

## PERIODIC MAGNETIC CURVES IN BERGER SPHERES

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**Abstract.** It is an interesting question whether a given equation of motion has a periodic solution or not, and in the positive case to describe it. We investigate periodic magnetic curves in elliptic Sasakian space forms and we obtain a quantization principle for periodic magnetic flowlines on Berger spheres. We give a criterion for periodicity of magnetic curves on the unit sphere  $\mathbb{S}^3$ .

**1. Introduction.** It has been a long-standing problem to find closed trajectories in dynamical systems on manifolds. The existence of closed trajectories turns out to be subtle and closely related to topology and geometric structures of the manifolds. For example, the following fundamental existence theorem of closed geodesics is well known. For any compact Riemannian manifold  $(M, g)$ , in each element of the fundamental group  $\pi_1(M)$ , there exists a closed geodesic which attains the minimum of the energy functional in the homotopy class.

An interesting generalization of the notion of geodesic is that of magnetic curve. In classical mechanics, motions of particles in the Euclidean 3-space are described by the Newton equation  $m\dot{\gamma}(t) = F(\gamma(t))$ , where  $m$  is the mass and the vector field  $F$  represents the force. Take an immersed surface  $M$  in the Euclidean 3-space  $\mathbb{E}^3$ . A motion of the particle constrained on  $M$  obeys the equation of motion:

$$\nabla_{\dot{\gamma}}\dot{\gamma} = 0.$$

Here  $\nabla$  is the Levi-Civita connection of  $M$ . This is nothing but the equation of geodesics.

The Newton equation is generalized to motions in curved spaces (semi-Riemannian manifolds) as follows:

$$m\nabla_{\dot{\gamma}}\dot{\gamma} = F_{\gamma}.$$

Hence, geodesics in Riemannian manifolds are mathematical models of motions of a particle with no force acting on it.

A free relativistic particle of mass  $m$  is described in terms of geodesics in Minkowski spacetime  $\mathbb{E}_1^4$  with the Lagrangian

$$L(x, \dot{x}) = -mc\sqrt{-\langle \dot{x}, \dot{x} \rangle}.$$

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Here  $c$  is the light speed. In general relativity, free falls are regarded as geodesics in the spacetime, i.e. time-oriented Lorentzian 4-manifolds.

Magnetic curves represent, in physics, the trajectories of charged particles moving on a Riemannian manifold under the influence of magnetic forces. The Landau-Hall problem means the study of the motion of charged particles on a Riemannian surface in the presence of a constant and static magnetic field. In this context, in the absence of any electric field, a particle of mass  $m$  and charge  $q$  moves with the velocity  $v(t)$  satisfying the Lorentz force law

$$\frac{dP(t)}{dt} = \frac{q}{c}v(t) \times \mathbf{B},$$

where  $\mathbf{B}$  is the magnetic field,  $P = (\varepsilon/c^2)v$  is the momentum and  $\varepsilon = mc^2/\sqrt{1 - |v|^2/c^2}$  is the energy of the particle. Geodesics are magnetic curves with vanishing magnetic field.

In Euclidean 3-space, if  $\mathbf{B}$  is stationary, i.e. independent of  $t$ , then the motion of the particle is described by a circular helix around  $\mathbf{B}$ . As a particular situation, the magnetic trajectory of the particle may be a circle, hence a periodic curve.

Moreover, magnetic curves are derived from the variational problem of the Landau-Hall functional. See e.g. [6, 17]. In the absence of a magnetic field, the Landau-Hall functional coincides with the kinetic energy functional. It is well known that geodesics are characterized as critical points of the energy functional. From variational point of view, the existence of global potential for magnetic fields is a natural assumption. Based on these observations, we study magnetic curves in 3-dimensional contact metric manifolds, especially in Sasakian manifolds.

More generally, in any 3-dimensional Riemannian manifold  $(M, g)$ , magnetic fields of nonzero constant length are in one to one correspondence to almost contact structures compatible to the metric  $g$  (see [3] and Section 2.4 of the present paper). This fact motivates us to study magnetic curves in almost contact metric 3-manifolds (with closed fundamental 2-form).

In [21], Taubes proved the generalized Weinstein conjecture in dimension 3, namely, *on a compact, orientable, contact 3-manifold the Reeb vector field  $\xi$  has at least one closed integral curve*. In conjunction with this problem it is important to explore the existence of periodic magnetic trajectories of the contact magnetic field defined by  $\xi$  in Sasakian manifolds. It is an important fact that the study of magnetic curves in Sasakian manifolds with constant  $\varphi$ -sectional curvature (known as *Sasakian space forms*) of arbitrary dimension reduces to their investigation in dimension 3. See [9]. Berger spheres are realized as Sasakian space forms of constant  $\varphi$ -sectional curvature  $c > -3$ . Such a Sasakian space form is called an *elliptic Sasakian space form*.

In 2009 Cabrerizo et al. [8] have been looked for periodic orbits of the contact magnetic field on the unit sphere  $\mathbb{S}^3$ . See also [2]. In this paper we study closed magnetic curves in arbitrary elliptic Sasakian space forms and we obtain a quantization principle for periodic magnetic flowlines on Berger spheres. We prove that *the set of all periodic magnetic curves of arbitrary strength on the Sasakian space form  $\mathcal{M}^3(c)$ , ( $c > -3$ ) can be quantized in the set*

of rational numbers (**Theorem 5.1**) and we write the quantization principle generalizing the result of Cabrerizo et al. Finally, we give a criterion for periodicity of magnetic curves on the unit sphere  $\mathbb{S}^3$  (**Theorem 6.1**).

To close Introduction, we mention some symplectic geometric motivations for the study of contact magnetic curves. During the study of area minimization problem among Lagrangian surfaces in Kähler surfaces, Schoen and Wolfson completely classified admissible singular points of area minimizing Lagrangian surfaces. These singularities are locally modelled by Hamilton-minimal Lagrangian cones in complex Euclidean plane  $\mathbb{C}^2$  (see [19, Theorem 7.1]). In particular, Schoen and Wolfson classified Hamilton-minimal Lagrangian cones in  $\mathbb{C}^2$ . Those Lagrangian cones are realized as cones over  $L$ -minimal Legendre curves in  $\mathbb{S}^3$ . As a result, closed  $L$ -minimal Legendre curves in the unit 3-sphere  $\mathbb{S}^3$  are classified (see [19] and also [13]). Here a Legendre curve in a 3-dimensional Sasakian manifold is said to be  $L$ -minimal if it is a critical point of the length functional under Legendre variations.

One can see that every closed  $L$ -minimal Legendre curve in  $\mathbb{S}^3$  is a contact magnetic curve. Conversely, every closed contact magnetic curve in  $\mathbb{S}^3$  which is Legendre with respect to the canonical Sasakian structure is  $L$ -minimal and hence induces a Hamiltonian-minimal Lagrangian cone in  $\mathbb{C}^2$ .

**2. Preliminaries.**

**2.1. Magnetic curves.** The motion of the charged particles in a Riemannian manifold under the action of the magnetic fields are known as magnetic curves. More precisely, a magnetic field  $F$  on a Riemannian manifold  $(M, g)$  is a closed 2-form  $F$  and the Lorentz force associated to  $F$  is a tensor field  $\phi$  of type  $(1, 1)$  such that

$$(1) \quad F(X, Y) = g(\phi X, Y), \quad X, Y \in \mathfrak{X}(M).$$

A curve  $\gamma$  on  $M$  that satisfies the Lorentz equation

$$(2) \quad \nabla_{\dot{\gamma}} \dot{\gamma} = \phi(\dot{\gamma}),$$

is called a magnetic trajectory of  $F$ , or simply a magnetic curve. Here  $\nabla$  denotes the Levi-Civita connection associated to the metric  $g$ . The Lorentz equation generalizes the equation of geodesics under arc length parametrization, namely,  $\nabla_{\dot{\gamma}} \dot{\gamma} = 0$ . A magnetic field  $F$  is said to be uniform if  $\nabla F = 0$ .

It is well-known that the magnetic trajectories have constant speed. When the magnetic curve  $\gamma(s)$  is arc length parametrized, it is called a normal magnetic curve.

The case of dimension 3 is rather special, since it allows us to identify 2-forms with vector fields via the Hodge  $\star$  operator and the volume form  $dv_g$  of the (oriented) manifold. In this way, magnetic fields may be identified with divergence free vector fields by

$$F_V = dv_g(V, \cdot, \cdot).$$

Magnetic fields  $F$  corresponding to Killing vector fields are usually known as Killing magnetic fields. Their trajectories, called Killing magnetic curves, are of great importance since they are related to the Kirchhoff elastic rods. See e.g. [3, 5].

**2.2. Sasakian manifolds.** A  $(\varphi, \xi, \eta)$  structure on a manifold  $M$  is defined by a field  $\varphi$  of endomorphisms of tangent spaces, a vector field  $\xi$  and a 1-form  $\eta$  satisfying

$$\eta(\xi) = 1, \quad \varphi^2 = -I + \eta \otimes \xi, \quad \varphi\xi = 0, \quad \eta \circ \varphi = 0.$$

If  $(M, \varphi, \xi, \eta)$  admits a compatible Riemannian metric  $g$ , namely

$$g(\varphi X, \varphi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \text{for all } X, Y \in \mathfrak{X}(M),$$

then  $M$  is said to have an *almost contact metric structure*, and  $(M, \varphi, \xi, \eta, g)$  is called an *almost contact metric manifold*. Consequently, we have that  $\xi$  is unitary and  $\eta(X) = g(\xi, X)$ , for any  $X \in \mathfrak{X}(M)$ .

We define a 2-form  $\Omega$  on  $(M, \varphi, \xi, \eta, g)$  by

$$(3) \quad \Omega(X, Y) = g(\varphi X, Y), \quad \text{for all } X, Y \in \mathfrak{X}(M),$$

called the *fundamental 2-form* of the almost contact metric structure  $(\varphi, \xi, \eta, g)$ .

If  $\Omega = d\eta$ , then  $(M, \varphi, \xi, \eta, g)$  is called a *contact metric manifold*. Here  $d\eta$  is defined by  $d\eta(X, Y) = \frac{1}{2}(X\eta(Y) - Y\eta(X) - \eta([X, Y]))$ , for any  $X, Y \in \mathfrak{X}(M)$ . On a contact metric manifold  $M$ , the 1-form  $\eta$  is a contact form (see Blair's book [7]). The vector field  $\xi$  is called the *Reeb vector field* of  $M$  and it is characterized by  $\iota_\xi \eta = 1$  and  $\iota_\xi d\eta = 0$ . Here  $\iota$  denotes the interior product. In analytical mechanics  $\xi$  is traditionally called the *characteristic vector field* of  $M$ .

An almost contact metric manifold  $M$  is said to be *normal* if the normality tensor  $S = N_\varphi + 2d\eta \otimes \xi$  vanishes, where  $N_\varphi$  is the *Nijenhuis torsion* of  $\varphi$  defined by  $N_\varphi(X, Y) = [\varphi X, \varphi Y] + \varphi^2[X, Y] - \varphi[\varphi X, Y] - \varphi[X, \varphi Y]$ , for any  $X, Y \in \mathfrak{X}(M)$ .

A *Sasakian manifold* is defined as a normal contact metric manifold. Denoting by  $\nabla$  the Levi-Civita connection associated to  $g$ , the Sasakian manifold  $(M, \varphi, \xi, \eta, g)$  is characterized by

$$(\nabla_X \varphi)Y = -g(X, Y)\xi + \eta(Y)X, \quad \text{for any } X, Y \in \mathfrak{X}(M).$$

As a consequence, we have

$$(4) \quad \nabla_X \xi = \varphi X, \quad \forall X \in \mathfrak{X}(M).$$

A contact metric structure  $(\varphi, \xi, \eta, g)$  is called *K-contact* if  $\xi$  is a Killing vector field. Due to (4) and the fact that  $\varphi$  is skew-symmetric, it follows that a Sasakian manifold is *K-contact*. The converse is not true in general. Yet, a 3-dimensional manifold is Sasakian if and only if it is *K-contact*.

A plane section  $\Pi$  at  $p \in M^{2n+1}$  is called a  $\varphi$ -section if it is invariant under  $\varphi_p$ . The sectional curvature  $k(\Pi)$  of a  $\varphi$ -section is called the  $\varphi$ -sectional curvature of  $M^{2n+1}$  at  $p$ . A Sasakian manifold  $(M^{2n+1}, \varphi, \xi, \eta, g)$  is said to be a *Sasakian space form* if it has constant  $\varphi$ -sectional curvature.

A Sasakian space form  $M(c)$  of constant  $\varphi$ -sectional curvature  $c$  is said to be *elliptic*, *parabolic* or *hyperbolic* if  $c > -3$ ,  $c = -3$  or  $c < -3$ , respectively. See e.g. [11].

Let us start with a Sasakian manifold  $(M, \varphi, \xi, \eta, g)$ . We take a positive constant  $a$  and we define a new Sasakian structure  $(\varphi, \hat{\xi}, \hat{\eta}, \hat{g})$  on  $M$  by

$$\hat{\xi} := \frac{1}{a}\xi, \quad \hat{\eta} := a\eta, \quad \hat{g} := ag + a(a - 1)\eta \otimes \eta.$$

This structure is called a *D-homothetic deformation* of  $(\varphi, \xi, \eta, g)$ . In particular, if  $M(c)$  is a Sasakian space form, then deforming the structure, we obtain also a Sasakian space form  $M(\hat{c})$ , where  $\hat{c} = \frac{c+3}{a} - 3$ . For every value of  $c$  there exist Sasakian space forms, as follows: the elliptic Sasakian space forms, also known as the *Berger spheres* if  $c > -3$ , the Heisenberg space  $\mathbb{R}^{2n+1}(-3)$ , if  $c = -3$ , and  $B^{2n} \times \mathbb{R}$  when  $c < -3$ . See also [7, Theorem 7.15]. Note that the case  $c > -3$  includes the standard unit sphere  $\mathbb{S}^{2n+1}(1)$ .

**2.3. Magnetic curves in Sasakian manifolds.** Let  $(M, \varphi, \xi, \eta, g)$  be a contact metric manifold and let  $\Omega$  be the fundamental 2-form defined by (3). Since  $\Omega = d\eta$  on a contact metric manifold,  $\Omega$  is a closed 2-form, thus we can define a magnetic field on  $M$  by

$$(5) \quad F_q(X, Y) = q\Omega(X, Y),$$

where  $X, Y \in \mathfrak{X}(M)$  and  $q$  is a real constant. We call  $F_q$  the *contact magnetic field* with the *strength*  $q$ . Notice that if  $q = 0$ , then the contact magnetic field vanishes identically and the magnetic curves are the geodesics of  $M$ . In the sequel we assume  $q \neq 0$ .

The *Lorentz force*  $\phi_q$  associated to the contact magnetic field  $F_q$  may be easily determined combining (3) and (1), namely

$$(6) \quad \phi_q = q\varphi,$$

where  $\varphi$  is the field of endomorphisms of the contact metric structure.

In this setting, the Lorentz equation (2) can be written as

$$(7) \quad \nabla_{\gamma'}\gamma' = q\varphi\gamma',$$

where  $\gamma : I \subseteq \mathbb{R} \rightarrow M^{2n+1}$  is a smooth curve parametrized by arc length. Here by  $\gamma'$  we denote the derivative of  $\gamma$  with respect to its arc length. The solutions of (7) are called *normal magnetic curves* or *trajectories* for  $F_q$ .

REMARK 2.1. Contact magnetic trajectories can be obtained as solutions of a variational problem. More precisely, they are the critical points of the Landau Hall functional:

$$\text{LH}(\gamma) = \int \left( \frac{1}{2}g(\dot{\gamma}(t), \dot{\gamma}(t)) + q\eta(\dot{\gamma}(t)) \right) dt.$$

Here, the contact 1-form  $\eta$  is regarded as the *potential 1-form*.

**2.4. Magnetic curves in 3-dimensional Riemannian manifolds.** Let  $F = F_V$  be a magnetic field with corresponding divergence free vector field  $V$  on an oriented 3-dimensional Riemannian manifold  $(M, g)$ .

Then one can check that the Lorentz force  $\phi$  of  $F$  satisfies ([4], page 7):

$$\phi^2 X = -g(V, V)X + g(V, X)V.$$

Moreover, we have

$$\begin{aligned} V^b(\phi X) &= g(\phi X, V) = g(V \times X, V) = dv_g(V, X, V) = 0, \\ g(\phi X, \phi Y) &= g(V, V)g(X, Y) - V^b(X)V^b(Y). \end{aligned}$$

Here  $V^b$  denotes the 1-form metrically dual to  $V$ . Thus, if  $V$  has constant length  $q > 0$ , then  $(\varphi, \xi, \eta) = (\phi/q, V/q, V^b/q)$  defines a  $(\varphi, \xi, \eta)$ -structure compatible to the Riemannian structure  $g$  satisfying  $\text{div}\xi = 0$ . The magnetic field  $F$  is represented by  $F = q\Omega$  as a contact magnetic field on the resulting almost contact metric manifold  $(M, \varphi, \xi, \eta, g)$ .

**3. Sasakian structures on the Berger spheres.** In this section, we give an explicit matrix group model of the 3-dimensional Berger spheres  $\mathcal{M}^3(c)$  equipped with a natural Sasakian structure of constant  $\varphi$ -sectional curvature  $c > -3$ .

As it is well known, the unit 3-sphere  $(\mathbb{S}^3; \eta_1, \xi_1, \varphi_1, g_1)$  is identified with the special unitary group  $G = \text{SU}(2)$  with bi-invariant metric.

Let us denote by  $\mathfrak{g}$  the Lie algebra of  $G$ . The bi-invariant metric  $g_1$  of constant curvature 1 on  $G$  is induced by the following inner product  $\langle \cdot, \cdot \rangle_1$  on  $\mathfrak{g}$ :

$$\langle X, Y \rangle_1 = -\frac{1}{2} \text{tr}(XY), \quad X, Y \in \mathfrak{g}.$$

Take a quaternionic basis of  $\mathfrak{g}$ :

$$\mathbf{i} = \begin{pmatrix} 0 & \sqrt{-1} \\ \sqrt{-1} & 0 \end{pmatrix}, \quad \mathbf{j} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad \mathbf{k} = \begin{pmatrix} \sqrt{-1} & 0 \\ 0 & -\sqrt{-1} \end{pmatrix}.$$

By using this basis, the group  $\text{SU}(2)$  is described as

$$\text{SU}(2) = \left\{ \begin{pmatrix} x_0 + \sqrt{-1}x_3 & -x_2 + \sqrt{-1}x_1 \\ x_2 + \sqrt{-1}x_1 & x_0 - \sqrt{-1}x_3 \end{pmatrix} \mid x_0^2 + x_1^2 + x_2^2 + x_3^2 = 1 \right\}.$$

In the spinor representation of the Euclidean 3-space  $\mathbb{E}^3$ , we identify  $\mathbb{R}^3$  with  $\mathfrak{g} = \mathfrak{su}(2)$  via the correspondence

$$(x_1, x_2, x_3) \longleftrightarrow x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k} = \begin{pmatrix} \sqrt{-1}x_3 & -x_2 + \sqrt{-1}x_1 \\ x_2 + \sqrt{-1}x_1 & -\sqrt{-1}x_3 \end{pmatrix}.$$

Denote by  $\{E_1, E_2, E_3\}$  the left translated vector fields of  $\{\mathbf{i}, \mathbf{j}, \mathbf{k}\}$ . Then a left invariant Sasakian structure of  $G$  is given by

$$(8) \quad \begin{aligned} \xi_1 &:= E_3, \quad \eta_1 = g_1(E_3, \cdot), \\ \varphi_1(E_1) &= -E_2, \quad \varphi_1(E_2) = E_1, \quad \varphi_1(E_3) = 0. \end{aligned}$$

Note that the commutation relations of  $\{E_1, E_2, E_3\}$  are

$$[E_1, E_2] = 2E_3, \quad [E_2, E_3] = 2E_1, \quad [E_3, E_1] = 2E_2.$$

The Lie group  $G$  acts isometrically on the Lie algebra  $\mathfrak{g}$  by the Ad-action.

$$\text{Ad} : G \times \mathfrak{g} \rightarrow \mathfrak{g}; \quad \text{Ad}(a)X = aXa^{-1}, \quad a \in G, \quad X \in \mathfrak{g}.$$

The Ad-orbit of  $\mathbf{k}/2$  is a sphere of radius  $1/2$  in the Euclidean 3-space  $\mathbb{E}^3 = \mathfrak{g}$ . The Ad-action of  $G$  on  $\mathbb{S}^2(1/2)$  is isometric and transitive. The isotropy subgroup of  $G$  at  $\mathbf{k}/2$  is

$$U(1) = \left\{ a_t = \begin{pmatrix} e^{\sqrt{-1}t} & 0 \\ 0 & e^{-\sqrt{-1}t} \end{pmatrix} \mid t \in \mathbb{R} \right\}.$$

Hence  $\mathbb{S}^2(1/2)$  is represented by  $SU(2)/U(1)$  as a Riemannian symmetric space. The natural projection

$$\pi_1 : \mathbb{S}^3 \rightarrow \mathbb{S}^2(1/2), \quad \pi_1(a) = \text{Ad}(a)(\mathbf{k}/2)$$

is a Riemannian submersion and defines a principal  $U(1)$ -bundle over  $\mathbb{S}^2(1/2)$ .

Since the Sasakian structure  $(\eta_1, \xi_1, \varphi_1, g_1)$  is left invariant, its  $D$ -homothetic deformation is also left invariant. Hence the elliptic Sasakian space form  $\mathcal{M}^3(c)$  is identified with  $SU(2)$  endowed with the left invariant contact metric structure:

$$\eta := \alpha\eta_1, \quad \xi := \xi_1/\alpha, \quad \varphi := \varphi_1,$$

$$g(X, Y) = \alpha g_1(X, Y) + \alpha(\alpha - 1)\eta_1(X)\eta_1(Y), \quad c = 4/\alpha - 3,$$

where  $\alpha$  is a positive real number. The Reeb vector field  $\xi$  generates a one parameter group of transformations on  $\mathcal{M}^3(c)$ . Since  $\xi$  is a Killing vector field, this transformation group acts isometrically on  $G$ . The transformation group generated by  $\xi$  is identified with the following Lie subgroup  $K$  of  $G$ :

$$K = \left\{ \begin{pmatrix} e^{\frac{\sqrt{-1}}{\alpha}t} & 0 \\ 0 & e^{-\frac{\sqrt{-1}}{\alpha}t} \end{pmatrix} \mid t \in \mathbb{R} \right\} \cong U(1).$$

Furthermore, the action of the transformation group generated by  $\xi$  corresponds to the natural right action of  $K$  on  $G$ :

$$G \times K \rightarrow G; \quad (a, k) \mapsto ak.$$

By using the curvature formula due to O'Neill, one can see that the orbit space  $G/K$  is a 2-sphere of radius  $\sqrt{\alpha}/2$ , namely the constant curvature  $(c+3)$ -sphere. The Riemannian metric  $g$  is not only  $G$ -left invariant but also  $K$ -right invariant. Hence  $G \times K$  acts isometrically on  $G$ . The elliptic Sasakian space form  $\mathcal{M}^3(c)$  is represented by  $G \times K/K = G$  as a naturally reductive homogeneous space. For  $c \neq 1$ ,  $\mathcal{M}^3(c)$  has 4-dimensional isometry group.

In particular,  $g$  is  $G$ -bi-invariant if and only if  $c = 1$ . In this case  $\mathcal{M}^3(1)$  is represented by  $G \times G/G$  as a Riemannian symmetric space. Note that  $\mathcal{M}^3(1)$  has 6-dimensional isometry group.

Now we shall take an orthonormal frame field  $\{e_1, e_2, e_3\}$  of  $\mathcal{M}^3(c)$  by

$$e_1 := \frac{1}{\sqrt{\alpha}}E_1, \quad e_2 := \frac{1}{\sqrt{\alpha}}E_2, \quad e_3 := \frac{1}{\alpha}\xi_1.$$

Then the commutation relations of this basis are

$$[e_1, e_2] = 2e_3, \quad [e_2, e_3] = \frac{c+3}{2}e_1, \quad [e_3, e_1] = \frac{c+3}{2}e_2.$$

The Levi-Civita connection  $\nabla$  of  $(\mathcal{M}^3(c), g)$  is described by

$$\begin{aligned}\nabla_{e_1}e_1 &= 0, & \nabla_{e_1}e_2 &= e_3, & \nabla_{e_1}e_3 &= -e_2, \\ \nabla_{e_2}e_1 &= -e_3, & \nabla_{e_2}e_2 &= 0, & \nabla_{e_2}e_3 &= e_1, \\ \nabla_{e_3}e_1 &= \frac{c+1}{2}e_2, & \nabla_{e_3}e_2 &= -\frac{c+1}{2}e_1, & \nabla_{e_3}e_3 &= 0.\end{aligned}$$

The Riemannian curvature tensor field  $R$  of  $(\mathcal{M}^3(c), g, \nabla)$  is described by

$$R_{1212} = c, \quad R_{1313} = R_{2323} = 1,$$

and the sectional curvatures are:

$$K_{12} = c, \quad K_{13} = K_{23} = 1.$$

The Ricci tensor  $\text{Ric}$  and the scalar curvature  $\text{scal}$  are computed to be

$$\text{Ric}_{11} = \text{Ric}_{22} = c + 1, \quad \text{Ric}_{33} = 2, \quad \text{scal} = 2(c + 2).$$

**4. Magnetic trajectories in 3-dimensional Berger spheres.** Let us consider a normal magnetic curve with respect to the magnetic field  $F_q = q\Omega$ , where  $q \in \mathbb{R}$  is the strength and  $\Omega$  is the fundamental 2-form defined by (3) on the Berger sphere  $\mathcal{M}^3(c)$ . Thus, the Lorentz force is  $\phi = q\varphi$ .

If we expand the tangent vector field  $T(s) = \gamma'(s)$  as  $T(s) = T_1(s)e_1 + T_2(s)e_2 + T_3(s)e_3$ , then the acceleration vector field  $\nabla_{\gamma'}\gamma'$  is

$$\nabla_{\gamma'}\gamma' = \left\{ T_1' - \frac{1}{2}(c-1)T_2T_3 \right\} e_1 + \left\{ T_2' + \frac{1}{2}(c-1)T_1T_3 \right\} e_2 + T_3'e_3.$$

Hence, the magnetic equation  $\nabla_{\gamma'}\gamma' = q\varphi T$  together with (8) yield  $T_3 = \text{constant}$  and

$$T_1'(s) = \tilde{q} T_2(s),$$

$$T_2'(s) = -\tilde{q} T_1(s),$$

where we put  $\cos \theta := T_3$  and  $\tilde{q} = q + \frac{1}{2}(c-1)\cos \theta$ .

Integrating, we obtain

$$T_1(s) = \cos(\tilde{q}s) T_1(0) + \sin(\tilde{q}s) T_2(0),$$

$$T_2(s) = \sin(\tilde{q}s) T_1(0) - \cos(\tilde{q}s) T_2(0),$$

$$T_3(s) = \cos \theta.$$

One can see that the curvature of  $\gamma$  is  $\kappa = |q| \sin \theta$ .

If we consider  $\alpha = 1$ , then  $c = 1$  and  $\tilde{q} = q$ .

**REMARK 4.1.** For an arc length parametrized curve  $\gamma(s)$  in a 3-dimensional contact metric manifold, its contact angle  $\theta(s)$  is defined by  $\cos \theta(s) = \eta(\gamma'(s))$ . An arc length parametrized curve  $\gamma(s)$  is said to be *Legendre* if  $\eta(\gamma') = 0$ . More generally,  $\gamma(s)$  is said to be *slant* if its contact angle is constant. Thus, every normal contact magnetic curve in  $\mathcal{M}^3(c)$  is slant.

REMARK 4.2. If  $\gamma(s)$  is a normal magnetic curve in  $\mathcal{M}^3(1)$ , for a certain strength  $q$ , then, after an affine change of the parameter,  $\gamma$  becomes a normal magnetic curve in  $\mathcal{M}^3(c)$  with a different strength  $\hat{q}$ . See for details [9].

As we have seen in Section 3 we have a circle bundle

$$\begin{array}{ccc} \mathrm{U}(1) & \longrightarrow & \mathcal{M}^3(c) \\ & & \downarrow \pi \\ & & \mathbb{S}^2(r) \end{array}$$

where the projection  $\pi$  is defined by  $\pi(a) = \mathrm{Ad}(a)(r\mathbf{k})$ . If  $\mathbb{S}^2(r) \subset \mathfrak{su}(2) \equiv \mathbb{E}^3$  is endowed with the induced metric from  $\langle \cdot, \cdot \rangle_1$  then  $\pi$  becomes a Riemannian submersion from  $(\mathcal{M}^3(c), g)$  onto  $\mathbb{S}^2(r)$  if and only if  $r = \sqrt{\alpha}/2$ .

The correspondence  $a \mapsto \ker \eta_a, a \in \mathcal{M}^3(c)$  defines a connection in this principal  $\mathrm{U}(1)$  bundle, hence if  $A \in \ker \eta_a$  then  $|\pi_{*,a}(A)| = |A|$ . Take  $\eta_a$  as the connection form (the standard choice).

Consider a regular curve  $\beta : \mathbb{R} \longrightarrow \mathbb{S}^2(r) \subset (\mathfrak{su}(2), \langle \cdot, \cdot \rangle_1), u \mapsto \beta(u)$ . As usual,  $\beta$  will be parametrized by the arc length and let  $\hat{\beta}$  be a horizontal lift of  $\beta$ . This means that  $\pi(\hat{\beta}(u)) = \beta(u)$  for all  $u \in \mathbb{R}$  and  $\langle \hat{\beta}(u)^* \hat{\beta}'(u), \mathbf{k} \rangle = 0$ . Here  $\hat{\beta}(u)^* = \overline{\hat{\beta}(u)^T}$ , and  $a^T$  denotes the transpose of the matrix  $a$ . The inner product on  $\mathfrak{su}(2)$ , the tangent space of  $\mathcal{M}^3(c)$  at the identity, is  $\langle X, Y \rangle = \alpha \langle X, Y \rangle_1 + \alpha(\alpha - 1) \langle X, \mathbf{k} \rangle_1 \langle Y, \mathbf{k} \rangle_1$ .

The complete lift of  $\beta$ , namely  $\pi^{-1}(\beta)$  is a flat surface in  $\mathcal{M}^3(c)$  and it is usually called the Hopf tube over  $\beta$ .

Denote  $\pi^{-1}(\beta)$  by  $H_\beta$ . It can be naturally parametrized by

$$F : \mathbb{R} \times \mathbb{R} \longrightarrow \mathcal{M}^3(c), \quad (t, u) \longmapsto F(t, u) = \hat{\beta}(u)a_t.$$

In fact, we can show that the metric  $g$  restricted to  $H_\beta$  can be expressed as

$$g_{H_\beta} = \alpha^2 dt^2 + du^2.$$

Hence, we have proved the following:

PROPOSITION 4.3. *If  $\beta$  is a curve on  $\mathbb{S}^2(r)$  of length  $L$ , then the corresponding Hopf tube  $H_\beta$  is isometric to  $\mathbb{S}^1(\alpha) \times [0, L]$ , where  $\mathbb{S}^1(\alpha)$  is the unit circle endowed with the metric  $\alpha^2 dt^2$ . Moreover, its mean curvature in  $\mathcal{M}^3(c)$  is  $H = \frac{1}{2}\kappa_\beta$ , where  $\kappa_\beta$  is the geodesic curvature of  $\beta$  in  $\mathbb{S}^2(r)$ .*

If  $\beta$  is a closed curve, i.e.  $\beta(u + L) = \beta(u)$  for all  $u \in \mathbb{R}$ , then the relation  $F(t, u) = \hat{\beta}(u)a_t$  defines a covering of the  $(t, u)$  plane onto the immersed torus in  $\mathcal{M}^3(c)$ , called the Hopf torus corresponding to  $\beta$ . One can easily see that, if  $\beta$  is a great circle in  $\mathbb{S}^2(r)$ , then the Hopf torus  $H_\beta$  is minimal in  $\mathcal{M}^3(c)$ . In [15], a method for constructing all flat tori in  $\mathbb{S}^3$  is given. See also [16].

PROPOSITION 4.4. *The magnetic curve  $\gamma$  is a geodesic of the Hopf tube  $H_\beta$ .*

PROOF. We know that

$$\nabla_{\gamma'} \gamma' = q(T_2 e_1 - T_1 e_2),$$

which is orthogonal both to  $\xi$  and  $\gamma'$ . It follows that  $\nabla_{\gamma'} \gamma'$  is collinear to the normal vector to  $H_\beta$ . Hence,  $\gamma$  is a geodesic on  $H_\beta$ .  $\square$

**5. Periodic magnetic trajectories on the 3-dimensional Berger spheres.** In this section we show how the set of all periodic magnetic trajectories on  $\mathcal{M}^3(c)$  can be quantized in the set of rational numbers.

First of all, we find a relation between the intrinsic geometry of the magnetic curve and those of its projection on the sphere  $\mathbb{S}^2(r)$ . We state the following result.

PROPOSITION 5.1. *If  $\gamma$  is a magnetic curve on  $\mathcal{M}^3(c)$ , then the projection curve  $\beta$  is a circle on  $\mathbb{S}^2(r)$  of geodesic curvature*

$$\kappa_\beta = \frac{\kappa^2 + \tau^2 - 1}{\kappa},$$

where  $\kappa$  and  $\tau$  are the (constant) curvature and torsion of  $\gamma$  in  $\mathcal{M}^3(c)$ .

PROOF. For any  $a \in \mathcal{M}^3(c) \equiv \text{SU}(2)$  we have

$$e_1|_a = \frac{1}{\sqrt{\alpha}} a\mathbf{i}, \quad e_2|_a = \frac{1}{\sqrt{\alpha}} a\mathbf{j}, \quad e_3|_a = \frac{1}{\alpha} a\mathbf{k}.$$

Let  $\gamma$  be a magnetic curve on  $\mathcal{M}^3(c)$  such that  $\dot{\gamma} = T_1 e_1 + T_2 e_2 + \cos \theta e_3$ , as in the previous Section. If  $\beta = \pi(\gamma)$ , then

$$\beta' = \pi_{*,\gamma}(\dot{\gamma}) = r(\dot{\gamma}\mathbf{k}\gamma^* + \gamma\mathbf{k}\dot{\gamma}^*) = \frac{2r}{\sqrt{\alpha}}(-T_1\gamma\mathbf{j}\gamma^* + T_2\gamma\mathbf{i}\gamma^*).$$

Since  $r = \sqrt{\alpha}/2$  we obtain

$$\beta' = \gamma(-T_1\mathbf{j} + T_2\mathbf{i})\gamma^*.$$

In the same manner we can compute

$$\beta'' = \gamma V(s)\gamma^*,$$

where

$$V(s) = \left(T_2' + \frac{2}{\alpha} \cos \theta T_1\right)\mathbf{i} + \left(-T_1' + \frac{2}{\alpha} \cos \theta T_2\right)\mathbf{j} - \frac{2}{\sqrt{\alpha}} \sin^2 \theta \mathbf{k}.$$

If  $\overset{2}{\nabla}$  denotes the Levi-Civita connection on  $\mathbb{S}^2(r)$  we have

$$\beta'' = \overset{2}{\nabla}_{\beta'} \beta' - \frac{\sin^2 \theta}{r^2} \beta.$$

Since  $T_1' = \tilde{q} T_2$  and  $T_2' = -\tilde{q} T_1$ , we get

$$(9) \quad \overset{2}{\nabla}_{\beta'} \beta' = (-q + 2 \cos \theta)\gamma(T_1\mathbf{i} + T_2\mathbf{j})\gamma^*.$$

If  $\nu_1$  denotes the first (unit) normal of  $\gamma$ , we have

$$\nu_1 = \frac{\varepsilon}{\sqrt{\alpha} \sin \theta} \gamma(T_2 \mathbf{i} - T_1 \mathbf{j}),$$

where  $\varepsilon = \text{sgn}(q)$ .

Thus, the unit normal to  $\beta$  can be considered

$$\nu_\beta := \pi_{*,\gamma}(\nu_1) = -\frac{\varepsilon}{\sin \theta} \gamma(T_1 \mathbf{i} + T_2 \mathbf{j}) \gamma^*.$$

Combing this with (9), we obtain

$$\overset{2}{\nabla}_{\beta'} \beta' = \varepsilon \sin \theta (q - 2 \cos \theta) \nu_\beta.$$

Therefore, the signed geodesic curvature of  $\beta$  is

$$\kappa_\beta = \frac{q - 2 \cos \theta}{\varepsilon \sin \theta}.$$

Concerning the magnetic curve  $\gamma$ , we know [9] that its curvature (the first curvature) is  $\kappa = \varepsilon q \sin \theta$  and its torsion (the signed second curvature) is  $\tau = q \cos \theta - 1$ .

Since  $q^2 = \kappa^2 + (\tau + 1)^2$ , the conclusion follows immediately.  $\square$

Consider now a periodic magnetic trajectory  $\gamma$  on  $\mathcal{M}^3(c)$ , i.e.  $\gamma(s + L) = \gamma(s)$  for all  $s \in \mathbb{R}$ . Then, the projection  $\beta$  is a circle on the sphere  $\mathbb{S}^2(r)$ .

Conversely, take  $\beta$  a closed curve on  $\mathbb{S}^2(r)$  of length  $L$  enclosing an oriented area  $A$ ,  $A \in [-2\pi r^2, 2\pi r^2]$ . Let  $\hat{\beta}$  be a horizontal lift of  $\beta$ . Since  $\hat{\beta}(0)$  and  $\hat{\beta}(L)$  belong to the same fibre, we have

$$\hat{\beta}(L) = \hat{\beta}(0) a_\delta, \quad \delta \in \mathbb{R}.$$

Usually,  $\delta$  is called the holonomy number of the canonical principal connection defined in the circle bundle. See e.g. [18, 1]. For the classical Hopf fibration,  $\hat{\beta}$  closes up if and only if there exists a positive integer  $m$  such that, after  $m$  consecutive liftings of  $\beta$  we get  $\hat{\beta}(mL) = \hat{\beta}(0)$  and hence  $m\delta = 2\pi p$ , with  $p \in \mathbb{Z}$ .

We have seen that the Hopf torus  $H_\beta$  is isometric to  $\mathbb{S}^1(\alpha) \times [0, L]$ . The isometry type depends either on the length  $L$  of  $\beta$ , or on the area  $A$  enclosed by  $\beta$  on  $\mathbb{S}^2(r)$ , namely  $A = \iint_D dA$ , where  $dA$  is the area element of  $\mathbb{S}^2(r)$ , and  $D$  is a domain on  $\mathbb{S}^2(r)$  such that  $\partial D = \beta$ . Notice that  $H_2(\mathbb{S}^2(r)) = \mathbb{Z}$  and  $\text{area}(\mathbb{S}^2(r)) = 4\pi r^2$ .

We have:

**PROPOSITION 5.2.** *Let  $\beta$  be a closed curve on  $\mathbb{S}^2(r)$  of length  $L$  enclosing an oriented area  $A$ . Then, the corresponding Hopf torus  $H_\beta$  is isometric to  $\mathbb{R}^2/\Gamma$ , where the lattice  $\Gamma$  is generated by the vectors  $(2\pi\alpha, 0)$  and  $(\frac{A}{2r^2}, L)$ .*

PROOF. With previous notations we have that  $\hat{\beta}(L) = \hat{\beta}(0)a_\delta$ , for some  $\delta$ . It is clear (due to the form of the metric on  $H_\beta$ ) that the group of deck transformations for the covering mentioned above is generated by the translations  $(2\pi\alpha, 0)$  and  $(\delta, L)$ .

We have to find  $\delta$ . It is known that  $\delta = \iint_D \Omega$ , where  $\Omega$  is the curvature 2-form of the connection in the principal circle bundle  $\pi : \mathcal{M}^3(c) \rightarrow \mathbb{S}^2(r)$ . It should be a multiple of the area element on  $\mathbb{S}^2(r)$ , and one can compute  $\Omega = \frac{1}{2r^2}dA$ . Hence  $\delta = \frac{A}{2r^2}$ .  $\square$

If  $\beta$  is the projection of a magnetic curve on  $\mathcal{M}^3(c)$ , then it is a circle on  $\mathbb{S}^2(r)$ . Denote by  $R$  its radius,  $R \leq r$ . We have

$$\kappa_\beta = \frac{\sqrt{r^2 - R^2}}{rR}, \quad L = 2\pi R, \quad A = 2\pi r(r - \sqrt{r^2 - R^2}).$$

Since  $\gamma$  is a periodic (closed) geodesic on the Hopf torus  $H_\beta$ , it corresponds to a segment in  $\mathbb{R}^2$  (with identified ends). This segment is in fact the diagonal of a parallelogram constructed by taking  $m$  vectors in the fibre, hence  $m$  times  $(2\pi\alpha, 0)$  and  $n$  vectors in the horizontal direction, i.e.  $n$  times  $(\frac{A}{2r^2}, L)$ ,  $n \in \mathbb{N}$ . Thus, the direction of the magnetic trajectory  $\gamma$  is given by

$$\left( 2\pi m\alpha + n\pi \left( 1 - \sqrt{1 - \frac{R^2}{r^2}} \right), 2\pi nR \right).$$

If we put  $\sigma = \cot \theta$  (here  $\theta$  is the contact angle of the curve  $\gamma$ ) and call this quantity the *slope* of  $\gamma$ , we have

$$\sigma = \frac{2m\alpha + n \left( 1 - \sqrt{1 - \frac{R^2}{r^2}} \right)}{2nR}.$$

Hence we get

$$R\sigma + \frac{1}{2}\sqrt{1 - \frac{R^2}{r^2}} = \frac{m}{n}\alpha + \frac{1}{2}.$$

We can state the following result.

**THEOREM 5.3.** *The set of all periodic magnetic curves of arbitrary strength on the Sasakian space form  $\mathcal{M}^3(c)$  can be quantized in the set of rational numbers.*

PROOF. Taking into account that  $r = \sqrt{\alpha/2}$  we conclude that the slope of a periodic magnetic trajectory corresponding to the circle  $\beta$  of radius  $R$  on the sphere  $\mathbb{S}^2(r)$  satisfies the following quantization principle

$$(10) \quad R\sigma + \frac{1}{2}\sqrt{1 - \frac{4R^2}{\alpha}} \in \mathbb{Q}\alpha + \frac{1}{2}.$$

$\square$

**6. Periodic magnetic curves on  $\mathbb{S}^3$ .** Let  $\mathbb{S}^3 = \{(z, w) \in \mathbb{C} \times \mathbb{C} : |z|^2 + |w|^2 = 1\}$  be the 3-dimensional sphere. It is known that  $\mathbb{S}^3$  is parallelizable by the following three global vector fields  $\xi = (iz, iw)$  (the Reeb vector field),  $\zeta = (-\bar{w}, \bar{z})$  and  $\varsigma = (-i\bar{w}, i\bar{z})$ . The group  $\mathbb{S}^1$  acts (simply transitively) on  $\mathbb{S}^3$  by

$$\mathbb{S}^3 \times \mathbb{S}^1 \longrightarrow \mathbb{S}^3 \quad : \quad (z, w; e^{it}) \longmapsto (e^{it}z, e^{it}w),$$

and the Hopf projection

$$\pi : \mathbb{S}^3 \longrightarrow \mathbb{S}^2 \quad : \quad (z, w) \longmapsto (z\bar{w}, \frac{1}{2}(|z|^2 - |w|^2))$$

from the 3-sphere  $\mathbb{S}^3$  to the 2-sphere  $\mathbb{S}^2(\frac{1}{2}) \subset \mathbb{C} \times \mathbb{R}$  has a structure of principal circle bundle.

We know that magnetic curves in Sasakian manifolds are helices of order 3 (see [9]). In particular, for the unit sphere  $\mathbb{S}^3$  we have a model helix given by

$$(11) \quad (\cos \psi \cos(as), \cos \psi \sin(as), \sin \psi \cos(bs), \sin \psi \sin(bs)),$$

where  $s$  is the arc length parameter and  $a, b$  and  $\psi$  satisfy

$$a^2 \cos^2 \psi + b^2 \sin^2 \psi = 1.$$

See e.g. [10, 20]. This helix lies on a torus whose mean curvature is constant  $H = \cot(2\psi)$ . In fact, every helix with both curvature and torsion different from zero is congruent to one of these helices.

It is not difficult to show that the helix (11) is periodic if and only if

$$a = 1/\sqrt{p^2 \sin^2 \psi + \cos^2 \psi}, \quad b = pa,$$

where  $p$  is a rational number. Moreover, all these helices project (via Hopf fibration) to (small) horizontal circles on  $\mathbb{S}^2(\frac{1}{2})$ .

In the following we give an example of a periodic magnetic trajectory on the unit sphere  $\mathbb{S}^3$  and we draw the picture of its stereographic projection on  $\mathbb{R}^3$ .

Consider the curve  $\gamma : I \longrightarrow \mathbb{S}^3$  defined by

$$(12) \quad \gamma(s) = (x_0(s), x_1(s), x_2(s), x_3(s)),$$

with

$$\begin{aligned} x_0(s) &= \cos(\frac{s}{2}) \cos(\omega s) - \frac{1}{\omega} (\cos \theta - \frac{1}{2}) \sin(\frac{s}{2}) \sin(\omega s) \\ x_1(s) &= \sin(\frac{s}{2}) \cos(\omega s) + \frac{1}{\omega} (\cos \theta - \frac{1}{2}) \cos(\frac{s}{2}) \sin(\omega s) \\ x_2(s) &= \frac{\sin \theta}{\omega} \cos(\frac{s}{2}) \sin(\omega s) \\ x_3(s) &= \frac{\sin \theta}{\omega} \sin(\frac{s}{2}) \sin(\omega s), \end{aligned}$$

where  $\omega = \sqrt{5/4 - \cos \theta}$  and  $\theta$  is constant. Then,  $\gamma$  is a normal magnetic curve corresponding to the contact magnetic field of  $\mathbb{S}^3$  and with strength 1, for which  $\theta$  expresses its contact angle. The curve above is slant with respect to the Reeb vector field induced from  $\mathbf{i}$  (and not from  $\mathbf{k}$  as in previous sections). According to [12], the curve  $\gamma$  is periodic if and only if  $\omega$  is

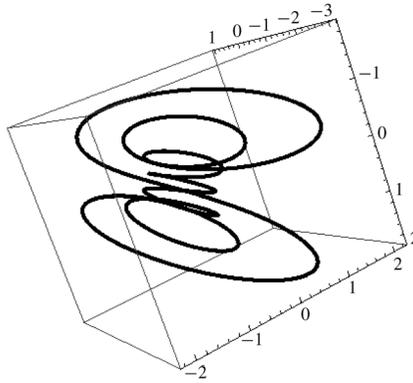


FIGURE 1.  $\cos \theta = 29/37$ .

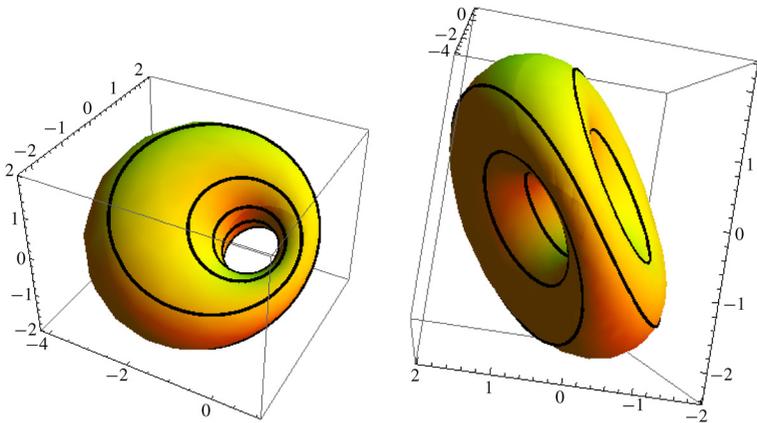


FIGURE 2. The curve and the tube after stereographic projection; two different view-points.

a rational number. In fact, the periodicity condition obtained by Ikawa in [12] is nothing but the quantization result obtained by Cabrerizo et al. in [8].

In the following we take  $\omega = \frac{2}{3}$  equivalently to  $\theta = \arccos \frac{29}{37}$ . Consider the stereographic projection of the sphere from its North pole. Then the image of  $\gamma$  is drawn in Figure 1.

We know that  $\gamma$  lies on a Hopf tube in  $\mathbb{S}^3$ . In Figure 2 we plot the image of this tube under the stereographic projection we have mentioned before.

We conclude this paper with a criterion of periodicity for magnetic curves in  $\mathbb{S}^3$ . As a consequence of Theorem 3.13 in [12] we may state the following:

**THEOREM 6.1.** *Let  $\gamma$  be a normal magnetic curve on the unit sphere  $\mathbb{S}^3$ . Then  $\gamma$  is periodic if and only if*

$$\frac{q}{\sqrt{q^2 - 4q \cos \theta + 4}} \in \mathbb{Q},$$

where  $q$  is the strength and  $\theta$  is the constant contact angle of  $\gamma$ .

**REMARK 6.2.** As we have mentioned in Introduction, every closed  $L$ -minimal Legendre curve in  $\mathbb{S}^3$  is a magnetic curve. Closed  $L$ -minimal Legendre curves in  $\mathbb{S}^3$  are classified as follows:

**THEOREM 6.3** ([13], [19]). *All of closed  $L$ -minimal Legendre curves in  $\mathbb{S}^3$  are parametrized as*

$$\frac{1}{\sqrt{m+n}} \left( \sqrt{n} \exp \left( i \sqrt{\frac{m}{n}} s \right), i \sqrt{m} \exp \left( i \sqrt{\frac{n}{m}} s \right) \right) \subset \mathbb{S}^3 \subset \mathbb{C}^2, \quad 0 \leq s \leq 2\pi \sqrt{mn},$$

where  $(m, n)$  is a pair of relatively prime positive integers. These are so-called torus knots of type  $(m, n)$ .

One can see that these Legendre knots are contact magnetic curves satisfying  $|q| = \frac{|n-m|}{\sqrt{mn}}$ . Kajigaya [14] showed that these Legendre knots are  $L$ -unstable.

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