')) = |\alpha|^{-n}K_{n,|\alpha|}(z,z')D^n_\alpha\left(\sum_{k\in\mathbb{Z}^m_{\geq 0}} a_k\prod_{r=1}^m e^{\alpha_r k_r t}(\zeta_r\overline{\zeta'}_r)^{k_r}\right)\Bigg|_{t=\langle z,z'\rangle}\,.$$

Since D^m_P is a Reinhardt domain, the set of all normalized monomials forms a complete orthonormal basis of $A^2(D^m_P)$. Therefore, the Bergman kernel has the following form:

$$K_{D^m_P}(\zeta,\zeta') = \sum_{k\in\mathbb{Z}^m_{\geq 0}} a_k(\zeta\overline{\zeta'})^k\,.$$

Thus we finally conclude that

$$K_{D^{\varphi}_{P,m}}((z,\zeta),(z',\zeta')) = |\alpha|^{-n} K_{n,|\alpha|}(z,z') D^n_\alpha K_{D^m_P}(F_\alpha(t,\zeta),\zeta')\Big|_{t=\langle z,z'\rangle} .$$

We have thus proved the theorem. □

This theorem gives an answer for Question 2.5 when $\Omega_1 = \mathbb{C}^n \times \{0\} \simeq \mathbb{C}^n$, $\Omega_2 = D^{\varphi}_{P,m}$ and $\varphi(z) = e^{-\|z\|^2}$.

2.3. Special case. In this section we restrict our attention to the following special case:

$$\mathscr{E}_p = \{(z,\zeta_1,\zeta_2) \in \mathbb{C}^3 : |\zeta_1|^2 + |\zeta_2|^{2p} < e^{-s|z|^2}\} .$$

The aim of this section is to show that the Bergman kernel of this domain is expressed in terms of the Fock-Bargmann kernel and a certain function M, which is defined from the polylogarithm function Li_{-n}. Let us begin with recalling some preliminary facts on the Thullen domain $\mathscr{D}_p$:

$$\mathscr{D}_p = \{(\zeta_1,\zeta_2) \in \mathbb{C}^2 : |\zeta_1|^2 + |\zeta_2|^{2p} < 1\} .$$

Put $a_{k_1,k_2} = \|\zeta_1^{k_1}\zeta_2^{k_2}\|^{-2}_{L^2_a(\mathscr{D}_p)}$. By (2), we have

$$a_{k_1,k_2} = \frac{p\Gamma(2+k_1+(k_2+1)/p)}{\pi^2\Gamma(k_1+1)\Gamma((k_2+1)/p)} .$$

Let M be defined by

$$M(t_1,t_2) := \sum_{k_1,k_2\geq 0} a_{k_1,k_2} t_1^{k_1} t_2^{k_2} .$$

The next lemma asserts that the series M can be expressed in terms of the polylogarithm function Li_{-n} (see also Appendix A).

LEMMA 2.8. *The series M has the following closed form:*

$$M(t_1,t_2) = \frac{1}{\pi^2(1-t_1)^{2+1/p}} \sum_{j=0}^{2} c_j Li_{-j}(g(t_1,t_2)) ,$$

where $c_0 = \frac{1}{p}+1$, $c_1 = \frac{2}{p}+1$, $c_2 = \frac{1}{p}$ and $g(t_1,t_2) = t_2/(1-t_1)^{1/p}$.

PROOF. Let us first recall the following simple fact:

$$\sum_{i=0}^{\infty} \frac{\Gamma(i+a)}{\Gamma(i+1)} t^i = \frac{\Gamma(a)}{(1-t)^a} , \tag{7}$$

where $a > 0$ and $|t| < 1$. Then we have

$$M(t_1,t_2) = \sum_{k_1,k_2\geq 0} \frac{p\Gamma(2+k_1+(k_2+1)/p)}{\pi^2\Gamma(k_1+1)\Gamma((k_2+1)/p)} t_1^{k_1} t_2^{k_2}$$

$$= \frac{p}{\pi^2} \sum_{k_2=0}^{\infty} \frac{1}{\Gamma((k_2+1)/p)} t_2^{k_2} \sum_{k_1=0}^{\infty} \frac{p\Gamma(2+k_1+(k_2+1)/p)}{\pi^2 \Gamma(k_1+1)} t_1^{k_1}$$

$$= \frac{p}{\pi^2} \sum_{k_2=0}^{\infty} \frac{\Gamma(2+(k_2+1)/p) t_2^{k_2}}{\Gamma((k_2+1)/p)(1-t_1)^{2+(k_2+1)/p}}$$

$$= \frac{p}{\pi^2(1-t_1)^{2+1/p}} \sum_{k_2=0}^{\infty} \frac{\Gamma(2+(k_2+1)/p)}{\Gamma((k_2+1)/p)} \cdot g(t_1,t_2)^{k_2} .$$

Here, the third equality follows from (7). Since

$$\frac{\Gamma(2+(k_2+1)/p)}{\Gamma((k_2+1)/p)} = \frac{c_0 + c_1 k_2 + c_2 k_2^2}{p} ,$$

we obtain

$$M(t_1,t_2) = \frac{1}{\pi^2(1-t_1)^{2+1/p}} \sum_{k_2=0}^{\infty} (c_0 + c_1 k_2 + c_2 k_2^2) g(t_1,t_2)^{k_2}$$

$$= \frac{1}{\pi^2(1-t_1)^{2+1/p}} \sum_{j=0}^{2} c_j \mathrm{Li}_{-j}(g(t_1,t_2)) .$$

This completes the proof. □

Now we obtain a description of the Bergman kernel of $\mathscr{E}_p$.

THEOREM 2.9. *The Bergman kernel $K_{\mathscr{E}_p}$ of $\mathscr{E}_p$ has the following form:*

$$K_{\mathscr{E}_p}((z,\zeta),(z',\zeta')) = K_{1,1+\frac{1}{p}}(z,z') M(e^{s\langle z,z'\rangle}\zeta_1\overline{\zeta'}_1, e^{s\langle z,z'\rangle}\zeta_2\overline{\zeta'}_2) + \left(1+\frac{1}{p}\right)^{-1} K_{1,1+\frac{1}{p}}(z,z')\, \mathcal{R}M(t_1, t_2^{1/p})\Big|_{\substack{t_1 = e^{s\langle z,z'\rangle}\zeta_1\overline{\zeta'}_1 \\ t_2 = e^{s\langle z,z'\rangle}\left(\zeta_2\overline{\zeta'}_2\right)^p}},$$

where $\mathcal{R}f$ is the radial derivative of f (i.e. $\mathcal{R}f(t) = \sum_{i=1}^2 t_i \frac{\partial f}{\partial t_i}(t)$).

PROOF. By Theorem 2.6, we have

$$K_{\mathscr{E}_p}((z,\zeta),(z',\zeta')) = \left(1+\frac{1}{p}\right)^{-1} K_{1,1+\frac{1}{p}}(z,z') G((z,\zeta),(z',\zeta')) ,$$

where G is given by

$$G((z,\zeta),(z',\zeta')) = \sum_{k_1,k_2\geq 0} a_{k_1,k_2} \left(1 + k_1 + \frac{1+k_2}{p}\right) \times \left(e^{s\langle z,z'\rangle}\zeta_1\overline{\zeta'_1}\right)^{k_1} \left(e^{s\langle z,z'\rangle}\left(\zeta_2\overline{\zeta'_2}\right)^p\right)^{k_2/p} .$$

This can be rewritten as

$$G((z,\zeta),(z',\zeta')) = \left(1+\frac{1}{p}\right) M(e^{s\langle z,z'\rangle}\zeta_1\overline{\zeta'}_1, e^{s\langle z,z'\rangle}\zeta_2\overline{\zeta'}_2) + \mathcal{R}M(t_1, t_2^{1/p})\Big|_{\substack{t_1=e^{s\langle z,z'\rangle}\zeta_1\overline{\zeta'}_1 \\ t_2=e^{s\langle z,z'\rangle}\left(\zeta_2\overline{\zeta'}_2\right)^p}}.$$

Here the equality follows from

$$\mathcal{R}M(t_1, t_2^{1/p}) = \mathcal{R}\sum_{k_1,k_2\geq 0} a_k t_1^{k_1} t_2^{k_2/p} = \sum_{k_1,k_2\geq 0} a_k\left(k_1+\frac{k_2}{p}\right) t_1^{k_1} t_2^{k_2/p}.$$

This completes the proof. □

We finish this section with one remark on G.

REMARK 2.10. In the case of $p=1$, $\mathscr{E}_p$ is a Hartogs type domain:

$$\mathscr{E}_1 = \{(z,\zeta)\in\mathbb{C}\times\mathbb{C}^2 : \|\zeta\|^2 < e^{-s|z|^2}\}.$$

For this case, by (11), G is given as follows:

$$G((z,\zeta),(z',\zeta')) = \frac{1}{\pi^2}\sum_{k=0}^{\infty}(k+1)_2(k+2)\left(e^{sz\overline{z'}}\langle\zeta,\zeta'\rangle\right)^k = \frac{1}{\pi^2}\frac{\partial^2 \mathrm{Li}_{-1}(t)}{\partial t^2}\bigg|_{t=e^{sz\overline{z'}}\langle\zeta,\zeta'\rangle} = \frac{2(2+e^{sz\overline{z'}}\langle\zeta,\zeta'\rangle)}{\pi^2(1-e^{sz\overline{z'}}\langle\zeta,\zeta'\rangle)^4}.$$

More generally, it is also known that the Bergman kernel of

$$\mathscr{D}_{n,m} = \{(z,\zeta)\in\mathbb{C}^{n+m} : \|\zeta\|^2 < e^{-s\|z\|^2}\}$$

is expressed in terms of the polylogarithm function (cf. [31]):

$$K_{\mathscr{D}_{n,m}}((z,\zeta),(z',\zeta')) = \frac{s^n}{\pi^{m+n}} e^{ms\langle z,z'\rangle}\frac{d^m}{dt^m}\mathrm{Li}_{-n}(t)\bigg|_{t=e^{s\langle z,z'\rangle}\langle\zeta,\zeta'\rangle}.$$

3. Holomorphic automorphism groups. This section is devoted to give a new class of non-hyperbolic Reinhardt domains with non-compact automorphism groups. We begin this section with the simplest case $\mathscr{E}_{p_1,p_2,s}$.

$$\mathscr{E}_{p_1,p_2,s} = \{(z,\zeta_1,\zeta_2)\in\mathbb{C}\times\mathbb{C}\times\mathbb{C} : |\zeta_1|^{2p_1}+|\zeta_2|^{2p_2} < e^{-s|z|^2}\}.$$

For simplicity of our argument, let us assume that $p_1\neq p_2$ and $p_1, p_2 \neq 1$. Although our argument works more general class of domains which contains $\mathscr{E}_{p_1,p_2,s}$, the author believes that this example will help the readers to grasp the key idea.

3.1. Cartan's theorem for $\mathscr{E}_{p_1,p_2,s}$. We begin our study with Cartan's theorem (Theorem 1.4) for our domain $\mathscr{E}_{p_1,p_2,s}$. We first observe that $\mathscr{E}_{p_1,p_2,s}$ contains $\mathcal{U} = \{(z,0,0) \in \mathbb{C}^3\} \simeq \mathbb{C}$. It follows that $\mathscr{E}_{p_1,p_2,s}$ is not hyperbolic. Thus it is a non-trivial question to determine whether Cartan's theorem holds for $\mathscr{E}_{p_1,p_2,s}$.

The aim of this section is to prove Cartan's theorem for $\mathscr{E}_{p_1,p_2,s}$. In order to bypass the difficulties arising from the unboundedness and non-hyperbolicity, we make use of the theory of Bergman kernel. We note that our proof is carried out without using Theorem 1.3. Our study begin with the following observation in [16]:

PROPOSITION 3.1. *Let $D \subset \mathbb{C}^n$ be a circular domain (not necessarily bounded) with the Bergman kernel K_D and suppose that D contains the origin. Assume that the following two conditions hold:*

(i) $K_D(0,0) > 0$,

(ii) $T_D(0,0)$ *is positive definite,*

where $T_D(z,w)$ is an $n \times n$ matrix defined by

$$T_D(z,w) := \begin{pmatrix} \dfrac{\partial^2}{\partial \overline{w}_1 \partial z_1} \log K_D(z,w) & \cdots & \dfrac{\partial^2}{\partial \overline{w}_1 \partial z_n} \log K_D(z,w) \\ \vdots & \ddots & \vdots \\ \dfrac{\partial^2}{\partial \overline{w}_n \partial z_1} \log K_D(z,w) & \cdots & \dfrac{\partial^2}{\partial \overline{w}_n \partial z_n} \log K_D(z,w) \end{pmatrix}.$$

Then every origin-preserving automorphism of D is linear.

Let us first check that (i) holds for $\mathscr{E}_{p_1,p_2,s}$.

LEMMA 3.2. *The Bergman kernel of $\mathscr{E}_{p_1,p_2,s}$ satisfies (i).*

PROOF. By Theorem 2.6, we see that $K_{\mathscr{E}_{p_1,p_2,s}}(0,0) > 0$ is equivalent to the positivity of the Fock-Bargmann kernel $K_{n,s}(0,0)$. It is obvious to check that $K_{n,s}(0,0) > 0$ from Example 2.4. □

We next check that (ii) holds for $\mathscr{E}_{p_1,p_2,s}$.

LEMMA 3.3. *The Bergman kernel of $\mathscr{E}_{p_1,p_2,s}$ satisfies (ii).*

PROOF. By Theorem 2.6, the Bergman kernel of $\mathscr{E}_{p_1,p_2,s}$ has an form:

$$(8) \qquad K_{\mathscr{E}_{p_1,p_2,s}}((z,\zeta),(w,\zeta')) = \frac{s e^{s\left(\frac{1}{p_1}+\frac{1}{p_2}\right)z\overline{w}}}{\pi} \sum_{k_1,k_2 \geq 0} a_k |p^{-1}(1+k)| e^{s\left(\frac{k_1}{p_1}+\frac{k_2}{p_2}\right)z\overline{w}} \times \left(\zeta_1 \overline{\zeta'}_1\right)^{k_1} \left(\zeta_2 \overline{\zeta'}_2\right)^{k_2}.$$

For simplicity of notation, we put

$$F(z,w) = e^{s\left(\frac{1}{p_1}+\frac{1}{p_2}\right)z\overline{w}},$$

$$G((z,\zeta),(w,\zeta'))=\sum_{k_1,k_2\geq 0} a_k|p^{-1}(1+k)|e^{s\left(\frac{k_1}{p_1}+\frac{k_2}{p_2}\right)z\overline{w}}\left(\zeta_1\overline{\zeta'}_1\right)^{k_1}\left(\zeta_2\overline{\zeta'}_2\right)^{k_2}.$$

Then $\log K_{\mathscr{E}_{p_1,p_2}}$ is expressed as follows:

$$\log K_{\mathscr{E}_{p_1,p_2}}((z,\zeta),(w,\zeta'))=\log\left(\frac{s}{\pi}\right)+\log F(z,w)+\log G((z,\zeta),(w,\zeta')).$$

By a straightforward calculation, we see that

$$T_{\mathscr{E}_{p_1,p_2}}(0,0)=\begin{pmatrix}c_1 & 0 & 0\\ 0 & c_2 & 0\\ 0 & 0 & c_3\end{pmatrix},$$

where c_1, c_2, c_3 are positive constants. This completes the proof of the lemma. □

Thus the Bergman kernel of $\mathscr{E}_{p_1,p_2,s}$ satisfies (i) and (ii). Therefore, we obtain Cartan's theorem for our domain.

THEOREM 3.4. *Every origin-preserving automorphism of $\mathscr{E}_{p_1,p_2,s}$ is linear.*

3.2. Description of $\mathrm{Aut}(\mathscr{E}_{p_1,p_2,s})$**.** In this section, we apply Theorem 3.4 to obtain an explicit description of the holomorphic automorphism group of $\mathscr{E}_{p_1,p_2,s}$. We first show the invariance of $\mathcal{U}=\{(z,0,0)\in\mathbb{C}^3\}\subset\mathscr{E}_{p_1,p_2,s}$. Although the proof is similar to that of [16, Lemma 8], we give it here for the sake of completeness.

LEMMA 3.5. *Let f be an arbitrary automorphism of $\mathscr{E}_{p_1,p_2,s}$. Then the space $\mathcal{U}$ is invariant under f (i.e. $f(\mathcal{U})\subset\mathcal{U}$).*

PROOF. Let us put $f(z,0,0)=(f_1(z),f_2(z),f_3(z))$. Since f is an automorphism of $\mathscr{E}_{p_1,p_2,s}$, we have

$$|f_2(z)|^{2p_1}+|f_3(z)|^{2p_2}<e^{-s|f_1(z)|^2}\leq 1.$$

By the above, f_2 and f_3 are bounded holomorphic functions on $\mathbb{C}$, and thus they are constant functions by Liouville's theorem. Since f is an automorphism, f is not a constant function. It follows that f_1 is a non-constant entire function. In particular, f_1 is not bounded. Thus there exists a sequence $\{z_k\}_{k\geq 0}$ such that $|f_1(z_k)|\to\infty$ as $k\to\infty$. Therefore we conclude that $f_2, f_3\equiv 0$. □

Before proceeding, we pause to recall the following result about the complex ellipsoid (see [18]):

LEMMA 3.6. *Let p_1, p_2 be positive real numbers such that $p_1\neq p_2$ and $p_1, p_2\neq 1$. If f is a linear automorphism of the complex ellipsoid $\{|\zeta_1|^{2p_1}+|\zeta_2|^{2p_2}<1\}\subset\mathbb{C}^2$, then f is given by*

$$f=\begin{pmatrix}e^{i\theta_1} & 0\\ 0 & e^{i\theta_2}\end{pmatrix},$$

where θ_1, θ_2 are arbitrary real numbers.

This lemma will be used in the next lemma, which describes linear automorphisms of $\mathscr{E}_{p_1,p_2,s}$. Since it is not easy to generalize the proof used in [16, Lemma 9] for our case, we show the next lemma by using different method.

LEMMA 3.7. *Let g be an arbitrary linear automorphism of $\mathscr{E}_{p_1,p_2,s}$. Then g is given by*

$$g = \begin{pmatrix} e^{i\theta} & 0 & 0 \\ 0 & e^{i\theta_1} & 0 \\ 0 & 0 & e^{i\theta_2} \end{pmatrix},$$

where $\theta, \theta_1, \theta_2$ are arbitrary real numbers.

PROOF. Let us put

$$g(z,\zeta) = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} z \\ \zeta \end{pmatrix},$$

where $a \in \mathbb{C}, b \in \mathrm{Mat}_{1\times 2}(\mathbb{C}), c \in \mathrm{Mat}_{2\times 1}(\mathbb{C}), d \in \mathrm{Mat}_{2\times 2}(\mathbb{C})$. Then Lemma 3.5 implies that $c = 0$. Since f is an automorphism, we see that $a \neq 0$ and $\det d \neq 0$.

We next show that $|a| = 1, b = 0$. Recall the following transformation formula of T_D under the automorphism φ:

$$T_D(z,w) = \overline{{}^t J_\varphi(w)} T_D(\varphi(z), \varphi(w)) J_\varphi(z),$$

where $J_\varphi(z)$ is the Jacobi matrix of φ at z. Applying this formula to g, we obtain

$$T_{\mathscr{E}_{p_1,p_2,s}}(0,0) = \overline{{}^t J_g(0)} T_{\mathscr{E}_{p_1,p_2,s}}(0,0) J_g(0).$$

It is equivalent to

$$\begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} = {}^t\overline{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}} \begin{pmatrix} c_1 & 0 & 0 \\ 0 & c_2 & 0 \\ 0 & 0 & c_3 \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix}.$$

This relation gives us $c_1 = |a|^2 c_1$ and $\overline{a} c_1 b = 0$. Since $c_1 \neq 0$ and $a \neq 0$, we have $|a| = 1$ and $b = 0$ as desired.

Let us turn to the remaining case. Observe that

$$\{(0,\zeta) \in \mathscr{E}_{p_1,p_2,s}\} = \{0\} \times \{|\zeta_1|^{2p_1} + |\zeta_2|^{2p_2} < 1\},$$
$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & d \end{pmatrix} \begin{pmatrix} 0 \\ \zeta \end{pmatrix} = \begin{pmatrix} 0 \\ d\zeta \end{pmatrix}.$$

Thus g induces a linear automorphism of $\{|\zeta_1|^{2p_1} + |\zeta_2|^{2p_2} < 1\}$:

$$g_d : \{|\zeta_1|^{2p_1} + |\zeta_2|^{2p_2} < 1\} \to \{|\zeta_1|^{2p_1} + |\zeta_2|^{2p_2} < 1\}, \quad \zeta \mapsto d\zeta.$$

The above argument, together with Lemma 3.6, implies our desired conclusion. □

We are now ready to determine the automorphism group of our domain $\mathscr{E}_{p_1,p_2,s}$.

THEOREM 3.8. *The automorphism group of* $\mathscr{E}_{p_1,p_2,s}$ *is generated by*

$$\varphi_{\theta,\theta_1,\theta_2} : (z, \zeta_1, \zeta_2) \mapsto (e^{i\theta} z, e^{i\theta_1} \zeta_1, e^{i\theta_2} \zeta_2),$$

$$\varphi_{v,p_1,p_2} : (z, \zeta_1, \zeta_2) \mapsto \left(z + v, e^{-\frac{s|v|^2}{2p_1} - \frac{sz\overline{v}}{p_1}} \zeta_1, e^{-\frac{s|v|^2}{2p_2} - \frac{sz\overline{v}}{p_2}} \zeta_2\right),$$

where $\theta, \theta_1, \theta_2 \in \mathbb{R}$ *and* $v \in \mathbb{C}$.

PROOF. It is not difficult to verify that both $\varphi_{\theta,\theta_1,\theta_2}$ and φ_{v,p_1,p_2} are elements of $\mathrm{Aut}(\mathscr{E}_{p_1,p_2,s})$. Let φ be an arbitrary automorphism of $\mathscr{E}_{p_1,p_2,s}$. By Lemma 3.5, there exists $v_0 \in \mathbb{C}$ such that $\varphi(0, 0, 0) = (v_0, 0, 0)$. Then we have $\varphi_{-v_0,p_1,p_2} \circ \varphi(0, 0, 0) = (0, 0, 0)$. By Theorem 3.4 and Lemma 3.7, we have $\varphi_{-v_0,p_1,p_2} \circ \varphi = \varphi_{\theta,\theta_1,\theta_2}$. Namely, we conclude that $\varphi = \varphi_{v_0,p_1,p_2} \circ \varphi_{\theta,\theta_1,\theta_2}$. □

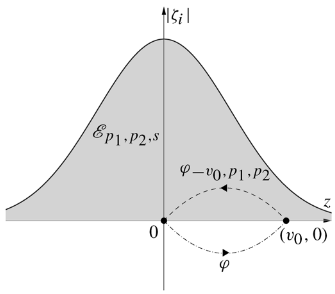


FIGURE 2.

3.3. General case. Let $(p_1, \ldots, p_\ell)$ be positive real numbers with $p_i \neq p_j$ for any $1 \leq i, j, \leq \ell$. In this section, we study the holomorphic automorphism group of $\mathscr{E}_{p,m,n,,s}$ which is defined by

$$\mathscr{E}_{p,m,n,s} := \left\{ (z, \zeta_{(1)}, \ldots, \zeta_{(\ell)}) \in \mathbb{C}^n \times \mathbb{C}^{m_1} \times \cdots \times \mathbb{C}^{m_\ell} : \sum_{j=1}^{\ell} \|\zeta_{(j)}\|^{2p_j} < e^{-s\|z\|^2} \right\}.$$

In the following, to simplify notation, we write $\widetilde{\mathscr{E}}$ instead of $\mathscr{E}_{p,m,n,s}$. As is already mentioned in the beginning of §3, the arguments for $\mathscr{E}_{p_1,p_2,s}$ are naturally generalized for $\widetilde{\mathscr{E}}$ after suitable modifications. Thus we will explain the key steps of our arguments and omit the details. We first show Cartan's theorem for $\widetilde{\mathscr{E}}$.

PROPOSITION 3.9. *Every origin-preserving automorphism of* $\widetilde{\mathscr{E}}$ *is linear.*

PROOF. It is enough to check (i), (ii) for our domain $\widetilde{\mathscr{E}}$. The argument for (i) is identical to that of Lemma 3.2. In this case $T_{\widetilde{\mathscr{E}}}(0, 0)$ has the following form

$$T_{\widetilde{\mathscr{E}}}(0, 0) = \begin{pmatrix} c_1 \mathrm{I}_n & 0 \\ 0 & D \end{pmatrix}.$$

Here c_1 is a positive number and D is an $|m| \times |m|$ diagonal matrix with positive entries. Thus $T_{\widetilde{\mathscr{E}}}(0,0)$ is positive definite. □

Our next task is to describe all linear automorphisms of $\widetilde{\mathscr{E}}$. In this case we need the following lemma instead of Lemma 3.6 (see [18]).

LEMMA 3.10. *Let $(p_1, \dots, p_\ell)$ be positive numbers with $p_i \neq p_j$ for any $1 \leq i, j, \leq \ell$. Let f be a linear automorphism of the generalized complex ellipsoid $\mathscr{E}_{p,m}$. Then f is given by*

$$f(\zeta_{(1)}, \dots, \zeta_{(\ell)}) = \begin{pmatrix} U_1 & & 0 \\ & \ddots & \\ 0 & & U_\ell \end{pmatrix} \begin{pmatrix} \zeta_{(1)} \\ \vdots \\ \zeta_{(\ell)} \end{pmatrix},$$

where $U_i \in U(m_i)$ for any $1 \leq i \leq \ell$.

Let us define $\widetilde{\mathcal{U}} \subset \widetilde{\mathscr{E}}$ by

$$\widetilde{\mathcal{U}} := \{(z, 0, \dots, 0) \in \mathbb{C}^n \times \mathbb{C}^{|m|}\}.$$

The next lemma gives the invariance of $\widetilde{\mathcal{U}}$ under the automorphisms. Since the proof is the same as that of Lemma 3.5, we omit it.

LEMMA 3.11. *Let f be an arbitrary automorphism of $\widetilde{\mathscr{E}}$. Then the space $\widetilde{\mathcal{U}}$ is invariant under f.*

We are now in position to describe all linear automorphisms of $\widetilde{\mathscr{E}}$.

LEMMA 3.12. *Let g be an arbitrary linear automorphism of $\widetilde{\mathscr{E}}$. Then g is given by*

$$g(z, \zeta_{(1)}, \dots, \zeta_{(\ell)}) = \begin{pmatrix} U & & & 0 \\ & U_1 & & \\ & & \ddots & \\ 0 & & & U_\ell \end{pmatrix} \begin{pmatrix} z \\ \zeta_{(1)} \\ \vdots \\ \zeta_{(\ell)} \end{pmatrix},$$

where $U \in U(n)$ and $U_i \in U(m_i)$ for any $1 \leq i \leq \ell$.

PROOF. By Lemma 3.11, we can put

$$g(z, \zeta) = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \begin{pmatrix} z \\ \zeta \end{pmatrix},$$

where $a \in \mathrm{Mat}_{n \times n}(\mathbb{C})$, $b \in \mathrm{Mat}_{n \times |m|}(\mathbb{C})$ and $d \in \mathrm{Mat}_{|m| \times |m|}(\mathbb{C})$. Applying the transformation formula of $T_{\widetilde{\mathscr{E}}}$ to g, we have the following relation:

$$\begin{pmatrix} c_1 \mathrm{I}_n & 0 \\ 0 & D \end{pmatrix} = {}^t\overline{\begin{pmatrix} a & b \\ 0 & d \end{pmatrix}} \begin{pmatrix} c_1 \mathrm{I}_n & 0 \\ 0 & D \end{pmatrix} \begin{pmatrix} a & b \\ 0 & d \end{pmatrix},$$

where $c_1 > 0$ and D is a diagonal matrix with positive entries in $\mathrm{Mat}_{|m| \times |m|}(\mathbb{C})$. Comparing the $(1, 1)$ block entry and $(1, 2)$ block entry, we have $c_1 \mathrm{I}_n = c_1 {}^t\overline{a} a$ and $c_1 {}^t\overline{a} b = 0$. It follows that $a \in U(n)$ and $b = 0$. By using the same argument as in Lemma 3.7, we see that d is a linear automorphism of the generalized complex ellipsoid $\mathscr{E}_{p,m}$ as desired. □

Using the same logic used in the proof of Theorem 3.8, we obtain the following theorem

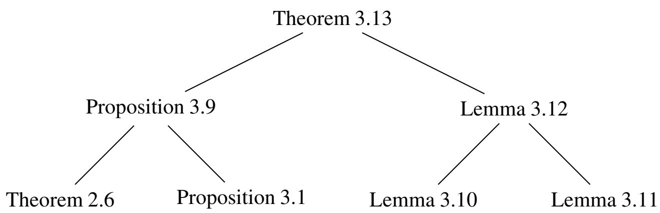
THEOREM 3.13. *The automorphism group of $\mathscr{E}_{p,m,n,s}$ is generated by*

$$\varphi_1 : (z, \zeta_{(1)}, \dots, \zeta_{(\ell)}) \mapsto (Uz, U_1\zeta_{(1)}, \dots, U_\ell\zeta_{(\ell)}),$$

$$\varphi_2 : (z, \zeta_{(1)}, \dots, \zeta_{(\ell)}) \mapsto \left(z + v, e^{-\frac{s\|v\|^2}{2p_1} - \frac{s\langle z,v\rangle}{p_1}}\zeta_{(1)}, \dots, e^{-\frac{s\|v\|^2}{2p_\ell} - \frac{s\langle z,v\rangle}{p_\ell}}\zeta_{(\ell)}\right),$$

where $U \in U(n)$, $U_k \in U(m_k)$ for any $1 \le k \le \ell$ and $v \in \mathbb{C}^n$.

3.4. A further generalization. Here we discuss about a further generalization of Theorem 3.13. Before doing so, let us pause to describe an outline of logical structure of §3.3.



We note that Proposition 3.9 remains true for

$$D_{P,m,s} := \{(z, \zeta) \in \mathbb{C}^n \times \mathbb{C}^m;\ P(|\zeta_1|^2, \dots, |\zeta_m|^2) < e^{-s\|z\|^2}\},$$

with the same assumptions as in §2.2. Indeed, it is verified by using Theorem 2.6 and Proposition 3.1.

PROPOSITION 3.14. *Every origin-preserving automorphism of $D_{P,m,s}$ is linear.*

On the other hand, the proofs of Lemmas 3.10 and 3.11 highly depend on special properties of the complex ellipsoid. We now assume the following condition:

(I) $\mathcal{V} = \{(z, 0) \in \mathbb{C}^n \times \mathbb{C}^m\} \subset D_{P,m,s}$ is invariant under the automorphisms of $D_{P,m,s}$.

We denote by $\mathrm{Iso}_0(D_{P,m,s})$ the set of all origin-preserving automorphisms. By Proposition 3.14, we see that

$$\mathrm{Iso}_0(D_{P,m,s}) = \{f \in \mathrm{Aut}(D_{P,m,s}) : f \text{ is linear}\}.$$

Then we can show the next lemma under (I), which is a generalization of Lemma 3.12:

LEMMA 3.15. *Let g be an arbitrary linear automorphism of $D_{P,m,s}$ and suppose that $D_{P,m,s}$ satisfies (I). Then g is given by*

$$g(z, \zeta) = \begin{pmatrix} U & 0 \\ 0 & A \end{pmatrix} \begin{pmatrix} z \\ \zeta \end{pmatrix},$$

where $U \in U(n)$ and $A \in \mathrm{Iso}_0(D_P^m)$.

We note that the condition (I) is used to show that the (2, 1) block entry of g is the zero matrix.

Let us define

$$\varphi_{v,\alpha}(z,\zeta) = (z+v, e^{-\frac{s\alpha_1\|v\|^2}{2} - s\alpha_1\langle z,v\rangle}\zeta_1, \ldots, e^{-\frac{s\alpha_m\|v\|^2}{2} - s\alpha_m\langle z,v\rangle}\zeta_m),$$

where $v \in \mathbb{C}^n$. Thanks to the quasi-homogeneity of P, we observe that $\varphi_{v,\alpha}$ is an automorphism of $D_{P,m,s}$. This observation, together with the above discussion, implies the next theorem.

THEOREM 3.16. *Suppose that $D_{P,m,s}$ satisfies the condition (I). Then the automorphism group of $D_{P,m,s}$ is generated by* $\mathrm{Iso}_0(D_{P,m,s})$ *and* $\{\varphi_{v,\alpha}\}_{v\in\mathbb{C}^n}$.

From this theorem, one can see that the automorphism group of $D_{P,m,s}$ is noncompact.

4. Cartan's theorem for finite volume Reinhardt domains.

Since our interest is to give a new family of non-hyperbolic Reinhardt domains with (i), (ii), (iii), we restricted our attention to some special domains in the previous sections. However, it might be worthwhile to reformulate Cartan's theorem (Theorem 1.4) for Reinhardt domains (possibly unbounded non-hyperbolic). In what follows, we always assume that a Reinhardt domain contains the origin.

THEOREM 4.1. *Let $D \subset \mathbb{C}^n$ be a Reinhardt domain (possibly unbounded non-hyperbolic). Suppose that* $\mathrm{Vol}(D) \lneq \infty$ *and $z_i \in A^2(D)$ for any $1 \le i \le n$. Then all automorphisms f with $f(0) = 0$ are linear.*

PROOF. It is enough to verify (i), (ii) of Proposition 3.1 for our D. Let us begin with a discussion of the Bergman kernel K_D. Put $S = \{z_1^{k_1}\cdots z_n^{k_n} : (k_1,\ldots,k_n) \in \mathbb{Z}^n_{\ge 0}\} \cap A^2(D)$. Since D is a Reinhardt domain containing the origin, the set S forms a complete orthogonal basis of $A^2(D)$ (cf. [1]). In particular, by the assumptions of the theorem, we see that the constant function $f \equiv 1$ and z_i are elements of S. Then, by (1), the Bergman kernel of D has the following form:

$$K_D(z,z') = c_0 + \sum_{r=1}^{n} c_{1r} z_r \overline{z'}_r + \sum_{j\in J} a_j (z_1\overline{z'}_1)^{j_1}\cdots(z_n\overline{z'}_n)^{j_n},$$

where $J := \{j = (j_1,\ldots,j_n) \in \mathbb{Z}^n_{\ge 0} : z_1^{j_1}\cdots z_n^{j_n} \in S\backslash\{f, z_1,\ldots,z_n\}\}$. By the assumption of the theorem, we have $c_0 > 0$ and $c_{1r} > 0$ for any $1 \le r \le n$. Especially we see that $K_D(0,0) > 0$. It follows that K_D satisfies (i) of Proposition 3.1. We next consider (ii) of Proposition 3.1. We first observe that

$$\frac{\partial^2 \log K_D(z,z')}{\partial\overline{z'}_i\partial z_j} = \frac{\left(\frac{\partial^2 K_D(z,z')}{\partial\overline{z'}_i\partial z_j}\right)K_D(z,w) - \left(\frac{\partial K_D(z,z')}{\partial\overline{z'}_i}\right)\left(\frac{\partial K_D(z,z')}{\partial z_j}\right)}{K_D(z,z')^2}.$$

This, together with the above expression of K_D, implies that

$$\frac{\partial^2 \log K_D(0,0)}{\partial \overline{z'}_i \partial z_j} = \begin{cases} \dfrac{c_{1i}}{c_0}\,, & \text{if } i = j \\ 0\,, & \text{otherwise}\,. \end{cases}$$

Thus $T_D(0,0)$ is a diagonal matrix with positive entries. Therefore $T_D(0,0)$ is a positive definite matrix. Hence, K_D and $T_D(0,0)$ satisfy (i) and (ii) of Proposition 3.1. This proves the theorem. □

The next two examples are non-hyperbolic unbounded Reinhardt domains in $\mathbb{C}^2$ satisfying the assumptions of this theorem.

EXAMPLE 4.2 (see [1]). Let $s, s' > 0$ and consider the following non-hyperbolic domain:

$$\widehat{\mathscr{D}_{1,1}} = \{(z_1, z_2) \in \mathbb{C}^2 : |z_2|^2 < e^{-s|z_1|^{\frac{2}{s'}}}\}\,.$$

Then one can check that $\mathbb{C}[z_1, z_2] \subset A^2(\widehat{\mathscr{D}_{1,1}})$. As an application of the above theorem, we see that all origin-preserving automorphism of $\widehat{\mathscr{D}_{1,1}}$ are linear. We note that this domain includes, as a special case, the domain $\mathscr{D}_{1,1}$ considered in Remark 2.10.

EXAMPLE 4.3 (see [14]). We next consider the following domain:

$$\Omega_W = \{(z, w) \in \mathbb{C}^2 : \log|w|^2 + |z|^2 + |w|^2 < 1\}\,.$$

It is known that all monomials $z^{k_1} w^{k_2}$ are elements of $A^2(\Omega_W)$. Thus Cartan's theorem also holds this non-hyperbolic unbounded Reinhardt domain.

If a Reinhardt domain D is bounded, then the assumption of this theorem is always verified. On the other hand, if a Reinhardt domain D is unbounded, then there is an example of finite volume Reinhardt domain D in $\mathbb{C}^2$ such that $z_1, z_2 \notin A^2(D)$:

EXAMPLE 4.4 (see [28]). Let us define

$$X_1 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_2| < \frac{1}{|z_1| \log|z_1|}, |z_1| > e\right\},$$
$$X_2 = \left\{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < \frac{1}{|z_2| \log|z_2|}, |z_2| > e\right\},$$
$$\Omega = X_1 \cup X_2 \cup \{(z_1, z_2) \in \mathbb{C}^2 : |z_1| < 2e, |z_2| < 2e\}.$$

Then it is known that the monomials contained in $A^2(\Omega)$ are precisely

$$a(z_1 z_2)^k, \quad k = 0, 1, 2, \dots\,.$$

Thus Ω is a finite volume Reinhardt domain with $z_1, z_2 \notin A^2(\Omega)$. In [28], by using this domain Ω, it is also shown that there is a family of domains $\{\Omega_k\}_{k \in \mathbb{Z}_{\geq 0}}$ in $\mathbb{C}^2$ such that $A^2(\Omega_k) = \operatorname{span}\{1, z_1 z_2, \dots, (z_1 z_2)^{k-1}\}$.

In view of the above theorem, it might be interesting to ask the following question.

QUESTION 4.5. Can we find a Reinhardt domain D in $\mathbb{C}^n$ such that $A^2(D) = \operatorname{span}\{1, z_1, z_2, \dots, z_n\}$?

We do not have any such examples at the time of writing this paper. To find such examples is interesting problem and it will be investigated in the future research. We conclude this article with the following theorem due to Engliš (see [8]).

THEOREM 4.6. *The Bergman space $A^2(D)$ is $\{0\}$ or is infinite-dimensional if D is a pseudoconvex Reinhardt domain in $\mathbb{C}^n$.*

Thus, if there exists a Reinhardt domain D as in the above question, then D must not be pseudoconvex.

Appendix A. Polylogarithm function. Here we collect basic properties of the polylogarithm function. Let us first recall that the following series expansion of the logarithm function:

$$-\log(1-z) = \sum_{k=1}^{\infty} \frac{z^k}{k}, \tag{9}$$

for $|z| < 1$. Replacing z^k/k by z^k/k^s, we define the polylogarithm function Li_s:

$$\mathrm{Li}_s(z) = \sum_{k=1}^{\infty} \frac{z^k}{k^s}, \tag{10}$$

for $|z| < 1$, $s \in \mathbb{C}$. The polylogarithm function $\mathrm{Li}_{-s}(z)$ is a rational function of z if $s \in \mathbb{Z}_{\geq 0}$. Indeed, it is verified from the following simple facts:

$$\mathrm{Li}_0(z) = \frac{z}{1-z},$$
$$\frac{\partial}{\partial z}\mathrm{Li}_s(z) = \frac{\mathrm{Li}_{s-1}(z)}{z}.$$

Let us list the first few cases.

EXAMPLE A.1. If $s = 0, -1, -2, -3, -4, -5$, then $\mathrm{Li}_s(z)$ is given as follows:

$$\mathrm{Li}_0(z) = \frac{z}{1-z}, \quad \mathrm{Li}_{-1} = \frac{z}{(1-z)^2}, \quad \mathrm{Li}_{-2}(z) = \frac{z(z+1)}{(1-z)^3},$$
$$\mathrm{Li}_{-3}(z) = \frac{z(z^2+4z+1)}{(1-z)^4}, \quad \mathrm{Li}_{-4}(z) = \frac{z(z^3+11z^2+11z+1)}{(1-z)^5},$$
$$\mathrm{Li}_{-5}(z) = \frac{z(z^4+26z^3+66z^2+26z+1)}{(1-z)^6}.$$

It is known that the polylogarithm function Li_s has the following form when $s = -n$ is negative integer:

$$\mathrm{Li}_{-n}(z) = \frac{z}{(1-z)^{n+1}} \sum_{j=0}^{n-1} A(n, j+1) z^j,$$

where $A(n,m)$ is the Eulerian number (cf. [4, Eq. (2.17)]):

$$A(n,m) = \sum_{k=0}^{m} (-1)^k \binom{n+1}{k} (m-k)^n .$$

By definition, it is easily seen that the m-th derivative of the polylogarithm is given by

$$\frac{\partial^m \mathrm{Li}_s(z)}{\partial z^m} = \sum_{k=0}^{\infty} \frac{(k+1)_m z^k}{(k+m)^s}. \tag{11}$$

There is a closed form of the m-th derivative of the polylogarithm:

$$\frac{\partial^m \mathrm{Li}_{-n}(z)}{\partial z^m} = \frac{m! \sum_{j=0}^{n} (-1)^{n+j} (m+1)_j S(1+n, 1+j)(1-z)^{n-j}}{(1-z)^{m+n+1}},$$

where $S(\cdot,\cdot)$ denotes the Stirling number of the second kind. This expression is a simple consequence of the following formula (cf. [4, Eq. 2.10c]):

$$\mathrm{Li}_{-n}(z) = \sum_{j=0}^{n} \frac{(-1)^{n+j} j! S(1+n, 1+j)}{(1-z)^{j+1}} .$$

The next table gives closed forms of $\frac{\partial^m \mathrm{Li}_{-n}(z)}{\partial z^m}$ for $1 \le m, n \le 3$.

m \ n	1	2	3
1	$-\frac{z+1}{(z-1)^3}$	$\frac{z^2+4z+1}{(z-1)^4}$	$-\frac{z^3+11z^2+11z+1}{(z-1)^5}$
2	$\frac{2(z+2)}{(z-1)^4}$	$-\frac{2(z^2+7z+4)}{(z-1)^5}$	$\frac{2(z^3+18z^2+33z+8)}{(z-1)^6}$
3	$-\frac{6(z+3)}{(z-1)^5}$	$\frac{6(z^2+10z+9)}{(z-1)^6}$	$-\frac{6(z^3+25z^2+67z+27)}{(z-1)^7}$

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