

ON LINEAR DEFORMATIONS OF BRIESKORN SINGULARITIES OF TWO VARIABLES INTO GENERIC MAPS

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Abstract. In this paper, we study deformations of Brieskorn polynomials of two variables obtained by adding linear terms consisting of the conjugates of complex variables and prove that the deformed polynomial maps have only indefinite fold and cusp singularities in general. We then estimate the number of cusps appearing in such a deformation. As a corollary, we show that a deformation of a complex Morse singularity with real linear terms has only indefinite folds and cusps in general and the number of cusps is 3.

1. Introduction. Let $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ be a complex polynomial map of two variables (z, w) with isolated singularity at the origin and $f_t : \mathbb{C}^2 \rightarrow \mathbb{C}$, $t \in [0, 1]$, be a family of polynomial maps such that $f_0 = f$, i.e., a deformation of f . A deformation of f decomposes the singularity of f at the origin into several smaller singularities, especially into complex Morse singularities for a generic choice of the deformation. Such a deformation is useful to understand the topology of the singularity of f at the origin. For example, from the deformations in [1, 2, 10, 11, 12] we can describe the Dynkin diagram of the singularity and obtain the monodromy matrix of its Milnor fibration explicitly.

We now regard f as a real polynomial map. A smooth map is called a *generic map* if it is transversal, in the jet-space, to certain submanifolds called Thom-Boardman singularities [4]. Usually, a normal crossing condition (Condition NC in [7, p. 157]) is required in addition. It is known that the set of generic maps is a dense subset in the space of smooth maps if the source manifold is compact, see [17]. In the case of smooth maps from \mathbb{R}^4 to \mathbb{R}^2 , a smooth map is generic with ignoring the normal crossing condition if and only if it has only fold and cusp singularities. In this paper, we are interested in deforming a smooth map into a generic one in this sense. To avoid confusion, we call such a map an *excellent map*, which is a terminology used in [21] for maps between 2-planes, and also used for maps from higher dimensional manifolds later, e.g. in [19].

We say that a deformation of f is *linear* if it is given in the form

$$(1.1) \quad f_t(z, w) = f(z, w) + a_1 z + b_1 w + a_2 \bar{z} + b_2 \bar{w},$$

where a_1, b_1, a_2, b_2 are analytic functions with variable $t \in \mathbb{R}$ which vanish at $t = 0$. The variables \bar{z} and \bar{w} are the complex conjugates of z and w respectively, which are needed since

we are studying real deformations. In this paper, we study linear deformations of Brieskorn polynomials of form (1.1) with $a_1 \equiv b_1 \equiv 0$.

THEOREM 1.1. *Let $f(z, w) = z^p + w^q$ be a Brieskorn polynomial with $p, q \geq 2$. If $a, b \in \mathbb{C}$ are generic then there exists a linear deformation $f_t(z, w)$ of f which consists of excellent maps for $t \in (0, 1]$ and satisfies $f_1(z, w) = f(z, w) + a\bar{z} + b\bar{w}$. Moreover, such an excellent map has only indefinite fold and cusp singularities.*

In the assertion, “ $a, b \in \mathbb{C}$ are generic” means that $a, b \in \mathbb{C}$ lie outside the union of zero-sets of a finite number of equations of the polar coordinates of a and b . See Remark 3.16 for more precise explanation.

Next, we observe the number of cusps of an excellent map obtained as a linear deformation of $f(z, w) = z^p + w^q$ in Theorem 1.1, whose existence is confirmed in that theorem.

THEOREM 1.2. *Let $f(z, w) = z^p + w^q$ be a Brieskorn polynomial with $p \geq q \geq 2$. Suppose that $f(z, w) + a\bar{z} + b\bar{w}$ is an excellent map. Then the number of cusps of this map belongs to $[(p+1)(q-1), (p-1)(q+1)]$. In particular, if $p = q$ then the number of cusps is $p^2 - 1$.*

There are related results about the number of cusps appearing in the versal deformations of complex isolated singularities, see [8, 9].

If $p = q = 2$ then equation (1.1) can be modified as

$$\left(z + \frac{a_1}{2}\right)^2 + \left(w + \frac{b_1}{2}\right)^2 + a_2\bar{z} + b_2\bar{w} - \frac{(a_1^2 + b_1^2)}{4}.$$

This is in the form $z^2 + w^2 + a\bar{z} + b\bar{w}$ up to translation of the coordinates and addition of a constant term. Therefore, the results in Theorems 1.1 and 1.2 hold for “any” linear deformations.

From Theorem 1.2, we can conclude that the number of cusps in a linear deformation of $f(z, w) = z^2 + w^2$ into an excellent map is 3. Furthermore, in this case, we can explicitly describe the image of the set of singularities of the deformed map as follows.

THEOREM 1.3. *Let f_t be a linear deformation of $f(z, w) = z^2 + w^2$ into excellent maps. Then the set of singular values of f_t , $t \in (0, 1]$, in \mathbb{R}^2 is a scaling and rotation of the curve given by the map $h : S^1 \rightarrow \mathbb{R}^2$ defined by*

$$h(\theta) = e^{2i\theta} + 2e^{-i\theta}.$$

This curve is a simple closed 3-cuspidal curve as shown in Figure 1. The map f_t , $t \in (0, 1]$, restricted to the set of singularities is injective and hence the number of cusps is 3.

We will use the higher differentials of H. Levine in [16] to prove Theorem 1.1, which give a criterion to determine if a smooth map is an excellent map or not. It is important to remark that the mappings studied in Theorem 1.1 are improper and we do not know what happens near the infinity. Therefore the ordinary stability analysis of differentiable mappings cannot be applied directly. It is very hard to apply Levine’s criterion to polynomial maps which are explicitly given. We could do this because we know, in advance, that their cusps

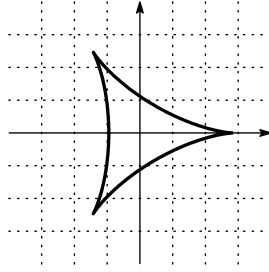


FIGURE 1. The curve given by $h(\theta) = e^{2i\theta} + 2e^{-i\theta}$.

are determined by the equation $T = 0$ which will be explained in Section 4. (Compare this equation with $\phi(z_0)$ in Proposition 3.9.) It may be possible to show the same claim using other techniques, but, since the equation $T = 0$ contains trigonometric functions, we guess that the difficulty of the calculation will be more or less the same. In this sense, our direct calculation performed to prove the theorem is indispensable. The assumption that the polynomial is of Brieskorn type is necessary to perform our calculation, otherwise the defining equation of cusps will become more complicated so that we cannot apply Levine's criterion practically.

Remark also that the topological studies of broken Lefschetz fibrations, initiated by the works of Auroux-Donaldson-Katzarkov [3] and O. Saeki [19], is related to our interest. For example, the linear deformation in Theorem 1.3 appears in [13, Move 4 in p.292] as a typical move which produces a broken Lefschetz fibration. The observation in this paper shows concretely that such a linear deformation is really an excellent map for a generic choice of the parameters.

In the recent work [5] of N. Dutertre and T. Fukui, they gave several formulas of Euler characteristics for singularities of stable maps under the assumption that the maps are locally trivial at infinity. Especially, it follows from their result that the number of cusps of stable and locally trivial at infinity maps appearing in a small deformation in our setting is constant modulo 2. This phenomenon can be seen in examples given in the last section in this paper.

The paper is organized as follows. Section 2 is for preliminaries. We briefly introduce the higher differentials of H. Levine in [16] in Section 2.1, and then explain how to use it in Section 2.2. This is the main technique in this paper, and the rest is full of hard calculation. We prove the first assertion in Theorem 1.1 in Section 3, Theorem 1.2 and the second assertion in Theorem 1.1 in Section 4, and finally prove Theorem 1.3 in Section 5. In Section 6, we close the paper with presenting two examples of linear deformations.

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2. Preliminaries.

2.1. Higher differentials. Let $f : X \rightarrow Y$ be a smooth map between an n_X -dimensional manifold X and an n_Y -dimensional manifold Y . Let $j^r f(p)$ denote the r -jet of f at $p \in X$ and let $J^r(f) : X \rightarrow J^r(X, Y)$ be the map which maps $p \in X$ to $j^r f(p)$, where $J^r(X, Y)$ is the set of r -jets. Denote by df the induced map from TX to $f^{-1}(TY)$ and $df_p = df|_{T_p X}$ for $p \in X$, where TX is the tangent bundle of X , $T_p X$ is the tangent space of X at p and $f^{-1}(TY)$ is the vector bundle over X whose fiber at $p \in X$ is $T_{f(p)}Y$.

In this section, we will introduce higher differentials of f according to [16], by which we interpret the transversality of $J^1(f)$ with $S_i(X, Y)$ and of $J^2(f)$ with $S_{i,h}(X, Y)$. Here $S_i(X, Y)$ is defined as

$$S_i(X, Y) = \{j^1 f(p) \in J^1(X, Y) \mid \text{rank } df_p = \min\{n_X, n_Y\} - i\}.$$

For the definition of $S_{i,h}(X, Y)$, see [14] and also [7]. We define

$$S_i(f) = \{p \in X \mid \text{rank } df_p = \min\{n_X, n_Y\} - i\}.$$

Note that $S_0(f)$ is the set of regular points of f and $S(f) = \bigcup_{i \geq 1} S_i(f)$ is the set of singular points of f , i.e., the points at which the differential of f is not surjective. Note also that the sets $S_{i_1, i_2, \dots, i_k}(f)$ play important role in the study of stable/generic maps. The sequence $\{i_1, i_2, \dots, i_k\}$ is called the *Boardman symbol of order k* . In this paper, since we are interested in determining if a singularity is a fold or a cusp, we only need the Boardman symbols up to order 3. The set $S_{i,h}(f)$ will be introduced after Proposition 2.2 and the set $S_{i,h,k}(f)$ will be introduced in the end of this subsection.

For $p \in X$ and $q \in Y$, let U and V be coordinate neighborhoods of p and q , respectively, such that $f(U) \subset V$. Since $TX|_U$ and $TY|_V$ are trivial we can choose bases $\{u_i\}$ and $\{v_j\}$ of the sections of these restricted bundles. Let $\{u_i^*\}$ and $\{v_j^*\}$ be the associated dual bases

$$\langle u_i, u_{i'}^* \rangle = \delta_{ii'}, \quad \langle v_j, v_{j'}^* \rangle = \delta_{jj'},$$

where $\langle \cdot, \cdot \rangle$ is the pairing of a vector space with its dual.

Let $w_j = v_j \circ f$ and $w_j^* = v_j^* \circ f$. Since df is linear on each fiber, there are smooth functions a_{ij} , $i = 1, \dots, n_X$, $j = 1, \dots, n_Y$, on U such that

$$df = \sum_{i,j} a_{ij} u_i^* \otimes w_j,$$

where

$$(df(u_i))_p = \sum_{j=1}^{n_Y} a_{ij}(p) w_j(p).$$

Set $E = TX|_{S_i(f)}$ and $F = f^{-1}(TY)|_{S_i(f)}$. Define L and G by the exactness of the sequence

$$0 \longrightarrow L \longrightarrow E \xrightarrow{df} F \xrightarrow{\pi_1} G \longrightarrow 0.$$

Note that L (resp. G) is an $(n_X - k)$ -dimensional (resp. $(n_Y - k)$ -dimensional) vector bundle over $S_i(f)$, where $k = \min\{n_X, n_Y\} - i$. We denote the fibers of L and G at $p \in S_i(f)$ by L_p and G_p respectively. Define the map $\varphi^1 : E \rightarrow L^* \otimes F$, for each $p \in S_i(f)$, by

$$(2.1) \quad \varphi_p^1(v)(l) = \sum_{i,j} \langle v, da_{ij}(p) \rangle \langle l, u_i^*(p) \rangle w_j(p),$$

with $v \in T_p X$ and $l \in L_p$. Then the second differential $d^2 f : E \rightarrow L^* \otimes G$ of f is defined as $d^2 f_p(v)(l) = \pi_1(\varphi_p^1(v)(l))$.

PROPOSITION 2.1 ([16]). *The map $d^2 f$ is a well-defined bundle map and $d^2 f|L$ is a section in $(L^* \odot L^*) \otimes G$, where \odot represents the symmetric product.*

The transversality of $J^1(f)$ to $S_i(X, Y)$ is determined by $d^2 f$ as follows.

PROPOSITION 2.2 ([14, 16]). *The map $J^1(f)$ is transversal to $S_i(X, Y)$ at $p \in S_i(f)$ if and only if the map*

$$d^2 f_p : T_p X \rightarrow L_p^* \otimes G_p$$

has rank $(n_X - k)(n_Y - k) = i(|n_X - n_Y| + i)$.

Let ψ be the restriction of $d^2 f_p$ to L_p , then ψ is represented by an $(n_X - k) \times (n_X - k)(n_Y - k)$ matrix. Assume that $J^1(f)$ is transversal to $S_i(X, Y)$ at $p \in S_i(f)$. Denote

$$S_{i,h}(f) = \{p \in X \mid \text{rank } \psi = n_X - k - h\}.$$

Note that if $p \in S_{i,0}(f)$ then p is a regular point of f restricted on $S_i(f)$. Using local coordinates we can check that in a neighborhood of p the set $S_i(f)$ is a submanifold of codimension $(n_X - k)(n_Y - k)$. Let U denote the neighborhood of p . Define the vector bundle T on $S_i(f) \cap U$ by the exactness of the following sequence of the vector bundles on $S_i(f) \cap U$:

$$0 \longrightarrow T \longrightarrow E \xrightarrow{d^2 f} L^* \otimes G \longrightarrow 0.$$

We can prove that T is the tangent bundle to $S_i(f) \cap U$.

The third differential is defined on the subset $S_{i,h}(f)$. Assume that $J^1(f)$ is transversal to $S_i(X, Y)$. Define the vector bundles R and K on $S_{i,h}(f)$ by the exact sequence

$$0 \longrightarrow R \longrightarrow L \xrightarrow{d^2 f} L^* \otimes G \xrightarrow{\pi_2} K \longrightarrow 0,$$

where $\dim R_p = h$ and $\dim K_p = h(n_Y - k) + (n_X - k - h)(n_Y - k - 1)$. Let R^\perp be the annihilator of R in L^* . Since R is a subbundle of L we have the exact sequence

$$0 \longrightarrow R^\perp \longrightarrow L^* \longrightarrow R^* \longrightarrow 0.$$

Hence

$$0 \longrightarrow R^\perp \otimes G \longrightarrow L^* \otimes G \longrightarrow R^* \otimes G \longrightarrow 0$$

is also exact. It is easy to check that $d^2f(L) \subset R^\perp \otimes G$. Then we have the commutative exact diagram

$$\begin{array}{ccccccc} L & \xrightarrow{d^2f} & L^* \otimes G & \xrightarrow{\pi_2} & K & \longrightarrow & 0 \\ \downarrow d^2f & & \downarrow = & & \downarrow \pi & & \\ R^\perp \otimes G & \longrightarrow & L^* \otimes G & \longrightarrow & R^* \otimes G & \longrightarrow & 0. \end{array}$$

The maps in the second square are surjective. That implies that

$$R^* \otimes K \longrightarrow R^* \otimes R^* \otimes G \longrightarrow 0$$

is exact. Consequently, we can define the bundle $R^* \odot K$ such that the sequence

$$0 \longrightarrow R^* \odot K \longrightarrow R^* \otimes K \xrightarrow{\gamma} R^* \wedge R^* \otimes G \longrightarrow 0$$

is exact, where γ is defined by the composition

$$R^* \otimes K \longrightarrow R^* \otimes R^* \otimes G \longrightarrow R^* \wedge R^* \otimes G.$$

Now, recall that we have the projection maps

$$\pi_1 : F \rightarrow G, \quad \pi_2 : L^* \otimes G \rightarrow K,$$

which induce

$$\begin{aligned} \pi'_1 &: R^* \otimes L^* \otimes F \rightarrow R^* \otimes L^* \otimes G, \\ \pi'_2 &: R^* \otimes L^* \otimes G \rightarrow R^* \otimes K. \end{aligned}$$

Let $\{z_1, \dots, z_{n_X-k}\}$ be a part of a basis of sections in E over the neighborhood U of p such that $\{z_j | S_i(f) \cap U\}$ is a basis for $L|U \cap S_i(f)$. We define $\varphi^2 : T \rightarrow R^* \otimes L^* \otimes F$ by

$$(2.2) \quad \varphi_x^2(t)(r \otimes l) = \sum_{i,j,m} \langle t, d(\langle z_i, da_{mj} \rangle)(x) \rangle \langle l, z_m^*(x) \rangle \langle r, z_i^*(x) \rangle w_j(x)$$

where $x \in U \cap S_i(f)$, $r \in R$ and $l \in L$. Then the third differential $d^3f : T \rightarrow R^* \otimes K$ of f is defined as $d^3f_x = \pi'_2 \circ \pi'_1 \circ \varphi_x^2$.

PROPOSITION 2.3 ([16]). *The map d^3f is well-defined over $S_{i,h}(f)$ and satisfies $d^3f(T) \subset R^* \odot K$.*

The following fact can be proved by calculating the differential of the jet map $J^2f : X \rightarrow J^2(X, Y)$ and checking the transversal condition of $J^2(f)$ with $S_{i,h}(X, Y)$ (see [16] and [7]).

PROPOSITION 2.4. *The map $J^2(f)$ is transversal to $S_{i,h}(X, Y)$ at p if and only if $d^3f_p : T_p \rightarrow R_p^* \odot K_p$ is surjective.*

If $J^2(f)$ is transversal to $S_{i,h}(X, Y)$ then $S_{i,h,k}(f)$ is defined as

$$S_{i,h,k}(f) = \{p \in S_{i,h}(f) \mid d(f)|_{S_{i,h}(f)} \text{ drops rank by } k\},$$

see [7, p.156].

2.2. Excellent maps. Suppose that $\dim X = 4$ and $\dim Y = 2$. In this setting, an excellent map is defined as follows.

DEFINITION 2.5. A smooth map $f : X \rightarrow Y$ is called an *excellent map* if for each point $p \in X$, there exist a system of local coordinates x_1, x_2, x_3, x_4 centered at p on X and a system of local coordinates y_1, y_2 centered at $f(p)$ on Y such that f is locally described in one of the following form:

- (1) $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2)$,
- (2) $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^2 + x_3^2 + x_4^2)$,
- (3) $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^2 + x_3^2 - x_4^2)$,
- (4) $(x_1, x_2, x_3, x_4) \mapsto (x_1, x_2^2 \pm x_3^2 + x_1x_4 + x_4^3)$.

The point in case (1) is a regular point. The point in cases (2), (3) and (4) is called a *definite fold*, an *indefinite fold* and a *cusps*, respectively.

Excellent maps actually exist “generically” in $C^\infty(X, Y)$ because of the following interpretation in terms of transversality. Let $S_1^2(X, Y)$ be the set of $j^2 f(p) \in J^2(X, Y)$ such that $j^1(f)(p) \in S_1(X, Y)$, $j^1(f)$ is transversal to $S_1(X, Y)$ at p , and $\text{rank } d(f|_{S_1(f)}) = \min\{\dim S_1(f), \dim Y\} - 1$, which is the notation in [16]. This set is nothing but $S_{1,1}(X, Y)$ in Section 2.1. Since we are studying the case where $\dim X = 4$ and $\dim Y = 2$, we have $\text{rank } d(f|_{S_1(f)}) = 0$.

PROPOSITION 2.6 ([15, 16]). *A smooth map $f : X \rightarrow Y$ is an excellent map if and only if*

- (a) $J^1(f)$ is transversal to $S_1(X, Y)$ and $S_2(X, Y)$, and
- (b) $J^2(f)$ is transversal to $S_1^2(X, Y)$.

The condition (a) is equivalent to that $d^2 f : E \rightarrow L^* \otimes G$ is surjective and $S_2(f) = \emptyset$. The condition (b) is equivalent to that, for any $p \in S(f)$, either (i) $p \in S_{1,0}(f)$ or (ii) $p \in S_{1,1}(f)$ and $d^3 f_p : T_p \rightarrow R_p^* \odot K_p$ is surjective. Note that the condition (ii) is equivalent to $p \in S_{1,1,0}(f)$. The following proposition gives a criterion to determine if a singular point is a fold, a cusp, or none of them.

PROPOSITION 2.7. *Let $f : X \rightarrow Y$ be a smooth map. Suppose that $p \in S_1(f)$ and $d^2 f_p : T_p X \rightarrow L_p^* \otimes G_p$ is surjective. Then p is a fold of f if and only if $p \in S_{1,0}(f)$, and p is a cusp of f if and only if $p \in S_{1,1}(f)$ and $d^3 f_p : T_p \rightarrow R_p^* \odot K_p$ is surjective.*

In the proofs of the main theorems, we will explicitly calculate the higher differentials by choosing suitable basis. Let $P = (Q, R) : \mathbb{R}^4 \rightarrow \mathbb{R}^2$ be a smooth map and $p \in S_1(P)$. We may choose local coordinates (x_1, x_2, x_3, x_4) of \mathbb{R}^4 at p and those of \mathbb{R}^2 at $P(p)$ such that $\text{grad } Q(p) = (1, 0, 0, 0)$ and $\text{grad } R(p) = (0, 0, 0, 0)$. We see that $\left\{ \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\}$ is a basis of sections of $T\mathbb{R}^4$ in the neighborhood of p . In this setting, $\dim L_p = 3$, $\dim G_p = 1$ and $d^2 P_p$ is represented by the matrix

$$M = \left(\frac{\partial^2 R}{\partial x_i \partial x_j} \right)_{i=1,2,3,4, j=2,3,4}.$$

Hence $d^2 P_p$ is surjective if and only if $\text{rank } M = 3$. We can check that the restriction $d^2 P_p|_{L_p}$ is represented by

$$H = \left(\frac{\partial^2 R}{\partial x_i \partial x_j} \right)_{i=2,3,4, j=2,3,4},$$

which is exactly the Hessian of R with variables (x_2, x_3, x_4) . In particular, $p \in S_{1,1}(P)$ if and only if $\text{rank } H = 2$.

If $p \in S_{1,1}(P)$ and $d^2 P_p$ is surjective, by choosing suitable coordinates (x_1, x_2, x_3, x_4) , we may further assume that $\frac{\partial^2 R}{\partial x_4 \partial x_j}(p) = 0$ for all $j = 2, 3, 4$. Then $\dim L_p^* = 3$, $\text{rank}(d^2 P_p|_{L_p}) = 2$, $\dim R_p^* = 1$ and $\dim K_p = 1$ imply $R_p^* \odot K_p = R_p^* \otimes K_p$. Let

$$\xi_j = -\frac{\partial Q}{\partial x_j} \frac{\partial}{\partial x_1} + \frac{\partial Q}{\partial x_1} \frac{\partial}{\partial x_j}$$

for $j = 2, 3, 4$.

LEMMA 2.8. $\{\xi_2, \xi_3, \xi_4\}$ is a basis of $L|_{S_1(P)} \cap U$ for some neighborhood U of p .

PROOF. For $j = 2, 3, 4$, we have $dQ(\xi_j) = 0$. Then $dR(\xi_j) = 0$ on $S_1(P)$ in this coordinate neighborhood. Thus ξ_j belongs to L . Since $\frac{\partial Q}{\partial x_1}(p) = 1$, they are independent in a small neighborhood of p . \square

Recall that $\left\{ \frac{\partial}{\partial x_2}, \frac{\partial}{\partial x_3}, \frac{\partial}{\partial x_4} \right\}$ is a basis of L_p . Since $\text{rank } M = 3$ we have $\frac{\partial^2 R}{\partial x_1 \partial x_4}(p) \neq 0$. Hence T_p is a one-dimensional space generated by $\frac{\partial}{\partial x_4}$.

In the following, the pairing $\langle \cdot, \cdot \rangle$ is defined with respect to the basis $\left\{ \frac{\partial}{\partial x_1}, \xi_2, \xi_3, \xi_4 \right\}$.

LEMMA 2.9. The map $d^3 P_p : T_p \rightarrow R_p^* \otimes K_p$ is given by

$$t \frac{\partial}{\partial x_4} \mapsto t \frac{\partial}{\partial x_4} (\xi_4(\xi_4(R)))(p) \langle \cdot, dx_4(p) \rangle w_2(p).$$

PROOF. Since $\frac{\partial Q}{\partial x_1}(p) = 1$ and $\frac{\partial Q}{\partial x_j}(p) = 0$ for $j = 2, 3, 4$, we have

$$\langle \cdot, \xi_j^*(p) \rangle = -\frac{\partial Q}{\partial x_j}(p) \langle \cdot, dx_1(p) \rangle + \frac{\partial Q}{\partial x_1}(p) \langle \cdot, dx_j(p) \rangle = \langle \cdot, dx_j(p) \rangle$$

for $j = 2, 3, 4$. The terms in (2.2) with $w_1(p)$ vanish by π'_1 . For $j = 2, 3$, $\langle \cdot, \xi_j^*(p) \rangle$ belongs to the image of $d^2 P_p$ and therefore their images by the map π'_2 vanish. Hence we have

$$d^3 P_p(t) = \left\langle t \frac{\partial}{\partial x_4}, d(\langle \xi_4, da_{42} \rangle)(p) \right\rangle \langle \cdot, dx_4(p) \rangle w_2(p).$$

To prove the assertion, it is enough to show that $\langle \xi_4, da_{42} \rangle = \xi_4(\xi_4(R))$. According to the definition of a_{42} we have $a_{42} = \xi_4(R)$. Thus $\langle \xi_4, da_{42} \rangle = \langle \xi_4, d(\xi_4(R)) \rangle$. Since $da_{42} = \frac{\partial a_{42}}{\partial x_1} dx_1 + \sum_{j=2,3,4} \xi_j(a_{42}) \xi_j^*$, we have $\langle \xi_4, da_{42} \rangle = \xi_4(a_{42}) = \xi_4(\xi_4(R))$. This is the equality which we want to have. \square

The map $d^3 P_p$ is surjective if and only if

$$\frac{\partial}{\partial x_4}(\xi_4(\xi_4(R)))(p) \neq 0.$$

Thus, in summary, we have the following criterion to check if a singularity is a fold, cusp or none of them.

LEMMA 2.10. *In the above setting, $p \in S_1(P)$ is a fold if and only if $\text{rank } H = 3$. It is a cusp if and only if $\text{rank } M = 3$, $\text{rank } H = 2$ and $\frac{\partial}{\partial x_4}(\xi_4(\xi_4(R)))(p) \neq 0$.*

3. Proof of the first assertion in Theorem 1.1. Throughout this section, for a function $h : \mathbb{R}^n \rightarrow \mathbb{R}$ with variables x_1, \dots, x_n , we denote the partial differentials $\frac{\partial h}{\partial x_i}$, $\frac{\partial^2 h}{\partial x_i \partial x_j}$ and $\frac{\partial^3 h}{\partial x_i \partial x_j \partial x_k}$ by h_{x_i} , $h_{x_i x_j}$ and $h_{x_i x_j x_k}$ respectively.

We are studying polynomial maps of the form $f(z, w) = z^p + w^q + a\bar{z} + b\bar{w}$. This polynomial has complex and complex-conjugate variables. Such a polynomial is called a *mixed polynomial* in [18]. Since the assertion in Theorem 1.1 is for generic a and b , we may assume that $ab \neq 0$. Let c_1 and c_2 be non-zero complex numbers satisfying $c_1^p = a\bar{c}_1$ and $c_2^q = b\bar{c}_2$, respectively. By changing the coordinates as $z = c_1 u$ and $w = c_2 v$ and setting $\mu = a\bar{c}_1/(b\bar{c}_2)$, we have

$$\begin{aligned} f(z, w) &= (c_1 u)^p + a\overline{c_1 u} + (c_2 v)^q + b\overline{c_2 v} \\ &= a\bar{c}_1(u^p + \bar{u}) + b\bar{c}_2(v^q + \bar{v}) \\ &= b\bar{c}_2(\mu(u^p + \bar{u}) + v^q + \bar{v}). \end{aligned}$$

Now we set

$$P(u, v; \mu) = \mu(u^p + \bar{u}) + v^q + \bar{v},$$

with $p, q \geq 2$ and $\mu \in \mathbb{C} \setminus \{0\}$.

LEMMA 3.1. *Suppose that P is an excellent map. Then there exists a linear deformation f_t of f which consists of excellent maps for $t \in (0, 1]$ and satisfies $f_1(z, w) = f(z, w) + a\bar{z} + b\bar{w}$. Moreover, if P has no definite fold then f_t does also for $t \in (0, 1]$.*

PROOF. Set $A(t) = at^{(p-1)q}$ and $B(t) = bt^{p(q-1)}$ and define the linear deformation $f_t(z, w)$ by $f_t(z, w) = f(z, w) + A(t)\bar{z} + B(t)\bar{w}$. Note that this satisfies $f_1(z, w) = f(z, w) + a\bar{z} + b\bar{w}$ as required. Since $c_1(t)^p = A(t)\bar{c}_1(t)$ and $c_2(t)^q = B(t)\bar{c}_2(t)$, we may choose $c_1(t) = c_{1,0}t^q$ and $c_2(t) = c_{2,0}t^p$, where $c_{1,0}$ and $c_{2,0}$ are non-zero complex numbers satisfying $c_{1,0}^p = a\bar{c}_{1,0}$ and $c_{2,0}^q = b\bar{c}_{2,0}$, such that

$$\mu = \frac{A(t)\bar{c}_1(t)}{B(t)\bar{c}_2(t)} = \frac{at^{(p-1)q} \cdot \bar{c}_{1,0}t^q}{bt^{p(q-1)} \cdot \bar{c}_{1,0}t^p} = \frac{a\bar{c}_{1,0}}{b\bar{c}_{1,0}}.$$

Therefore, for each $t \in (0, 1]$, the smooth map f_t can be replaced by P using this change of coordinates. \square

By Lemma 3.1, to prove Theorem 1.1, it is enough to show that $P(u, v; \mu)$ is an excellent map for a generic choice of μ . Hereafter we study the map P instead of f .

LEMMA 3.2. *A singular point $z_0 = (u_0, v_0)$ of P satisfies*

$$\begin{cases} p|u_0|^{p-1} = q|v_0|^{q-1} = 1, \\ \frac{p-1}{2} \arg u_0 + \arg \mu = \frac{q-1}{2} \arg v_0 + \kappa\pi, \end{cases}$$

where κ is some integer.

PROOF. By [18, Proposition 1], a singular point $z_0 = (u_0, v_0)$ of the mixed polynomial P satisfies

$$\overline{\frac{\partial P}{\partial u}(z_0)} = \alpha \frac{\partial P}{\partial \bar{u}}(z_0) \quad \text{and} \quad \overline{\frac{\partial P}{\partial v}(z_0)} = \alpha \frac{\partial P}{\partial \bar{v}}(z_0)$$

for a complex number α with $|\alpha| = 1$. Substituting $P(u, v; \mu) = \mu(u^p + \bar{u}) + v^q + \bar{v}$ and taking their conjugates, we have

$$\mu p u_0^{p-1} = \bar{\mu} \bar{\alpha} \quad \text{and} \quad q v_0^{q-1} = \bar{\alpha}.$$

The first equation in the assertion follows by taking the absolute values of these equations, and the second one follows by eliminating α and checking the arguments. \square

Set $Q(u, v; \mu)$ and $R(u, v; \mu)$ to be the real and imaginary part of $P(u, v; \mu)$ respectively, i.e., $P(u, v; \mu) = Q(u, v; \mu) + iR(u, v; \mu)$. Set $r_1 = |u|$, $\theta_1 = \arg u$, $r_2 = |v|$ and $\theta_2 = \arg v$, so that $(r_1, \theta_1, r_2, \theta_2)$ are regarded as the polar coordinates of \mathbb{C}^2 . Note that since any singular point $z_0 = (u_0, v_0)$ satisfies $u_0 \neq 0$ and $v_0 \neq 0$ by Lemma 3.2 and we only need to observe P in a small neighborhood of the set of singular points, we may assume that $r_1 \neq 0$ and $r_2 \neq 0$, that is, $\arg u$ and $\arg v$ are well-defined. Since

$$\begin{aligned} P &= Q + iR = \mu(u^p + \bar{u}) + (v^q + \bar{v}) \\ &= |\mu|r_1^p e^{i(p\theta_1 + \arg \mu)} + |\mu|r_1 e^{i(-\theta_1 + \arg \mu)} + r_2^q e^{iq\theta_2} + r_2 e^{-i\theta_2}, \end{aligned}$$

we have

$$\begin{cases} Q = |\mu|r_1^p \cos(p\theta_1 + \arg \mu) + |\mu|r_1 \cos(-\theta_1 + \arg \mu) + r_2^q \cos(q\theta_2) + r_2 \cos(-\theta_2), \\ R = |\mu|r_1^p \sin(p\theta_1 + \arg \mu) + |\mu|r_1 \sin(-\theta_1 + \arg \mu) + r_2^q \sin(q\theta_2) + r_2 \sin(-\theta_2). \end{cases}$$

To simplify the notation, hereafter we may omit μ in brackets. For example, we denote $Q(z_0)$ instead of $Q(z_0; \mu)$.

Now we calculate the gradient of the real part Q at a singular point $z_0 = (u_0, v_0)$ of P . Let (k_1, k_2, k_3, k_4) be the gradient at z_0 , i.e.,

$$(k_1, k_2, k_3, k_4) = (Q_{r_1}(z_0), Q_{\theta_1}(z_0), Q_{r_2}(z_0), Q_{\theta_2}(z_0)).$$

We set

$$\begin{aligned} \Theta_1 &= \frac{p+1}{2} \arg u_0, & \Theta_2 &= \frac{p-1}{2} \arg u_0 + \arg \mu, \\ \Theta_3 &= \frac{q+1}{2} \arg v_0, & \Theta_4 &= \frac{q-1}{2} \arg v_0. \end{aligned}$$

LEMMA 3.3.

$$\begin{aligned} k_1 &= 2|\mu| \cos \Theta_1 \cos \Theta_2, & k_2 &= -2|\mu||u_0| \sin \Theta_1 \cos \Theta_2, \\ k_3 &= 2 \cos \Theta_3 \cos \Theta_4, & k_4 &= -2|v_0| \sin \Theta_3 \cos \Theta_4. \end{aligned}$$

PROOF. By Lemma 3.2,

$$\begin{aligned} k_1 &= Q_{r_1}(z_0) = p|\mu||u_0|^{p-1} \cos(p \arg u_0 + \arg \mu) + |\mu| \cos(-\arg u_0 + \arg \mu) \\ &= |\mu| \{ \cos(p \arg u_0 + \arg \mu) + \cos(-\arg u_0 + \arg \mu) \} \\ &= 2|\mu| \cos\left(\frac{p+1}{2} \arg u_0\right) \cos\left(\frac{p-1}{2} \arg u_0 + \arg \mu\right) \\ &= 2|\mu| \cos \Theta_1 \cos \Theta_2. \end{aligned}$$

The equations for k_2, k_3 and k_4 are obtained similarly. \square

LEMMA 3.4. *The values $\Theta_1, \Theta_2, \Theta_3, \Theta_4$ satisfy the following relations:*

$$\begin{cases} \Theta_2 = \Theta_4 + \kappa\pi, \\ \Theta_1 = p \arg u_0 - \Theta_2 + \arg \mu = \arg u_0 + \Theta_2 - \arg \mu, \\ \Theta_3 = q \arg v_0 - \Theta_2 + \kappa\pi = \arg v_0 + \Theta_2 - \kappa\pi, \\ \frac{p-1}{p+1} \Theta_1 + \arg \mu = \frac{q-1}{q+1} \Theta_3 + \kappa\pi, \end{cases}$$

where κ is the integer given in Lemma 3.2.

PROOF. The assertion follows from Lemma 3.2 and the defining equations of Θ_i 's. \square

3.1. Fold singularities in case $k_1 \neq 0$. In this subsection, we always assume that $k_1 \neq 0$. The case $k_1 = 0$ will be studied later.

Let $(r'_1, \theta'_1, r'_2, \theta'_2)$ be new coordinates in a small neighborhood of z_0 defined by

$$(r'_1, \theta'_1, r'_2, \theta'_2) = (k_1 r_1 + k_2 \theta_1 + k_3 r_2 + k_4 \theta_2, \theta_1, r_2, \theta_2).$$

LEMMA 3.5. $(Q_{r'_1}(z_0), Q_{\theta'_1}(z_0), Q_{r'_2}(z_0), Q_{\theta'_2}(z_0)) = (1, 0, 0, 0)$.

PROOF. Since $r_1 = \frac{1}{k_1} r'_1 - \frac{k_2}{k_1} \theta'_1 - \frac{k_3}{k_1} r'_2 - \frac{k_4}{k_1} \theta'_2$ we have $Q_{r'_1}(z_0) = \frac{1}{k_1} Q_{r_1}(z_0) = 1$, $Q_{\theta'_1}(z_0) = -\frac{k_2}{k_1} Q_{r_1}(z_0) + Q_{\theta_1}(z_0) = 0$, $Q_{r'_2}(z_0) = -\frac{k_3}{k_1} Q_{r_1}(z_0) + Q_{r_2}(z_0) = 0$ and $Q_{\theta'_2}(z_0) = -\frac{k_4}{k_1} Q_{r_1}(z_0) + Q_{\theta_2}(z_0) = 0$. \square

Set $\hat{R} = R - sQ$, where $s = R_{r'_1}(z_0)$, so that $\hat{R}_{r'_1} = R_{r'_1} - sQ_{r'_1}$ vanishes at z_0 .

LEMMA 3.6. *The Hessian H of \hat{R} with variables $(\theta'_1, r'_2, \theta'_2)$ is*

$$H = \begin{pmatrix} k_2^2 A - 2k_2 B + C & k_3(k_2 A - B) & k_4(k_2 A - B) \\ k_3(k_2 A - B) & k_3^2 A + D & k_3 k_4 A + E \\ k_4(k_2 A - B) & k_3 k_4 A + E & k_4^2 A + F \end{pmatrix},$$

where

$$\begin{aligned} A &= \frac{1}{k_1^2} \hat{R}_{r_1 r_1}, & B &= \frac{1}{k_1} \hat{R}_{r_1 \theta_1}, & C &= \hat{R}_{\theta_1 \theta_1} \\ D &= \hat{R}_{r_2 r_2}, & E &= \hat{R}_{r_2 \theta_2}, & F &= \hat{R}_{\theta_2 \theta_2}. \end{aligned}$$

Its determinant is

$$\det H = (k_4^2 D - 2k_3 k_4 E + k_3^2 F)(AC - B^2) + (k_2^2 A - 2k_2 B + C)(DF - E^2).$$

PROOF. Since $r_1 = \frac{1}{k_1} r'_1 - \frac{k_2}{k_1} \theta'_1 - \frac{k_3}{k_1} r'_2 - \frac{k_4}{k_1} \theta'_2$, we have

$$\hat{R}_{\theta'_1} = -\frac{k_2}{k_1} \hat{R}_{r_1} + \hat{R}_{\theta_1}, \quad \hat{R}_{r'_2} = -\frac{k_3}{k_1} \hat{R}_{r_1} + \hat{R}_{r_2}, \quad \hat{R}_{\theta'_2} = -\frac{k_4}{k_1} \hat{R}_{r_1} + \hat{R}_{\theta_2},$$

and taking partial differentiation again, we have

$$\begin{aligned} \hat{R}'_{\theta'_1 \theta'_1} &= \frac{k_2^2}{k_1^2} \hat{R}_{r_1 r_1} - \frac{2k_2}{k_1} \hat{R}_{r_1 \theta_1} + \hat{R}_{\theta_1 \theta_1}, & \hat{R}'_{\theta'_1 r'_2} &= \frac{k_2 k_3}{k_1^2} \hat{R}_{r_1 r_1} - \frac{k_3}{k_1} \hat{R}_{r_1 \theta_1}, \\ \hat{R}'_{\theta'_1 \theta'_2} &= \frac{k_2 k_4}{k_1^2} \hat{R}_{r_1 r_1} - \frac{k_4}{k_1} \hat{R}_{r_1 \theta_1}, & \hat{R}'_{r'_2 r'_2} &= \frac{k_3^2}{k_1^2} \hat{R}_{r_1 r_1} + \hat{R}_{r_2 r_2}, \\ \hat{R}'_{r'_2 \theta'_2} &= \frac{k_3 k_4}{k_1^2} \hat{R}_{r_1 r_1} + \hat{R}_{r_2 \theta_2}, & \hat{R}'_{\theta'_2 \theta'_2} &= \frac{k_4^2}{k_1^2} \hat{R}_{r_1 r_1} + \hat{R}_{\theta_2 \theta_2}. \end{aligned}$$

Thus we obtain the Hessian. The calculation of the determinant is straightforward. \square

Now we calculate the Hessian of \hat{R} at a singular point $z_0 = (u_0, v_0)$ of P . We first determine the value s which we used in the above change of coordinates.

LEMMA 3.7. $s = \tan \Theta_2$.

PROOF. By Lemma 3.2,

$$R_{r_1}(z_0) = p|\mu||u_0|^{p-1} \sin(p \arg u_0 + \arg \mu) + |\mu| \sin(\arg \mu - \arg u_0) = 2|\mu| \sin \Theta_2 \cos \Theta_1.$$

$$\text{Hence } s = R_{r'_1}(z_0) = \frac{1}{k_1} R_{r_1}(z_0) = \tan \Theta_2. \quad \square$$

LEMMA 3.8. A singular point $z_0 = (u_0, v_0)$ of P satisfies

$$\begin{aligned} A(z_0) &= \frac{1}{k_1^2 \cos \Theta_2} p(p-1)|\mu||u_0|^{p-2} \sin \Theta_1, & B(z_0) &= \frac{1}{k_1 \cos \Theta_2} (p-1)|\mu| \cos \Theta_1, \\ C(z_0) &= -\frac{1}{\cos \Theta_2} (p-1)|\mu||u_0| \sin \Theta_1, & D(z_0) &= \frac{(-1)^\kappa}{\cos \Theta_2} q(q-1)|v_0|^{q-2} \sin \Theta_3, \\ E(z_0) &= \frac{(-1)^\kappa}{\cos \Theta_2} (q-1) \cos \Theta_3, & F(z_0) &= \frac{(-1)^{\kappa+1}}{\cos \Theta_2} (q-1)|v_0| \sin \Theta_3. \end{aligned}$$

PROOF. By Lemma 3.7 and Lemma 3.4,

$$\begin{aligned}
A(z_0) &= \frac{1}{k_1^2} \hat{R}_{r_1 r_1}(z_0) = \frac{1}{k_1^2} (R_{r_1 r_1}(z_0) - s Q_{r_1 r_1}(z_0)) \\
&= \frac{1}{k_1^2} \{p(p-1)|\mu||u_0|^{p-2} \sin(p \arg u_0 + \arg \mu) - sp(p-1)|\mu||u_0|^{p-2} \cos(p \arg u_0 + \arg \mu)\} \\
&= \frac{1}{k_1^2 \cos \Theta_2} p(p-1)|\mu||u_0|^{p-2} \{\sin(p \arg u_0 + \arg \mu) \cos \Theta_2 - \cos(p \arg u_0 + \arg \mu) \sin \Theta_2\} \\
&= \frac{1}{k_1^2 \cos \Theta_2} p(p-1)|\mu||u_0|^{p-2} \sin(p \arg u_0 + \arg \mu - \Theta_2) \\
&= \frac{1}{k_1^2 \cos \Theta_2} p(p-1)|\mu||u_0|^{p-2} \sin \Theta_1.
\end{aligned}$$

The other values $B(z_0)$, $C(z_0)$, $D(z_0)$, $E(z_0)$ and $F(z_0)$ can be obtained by similar calculation. \square

PROPOSITION 3.9. *At a singular point $z_0 = (u_0, v_0)$ of P with $k_1 \neq 0$, the determinant of the Hessian $H(z_0)$ is given by*

$$\det H(z_0) = -\frac{1}{k_1^2 \cos \Theta_2} 4(p-1)(q-1)|\mu|^2 \phi(z_0),$$

where

$$\phi(z_0) = (-1)^\kappa (p-1)|v_0| \sin \Theta_3 + (q-1)|\mu||u_0| \sin \Theta_1.$$

PROOF. By Lemma 3.3, Lemma 3.8 and the first equation in Lemma 3.2 we have

$$\begin{aligned}
(AC - B^2)(z_0) &= -\frac{1}{k_1^2 \cos^2 \Theta_2} (p-1)^2 |\mu|^2, \\
(k_2^2 A - 2k_2 B + C)(z_0) &= \frac{1}{k_1^2} 4(p-1)|\mu|^3 |u_0| \sin \Theta_1 \cos \Theta_2, \\
(DF - E^2)(z_0) &= -\frac{1}{\cos^2 \Theta_2} (q-1)^2, \\
(k_4^2 D - 2k_3 k_4 E + k_3^2 F)(z_0) &= (-1)^\kappa 4(q-1)|v_0| \sin \Theta_3 \cos \Theta_4.
\end{aligned}$$

Substituting these results to the equation in Lemma 3.6, we have the assertion. Note that the first equation in Lemma 3.4 implies that $\cos \Theta_2 = (-1)^\kappa \cos \Theta_4$ and this equality is used in the calculation. \square

PROPOSITION 3.10. *If a singular point $z_0 = (u_0, v_0)$ of P satisfies $k_1 \neq 0$ and $\phi(z_0) \neq 0$ then it is a fold.*

PROOF. By Lemma 3.2, $u_0, v_0 \in \mathbb{C} \setminus \{0\}$. Since $k_1 = 2|\mu| \cos \Theta_1 \cos \Theta_2 \neq 0$ and $\phi(z_0) \neq 0$ by assumption, we have $\det H(z_0) \neq 0$ by Proposition 3.9, i.e., $\text{rank } H(z_0) = 3$. Thus z_0 is a fold by Lemma 2.10. \square

3.2. Cusp singularities in case $k_1 \neq 0$ and $\hat{R}_{\theta'_1\theta'_1}(z_0) \neq 0$. We assume that $k_1 \neq 0$ as in the previous subsection. Further we assume that $\hat{R}_{\theta'_1\theta'_1}(z_0) \neq 0$. The case $\hat{R}_{\theta'_1\theta'_1}(z_0) = 0$ will be studied in the next subsection.

Let $(r''_1, \theta''_1, r''_2, \theta''_2)$ be new coordinates in a small neighborhood of z_0 defined by

$$(r''_1, \theta''_1, r''_2, \theta''_2) = (r'_1, \theta'_1 + \ell_1 r'_2 + \ell_2 \theta'_2, r'_2, \theta'_2),$$

where

$$\ell_1 = \frac{\hat{R}_{\theta'_1 r'_2}(z_0)}{\hat{R}_{\theta'_1 \theta'_1}(z_0)} \quad \text{and} \quad \ell_2 = \frac{\hat{R}_{\theta'_1 \theta'_2}(z_0)}{\hat{R}_{\theta'_1 \theta'_1}(z_0)}.$$

LEMMA 3.11. *Let z_0 be a singular point of P satisfying $k_1 \neq 0$, $\phi(z_0) = 0$ and $\hat{R}_{\theta'_1\theta'_1}(z_0) \neq 0$. Then the Hessian $H''(z_0)$ of \hat{R} with variables $(\theta''_1, r''_2, \theta''_2)$ at z_0 is*

$$H''(z_0) = \begin{pmatrix} \hat{R}_{\theta'_1\theta'_1}(z_0) & 0 & 0 \\ 0 & -4(p-1)^2|\mu|^2/(k_1^2\hat{R}_{\theta'_1\theta'_1}(z_0)) & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where $\hat{R}_{\theta'_1\theta'_1}(z_0) = \frac{1}{k_1^2}4(p-1)|\mu|^3|u_0|\sin\Theta_1\cos\Theta_2$.

PROOF. Using $\theta'_1 = \theta''_1 - \ell_1 r''_2 - \ell_2 \theta''_2$, we can easily check that $\hat{R}_{\theta''_1\theta''_1} = \hat{R}_{\theta'_1\theta'_1}$ and $\hat{R}_{\theta''_1 r''_2} = \hat{R}_{\theta'_1 r'_2} = 0$ at z_0 . By Lemma 3.6, $\hat{R}_{\theta'_1\theta'_1} = k_2^2 A - 2k_2 B + C$, and its value at z_0 had been calculated in the proof of Proposition 3.9.

By the change of coordinates, $\hat{R}_{r''_2 r''_2} = \ell_2^2 \hat{R}_{\theta'_1\theta'_1} - 2\ell_1 \hat{R}_{\theta'_1 r'_2} + \hat{R}_{r'_2 r'_2}$. Then the value at z_0 is calculated, by using Lemma 3.6, as

$$\begin{aligned} \hat{R}_{r''_2 r''_2}(z_0) &= \frac{-\hat{R}_{\theta'_1 r'_2}(z_0)^2 + \hat{R}_{r'_2 r'_2}(z_0)\hat{R}_{\theta'_1\theta'_1}(z_0)}{\hat{R}_{\theta'_1\theta'_1}(z_0)} \\ &= \frac{k_3^2(A(z_0)C(z_0) - B(z_0)^2) + D(z_0)(k_2^2 A(z_0) - 2k_2 B(z_0) + C(z_0))}{\hat{R}_{\theta'_1\theta'_1}(z_0)}. \end{aligned}$$

By substituting the values in Lemma 3.3 and Lemma 3.8 and using the first equation in Lemma 3.2 and $\phi(z_0) = 0$, we can verify that the numerator is $-\frac{4}{k_1^2}(p-1)^2|\mu|^2$.

By the change of coordinates, $\hat{R}_{\theta''_2\theta''_2} = \ell_1\ell_2\hat{R}_{\theta'_1\theta'_1} - \ell_1\hat{R}_{\theta'_1\theta'_2} - \ell_2\hat{R}_{\theta'_1 r'_2} + \hat{R}_{r'_2\theta'_2}$. Then the value at z_0 is calculated, by using Lemma 3.6, as

$$\begin{aligned} \hat{R}_{\theta''_2\theta''_2}(z_0) &= \frac{1}{\hat{R}_{\theta'_1\theta'_1}(z_0)} \left(-\hat{R}_{\theta'_1\theta'_2}(z_0)\hat{R}_{\theta'_1 r'_2}(z_0) + \hat{R}_{\theta'_1\theta'_1}(z_0)\hat{R}_{r'_2\theta'_2}(z_0) \right) \\ &= \frac{1}{\hat{R}_{\theta'_1\theta'_1}(z_0)} \{ -k_3 k_4 (k_2 A(z_0) - B(z_0))^2 \\ &\quad + (k_3 k_4 A(z_0) + E(z_0))(k_2^2 A(z_0) - 2k_2 B(z_0) + C(z_0)) \}. \end{aligned}$$

We can verify that the numerator vanishes by applying lemmas and $\phi(z_0) = 0$ as in the previous case. Thus $\hat{R}_{r''_2\theta''_2}(z_0) = 0$. Finally we have $\hat{R}_{\theta''_2\theta''_2}(z_0) = 0$ since $\hat{R}_{r''_2 r''_2}(z_0) \neq 0$ and $\det H''(z_0) = 0$. \square

Thus the new coordinates $(r_1'', \theta_1'', r_2'', \theta_2'')$ satisfy the setting of Lemma 2.10. Recall that in these coordinates

$$\xi_4 = -\frac{\partial Q}{\partial \theta_2''} \frac{\partial}{\partial r_1''} + \frac{\partial Q}{\partial r_1''} \frac{\partial}{\partial \theta_2''},$$

where the functions and derivatives are taken on the subset $S_1(P) \cap U$ of $S_1(P)$.

PROPOSITION 3.12. *Let z_0 be a singular point of P satisfying $k_1 \neq 0$, $\phi(z_0) = 0$ and $\hat{R}_{\theta_1'\theta_1'}(z_0) \neq 0$. Then*

$$\frac{\partial}{\partial \theta_2''} (\xi_4(\xi_4(\hat{R}))) (z_0) = \frac{1}{\sin^2 \Theta_1 \cos^2 \Theta_1 \cos^2 \Theta_2} \psi(z_0),$$

where $\psi(z_0)$ is a polynomial with variables $\sin \Theta_i$ and $\cos \Theta_i$, $i = 1, 2, 3$, which is not constant zero and does not depend on the value $|\mu|$.

PROOF. It is easy to verify that the value of $k_2(k_2A - B) - (k_2^2A - 2k_2B + C)$ at z_0 is 0. Since this appears as a factor of the numerators of

$$k_2\ell_1 - k_3 = \frac{k_2\hat{R}_{\theta_1'r_2'}(z_0) - k_3\hat{R}_{\theta_1'\theta_1'}(z_0)}{\hat{R}_{\theta_1'\theta_1'}(z_0)}$$

and

$$k_2\ell_2 - k_4 = \frac{k_2\hat{R}_{\theta_1'\theta_2'}(z_0) - k_4\hat{R}_{\theta_1'\theta_1'}(z_0)}{\hat{R}_{\theta_1'\theta_1'}(z_0)},$$

we have $k_2\ell_1 - k_3 = k_2\ell_2 - k_4 = 0$. Because of this property, the change of coordinates from $(r_1, \theta_1, r_2, \theta_2)$ to $(r_1'', \theta_1'', r_2'', \theta_2'')$ is given as

$$(r_1, \theta_1, r_2, \theta_2) = \left(\frac{1}{k_1} r_1'' - \frac{k_2}{k_1} \theta_1'', \theta_1'' - \ell_1 r_2'' - \ell_2 \theta_2'', r_2'', \theta_2'' \right).$$

We can verify that

$$\ell_2 = \frac{(-1)^\kappa |v_0| \sin \Theta_3}{|\mu| |u_0| \sin \Theta_1},$$

and if furthermore $\phi(z_0) = 0$ then $\ell_2 = -\frac{q-1}{p-1}$.

By checking the change of coordinates, we see that

$$\frac{\partial}{\partial \theta_2''} = -\ell_2 \frac{\partial}{\partial \theta_1} + \frac{\partial}{\partial \theta_2} \quad \text{and} \quad \frac{\partial}{\partial r_1''} = \frac{1}{k_1} \frac{\partial}{\partial r_1}.$$

By Lemma 3.2, r_1 is a constant on $S_1(P) \cap U$ and

$$\theta_1 = \frac{q-1}{p-1} \theta_2 + \frac{2}{p-1} (\kappa\pi - \arg \mu).$$

Hence we have

$$\frac{\partial}{\partial \theta_2''} = \frac{2(q-1)}{p-1} \frac{\partial}{\partial \theta_1} \quad \text{and} \quad \frac{\partial}{\partial r_1''} = 0.$$

Now we calculate the equation in the assertion. We can replace \hat{R} in the equation by R since

$$\begin{aligned}\xi_4(\hat{R}) &= -Q_{\theta_2''} \hat{R}_{r_1''} + Q_{r_1''} \hat{R}_{\theta_2''} = -Q_{\theta_2''} (R_{r_1''} - s Q_{r_1''}) + Q_{r_1''} (R_{\theta_2''} - s Q_{\theta_2''}) \\ &= -Q_{\theta_2''} R_{r_1''} + Q_{r_1''} R_{\theta_2''} = \xi_4(R).\end{aligned}$$

We can easily check that

$$\xi_4 = \frac{\partial Q}{\partial r_1''} \frac{\partial}{\partial \theta_2''} = \frac{4|\mu|(q-1)}{k_1(p-1)} c(\theta_1; \arg \mu) \frac{\partial}{\partial \theta_1},$$

where

$$c(\theta_1; \arg \mu) = \cos\left(\frac{p+1}{2}\theta_1\right) \cos\left(\frac{p-1}{2}\theta_1 + \arg \mu\right).$$

Thus we have

$$\frac{\partial}{\partial \theta_2''} (\xi_4(\xi_4(\hat{R}))) = \frac{32|\mu|^2}{k_1^2} \left(\frac{q-1}{p-1}\right)^3 \frac{\partial}{\partial \theta_1} \left(c(\theta_1; \arg \mu) \frac{\partial}{\partial \theta_1} \left(c(\theta_1; \arg \mu) \frac{\partial R}{\partial \theta_1} \right) \right).$$

We calculate the right-hand side and then take the value at $z_0 = (u_0, v_0)$. In particular, by this substitution, we have $c(\arg u_0; \arg \mu) = \cos \Theta_1 \cos \Theta_2$. After these calculation and substitution, we eliminate $|\mu|$ by using $\phi(z_0) = 0$. In this process, $\sin^2 \Theta_1$ appears in the denominator. Since k_1^2 is also in the denominator, $\cos^2 \Theta_1 \cos^2 \Theta_2$ also appears. No other $\sin \Theta_i$'s and $\cos \Theta_i$'s appear in the denominator. Thus we have the equation claimed in the assertion.

By direct calculation, we can see that $\phi(z_0)$ is not zero at the point defined by either (i) $\sin \Theta_2 = \sin \Theta_4 = 0$ and $\Theta_1 = \Theta_3 = \frac{\pi}{4}$ or (ii) $\sin \Theta_2 = \sin \Theta_4 = 0$ and $\Theta_1 = -\Theta_3 = \frac{\pi}{4}$. Hence $\phi(z_0)$ is not constant zero. \square

PROPOSITION 3.13. *For a generic choice of μ , every singular point z_0 of P with $k_1 \neq 0$ and $\hat{R}_{\theta_1''}(\hat{R}) \neq 0$ is either a fold or a cusp.*

PROOF. We first prove that $\phi(z_0) = 0$ and $\psi(z_0) = 0$ have no common solution for a generic choice of μ . We will use Chebyshev polynomials to represent these two functions. For $n \in \mathbb{N}$, let $T_n(t)$ and $U_n(t)$ be the Chebyshev polynomials defined by

$$\cos(n\theta) = T_n(\cos \theta) \quad \text{and} \quad \sin((n+1)\theta) = U_n(\cos \theta) \sin \theta.$$

By the definitions of Θ_i 's and Lemma 3.4,

$$\Theta_1 = \frac{p+1}{2} \arg u_0 \quad \text{and} \quad \Theta_3 = \frac{q+1}{2} \arg v_0 = \frac{q+1}{2} \left(\frac{p-1}{q-1} \arg u_0 + \frac{2(\arg \mu - k\pi)}{q-1} \right).$$

Since $|u_0|$ and $|v_0|$ are non-zero constant along the singular set $S(P)$, by setting $\theta = \frac{\arg u_0}{2(q-1)}$ we parametrize $S(P)$ by θ . Substituting this to the above Θ_1 and Θ_3 we have

$$\Theta_1 = (p+1)(q-1)\theta \quad \text{and} \quad \Theta_3 = (p-1)(q+1)\theta + \frac{(q+1)(\arg \mu - k\pi)}{q-1}.$$

We set $n_1, n_3 \in \mathbb{Z}$ and $m_\kappa \in \mathbb{R}$ such that $\Theta_1 = n_1\theta$ and $\Theta_3 = n_3\theta + m_\kappa$, where m_κ depends on $\arg \mu$. Set $t = \cos \theta$. Then $\phi(z_0)$ is written as

$$\begin{aligned} \phi(z_0) = & (-1)^k (p-1) |v_0| (\sin \theta \cos m_\kappa U_{n_3-1}(t) + \sin m_\kappa T_{n_3}(t)) \\ & + (q-1) |u_0| |\mu| \sin \theta U_{n_1-1}(t). \end{aligned}$$

Similarly, $\psi(z_0)$ is written as

$$\psi(z_0) = \sin \theta P_1(t; \arg \mu) + P_2(t; \arg \mu),$$

where P_i are some polynomials on t whose coefficients depend on $\arg \mu$. Remark that $\phi(z_0)$ and $\psi(z_0)$ are not constant zero. From the equation $\phi(z_0) = 0$ of θ , we obtain that

$$\Phi(t; \mu) = |\mu|^2 \Phi_1(t) + |\mu| \cos m_\kappa \Phi_2(t) + \cos^2 m_\kappa \Phi_3(t) + \Phi_4(t) = 0$$

and from $\psi(z_0) = 0$ we get $\Psi(t; \arg \mu) = 0$, where Φ_i are non-zero polynomials not depending on μ and Ψ is a non-zero polynomial on t whose coefficients depend on $\arg \mu$. If $\phi(z_0) = 0$ and $\psi(z_0) = 0$ have a common solution then $\Phi(t; \mu) = 0$ and $\Psi(t; \arg \mu) = 0$ also.

Assume that the four polynomials $\Phi_i(t)$ have common factors which provide solutions of $\Phi(t; \mu) = 0$. These solutions do not depend on μ since $\Phi_i(t)$ do not depend on. However, since $\sin \Theta_1 \neq 0$, any solution of $\phi(z_0) = 0$ depends on $|\mu|$. This means that the above solutions do not satisfy $\phi(z_0) = 0$.

Dividing $\Phi(t; \mu) = 0$ by these factors, we may assume that $\Phi_i(t)$ do not have a common factor. Let $S(|\mu|, \arg \mu)$ be the resultant of $\Phi(t; \mu)$ and $\Psi(t; \arg \mu)$ with respect to t . We will show that $S(|\mu|, \arg \mu) \neq 0$. Assume $S(|\mu|, \arg \mu) \equiv 0$, that means the polynomials $\Phi(t; \mu)$ and $\Psi(t; \arg \mu)$ have a common factor for any $(|\mu|, \arg \mu)$. Choose $\arg \mu$ such that $\cos^2 m_\kappa \Phi_3 + \Phi_4$ is not dividable by any non-constant common factor of Φ_1, Φ_2 . Since $\Phi(t; \mu)$ is dividable by some non-constant factor of $\Psi(t; \arg \mu)$ for all $|\mu|$, there are infinitely many $|\mu|$ such that $\Phi(t; \mu)$ share the same factor $\Delta(t)$. This implies that $\Delta(t)$ is a common factor of Φ_1, Φ_2 and hence of $\cos^2 m_\kappa \Phi_3 + \Phi_4$. This is a contradiction. Thus $\phi(z_0) = 0$ and $\psi(z_0) = 0$ have no common solution for generic μ .

Let z_0 be a singular point of P with $k_1 \neq 0$ and $\hat{R}_{\theta'_1 \theta'_1}(z_0) \neq 0$. Let μ be a generic value in \mathbb{C} chosen in the above paragraph. Furthermore, we may assume that $\sin \Theta_1 = 0$ and $\sin \Theta_3 = 0$ have no common solution. By direct calculation we have

$$\hat{R}_{r'_1 \theta'_2} = (k_2 \ell_2 - k_4) A - \ell_2 B + \frac{1}{k_1} \hat{R}_{r_1 \theta_2}.$$

As we mentioned in the proof of Proposition 3.12, $k_2 \ell_2 - k_4 = 0$. The term $\hat{R}_{r_1 \theta_2}$ also vanishes. Note that $k_1 = 2|\mu| \cos \Theta_1 \cos \Theta_2 \neq 0$ and $\hat{R}_{\theta'_1 \theta'_1} \neq 0$ by assumption. Therefore, $\hat{R}_{r'_1 \theta'_2} = -\ell_2 B$. By substituting the results in Lemma 3.3, Lemma 3.6 and Lemma 3.8, we have

$$\begin{aligned} \hat{R}_{r'_1 \theta'_2}(z_0) &= -\frac{\hat{R}_{\theta'_1 \theta'_2}(z_0)}{\hat{R}_{\theta'_1 \theta'_1}(z_0)} \frac{(p-1) |\mu| \cos \Theta_1}{k_1 \cos \Theta_2} = -\frac{k_4 (k_2 A(z_0) - B(z_0)) (p-1) |\mu| \cos \Theta_1}{\hat{R}_{\theta'_1 \theta'_1}(z_0) k_1 \cos \Theta_2} \\ &= \frac{k_4}{\hat{R}_{\theta'_1 \theta'_1}(z_0)} \frac{2(p-1)^2 |\mu|^3 \cos \Theta_1}{k_1^3 \cos \Theta_2}. \end{aligned}$$

Recall that $k_4 = -2|v_0| \sin \Theta_3 \cos \Theta_4$. If $\sin \Theta_3 = 0$ then $\phi(z_0) = 0$ implies $\sin \Theta_1 = 0$. This cannot happen because we had chosen μ such that they have no common solution. Thus $\hat{R}_{r_1''\theta_2''} \neq 0$. With this non-vanishing and the entries of the matrix $H''(z_0)$ in Lemma 3.11, we can conclude that the rank of the right-top 3×3 minor of the Hessian of \hat{R} with variables $(r_1'', \theta_1'', r_2'', \theta_2'')$ is 3. In particular, the rank of the matrix M in Lemma 2.10 is 3.

Now we apply Lemma 2.10. If $\phi(z_0) \neq 0$ then z_0 satisfies $\det H(z_0) \neq 0$ by Proposition 3.9. Hence it is a fold by Lemma 2.10. If $\phi(z_0) = 0$ then $\psi(z_0) \neq 0$. This is equivalent to $\frac{\partial}{\partial \theta_2''}(\xi_4(\xi_4(\hat{R})))_{(z_0)} \neq 0$ under the assumption $k_1 \neq 0$ and $\phi(z_0) = 0$. By Lemma 3.11, $\text{rank } H''(z_0) = 2$. Moreover, as mentioned above, the rank of the matrix M in Lemma 2.10 is 3. Hence z_0 is a cusp by Lemma 2.10. This completes the proof. \square

3.3. Non-existence of cusps with $\hat{R}_{\theta_1'\theta_1'}(z_0) = 0$ for generic μ .

LEMMA 3.14. *For a generic choice of μ , any singular point z_0 of P with $k_1 \neq 0$ and $\phi(z_0) = \hat{R}_{\theta_1'\theta_1'}(z_0) = 0$ is a fold.*

PROOF. Since $\hat{R}_{\theta_1'\theta_1'}(z_0) = 0$ and $k_1 \neq 0$, we have $\sin \Theta_1 = 0$. Then $\phi(z_0) = 0$ implies $\sin \Theta_3 = 0$. Hence the last equation in Lemma 3.4 has no solution if we choose $\arg \mu$ generic, that is, any singular point z_0 satisfies $\phi(z_0) \neq 0$ for a fixed generic μ . By Proposition 3.9, $\det H(z_0) \neq 0$. Hence z_0 is a fold by Lemma 2.10. \square

3.4. Non-existence of cusps with $k_1 = 0$ for generic μ .

LEMMA 3.15. *For a generic choice of μ , any singular point z_0 of P with $k_1 = 0$ is a fold.*

PROOF. Since $k_1 = 2|\mu| \cos \Theta_1 \cos \Theta_2$, either $\cos \Theta_1$ or $\cos \Theta_2$ is zero.

First we assume that $\cos \Theta_2 \neq 0$. In this case, $k_2 \neq 0$ because $\cos \Theta_1 = 0$ and $\sin \Theta_1 = \pm 1$. Let $(r_1', \theta_1', r_2', \theta_2')$ be new coordinates in a small neighborhood of z_0 defined by

$$(r_1', \theta_1', r_2', \theta_2') = (r_1, k_1 r_1 + k_2 \theta_1 + k_3 r_2 + k_4 \theta_2, r_2, \theta_2).$$

Then we have $(Q_{r_1'}(z_0), Q_{\theta_1'}(z_0), Q_{r_2'}(z_0), Q_{\theta_2'}(z_0)) = (0, 1, 0, 0)$. Set $\hat{R} = R - sQ$, where $s = R_{\theta_1'}(z_0)$, so that $\hat{R}_{\theta_1'} = R_{\theta_1'} - sQ_{\theta_1'}$ vanishes at z_0 . The constant s is calculated as

$$\begin{aligned} s &= \frac{1}{k_2} Q_{\theta_1}(z_0) = \frac{1}{k_2} \{p(|\mu||u_0|^p \cos(p \arg u_0 + \arg \mu) - |\mu||u_0| \cos(-\arg u_0 + \arg \mu))\} \\ &= \frac{1}{k_2} |u_0| \{\cos(p \arg u_0 + \arg \mu) - \cos(-\arg u_0 + \arg \mu)\} \\ &= -\frac{2}{k_2} |u_0| \sin\left(\frac{p-1}{2} \arg u_0 + \arg \mu\right) \sin\left(\frac{p+1}{2} \arg u_0\right) \\ &= \frac{\sin \Theta_2}{\cos \Theta_2}. \end{aligned}$$

This coincides with the s in the case $k_1 \neq 0$ obtained in Lemma 3.7. Therefore, \hat{R} above is exactly the same as in the case $k_1 \neq 0$.

Since $k_1 = 0$, by straightforward calculation, we have

$$\hat{R}_{r'_1 r'_2} = -\frac{k_3}{k_2} \hat{R}_{r_1 \theta_1} \quad \text{and} \quad \hat{R}_{r'_1 \theta'_2} = -\frac{k_4}{k_2} \hat{R}_{r_1 \theta_1}.$$

We can check that $\hat{R}_{r'_1 \theta'_2}(z_0) = (p-1)|\mu| \frac{\cos \Theta_1}{\cos \Theta_2} = 0$. Therefore both $\hat{R}_{r'_1 r'_2}$ and $\hat{R}_{r'_1 \theta'_2}$ vanish at z_0 and the Hessian of \hat{R} with variables (r'_1, r'_2, θ'_2) at z_0 is of the form

$$H(z_0) = \begin{pmatrix} \hat{R}_{r'_1 r'_1}(z_0) & 0 & 0 \\ 0 & \hat{R}_{r'_2 r'_2}(z_0) & \hat{R}_{r'_2 \theta'_2}(z_0) \\ 0 & \hat{R}_{r'_2 \theta'_2}(z_0) & \hat{R}_{\theta'_2 \theta'_2}(z_0) \end{pmatrix}.$$

By straightforward calculation we have

$$\hat{R}_{r'_1 r'_1}(z_0) = \hat{R}_{r_1 r_1}(z_0) = p(p-1)|\mu||u_0|^{p-2} \frac{\sin \Theta_1}{\cos \Theta_2} \neq 0.$$

We also have

$$\begin{aligned} \hat{R}_{r'_2 r'_2}(z_0) &= \hat{R}_{r_2 r_2} + \frac{k_3^2}{k_2^2} \hat{R}_{\theta_1 \theta_1} = \frac{1}{k_2^2 |v_0|} 4|\mu||u_0| \sin \Theta_1 \cos \Theta_2 \phi'(z_0), \\ \hat{R}_{r'_2 \theta'_2}(z_0) &= \hat{R}_{r_2 \theta_2} + \frac{k_3 k_4}{k_2^2} \hat{R}_{\theta_1 \theta_1} = \frac{(-1)^\kappa}{k_2^2} 4|\mu||u_0| \sin \Theta_1 \cos \Theta_2 \cos \Theta_3 \phi(z_0), \\ \hat{R}_{\theta'_2 \theta'_2}(z_0) &= \hat{R}_{\theta_2 \theta_2} + \frac{k_4^2}{k_2^2} \hat{R}_{\theta_1 \theta_1} = \frac{(-1)^{\kappa+1}}{k_2^2} 4|\mu||u_0||v_0| \sin \Theta_1 \sin \Theta_3 \cos \Theta_2 \phi(z_0), \end{aligned}$$

where

$$\phi'(z_0) = (-1)^\kappa (q-1)|\mu||u_0| \sin \Theta_1 \sin \Theta_3 - (p-1)|v_0| \cos^2 \Theta_3$$

and $\phi(z_0)$ is the equation in Proposition 3.9. Then the determinant of the Hessian H is calculated as

$$\det(H(z_0)) = 16(q-1)|\mu|^3 |u_0|^3 \sin \Theta_1 \cos \Theta_2 \phi(z_0) \hat{R}_{r'_1 r'_1}(z_0).$$

Therefore $\det(H(z_0)) = 0$ if and only if $\phi(z_0) = 0$. By the last equation in Lemma 3.4 and the assumption $\cos \Theta_1 = 0$, we see that if we fix a value of $\arg \mu$ then there are only finitely many possible values for Θ_3 . The values $|u_0|$ and $|v_0|$ are determined by p and q as seen in Lemma 3.2. Therefore if we choose $|\mu|$ generic then $\phi(z_0) = 0$ has no solution. Hence any singular point is a fold for generic μ by Lemma 2.10.

Next we assume that $\cos \Theta_2 = 0$. By Lemma 3.4, we also have $\cos \Theta_4 = 0$. Therefore the gradient (k_1, k_2, k_3, k_4) of the real part Q at the singular point z_0 of P vanishes. In this case, we consider the gradient of the imaginary part R at z_0 ;

$$(\hat{k}_1, \hat{k}_2, \hat{k}_3, \hat{k}_4) = (R_{r_1}(z_0), R_{\theta_1}(z_0), R_{r_2}(z_0), R_{\theta_2}(z_0)).$$

Each element is given as

$$\begin{aligned} \hat{k}_1 &= 2|\mu| \cos \Theta_1 \sin \Theta_2, & \hat{k}_2 &= -2|\mu||u_0| \sin \Theta_1 \sin \Theta_2, \\ \hat{k}_3 &= 2 \cos \Theta_3 \sin \Theta_4, & \hat{k}_4 &= -2|v_0| \sin \Theta_3 \sin \Theta_4. \end{aligned}$$

Since $\cos \Theta_2 = 0$, we set $\Theta_2 = \ell\pi + \frac{\pi}{2}$ with $\ell \in \mathbb{Z}$. Substituting this to the defining equation of Θ_1 , we obtain

$$(3.1) \quad \Theta_1 = \frac{p+1}{p-1} \left(\ell\pi + \frac{\pi}{2} - \arg \mu \right).$$

Therefore, by choosing $\arg \mu$ generic, we may assume that $\cos \Theta_1 \neq 0$.

The strategy of the calculation below is exactly the same as what we did in Section 3.1. First we apply the change of coordinates

$$(r'_1, \theta'_1, r'_2, \theta'_2) = (\hat{k}_1 r_1 + \hat{k}_2 \theta_1 + \hat{k}_3 r_2 + \hat{k}_4 \theta_2, \theta_1, r_2, \theta_2)$$

so that $(R_{r'_1}(z_0), R_{\theta'_1}(z_0), R_{r'_2}(z_0), R_{\theta'_2}(z_0)) = (1, 0, 0, 0)$. Then we have $Q_{r'_1}(z_0) = \frac{1}{\hat{k}_1} Q_{r_1}(z_0) = \frac{\hat{k}_1}{\hat{k}_1} = 0$. This means that we do not need to apply the change of coordinates of the target space \mathbb{R}^2 which we did by setting $\hat{R} = R - sQ$ between Lemma 3.5 and Lemma 3.6. Then the Hessian \hat{H} of Q with variables $(\theta'_1, r'_2, \theta'_2)$ is

$$\hat{H} = \begin{pmatrix} \hat{k}_2^2 \hat{A} - 2\hat{k}_2 \hat{B} + \hat{C} & \hat{k}_3(\hat{k}_2 \hat{A} - \hat{B}) & \hat{k}_4(\hat{k}_2 \hat{A} - \hat{B}) \\ \hat{k}_3(\hat{k}_2 \hat{A} - \hat{B}) & \hat{k}_3^2 \hat{A} + \hat{D} & \hat{k}_3 \hat{k}_4 \hat{A} + \hat{E} \\ \hat{k}_4(\hat{k}_2 \hat{A} - \hat{B}) & \hat{k}_3 \hat{k}_4 \hat{A} + \hat{E} & \hat{k}_4^2 \hat{A} + \hat{F} \end{pmatrix},$$

where

$$\begin{aligned} \hat{A} &= \frac{1}{\hat{k}_1^2} Q_{r_1 r_1}, & \hat{B} &= \frac{1}{\hat{k}_1} Q_{r_1 \theta_1}, & \hat{C} &= Q_{\theta_1 \theta_1}, \\ \hat{D} &= Q_{r_2 r_2}, & \hat{E} &= Q_{r_2 \theta_2}, & \hat{F} &= Q_{\theta_2 \theta_2}, \end{aligned}$$

and its determinant is

$$\det H = (\hat{k}_4^2 \hat{D} - 2\hat{k}_3 \hat{k}_4 \hat{E} + \hat{k}_3^2 \hat{F})(\hat{A} \hat{C} - \hat{B}^2) + (\hat{k}_2^2 \hat{A} - 2\hat{k}_2 \hat{B} + \hat{C})(\hat{D} \hat{F} - \hat{E}^2).$$

By direct calculation, we obtain

$$\det \hat{H}(z_0) = \frac{1}{\hat{k}_1^2} 4(p-1)(q-1)|\mu|^2 \sin \Theta_2 \phi(z_0),$$

where $\phi(z_0)$ is the equation in Proposition 3.9. Since $\cos \Theta_2 = 0$, it is enough to prove that $\phi(z_0) = 0$ has no solution for generic μ . Note that $|u_0|$ and $|v_0|$ are fixed by Lemma 3.2. Set $\Theta_2 = \ell\pi + \frac{\pi}{2}$ with $\ell \in \mathbb{Z}$ as before. By the first equation in Lemma 3.4, $\Theta_4 = (\ell - \kappa)\pi + \frac{\pi}{2}$. Substituting them to the defining equations of Θ_i 's, we obtain equation (3.1) and

$$\Theta_3 = \frac{q+1}{q-1} \left((\ell - \kappa)\pi + \frac{\pi}{2} \right).$$

The number of the possible values of the first term of $\phi(z_0)$ is finite, while the second term can vary according to μ . Therefore $\phi(z_0) = 0$ has no solution for generic $\arg \mu$. Hence any singular point is a fold for generic μ by Lemma 2.10. \square

REMARK 3.16. In summary, for $P(u, v; \mu)$ to be an excellent map, μ is chosen in the above proofs such that

- $S(|\mu|, \arg \mu) \neq 0$ (required in the proof of Proposition 3.13),

- $\arg \mu$ is not in a finite set in S^1 (required in the case $k_1 \neq 0$ and $\hat{R}_{\theta'_1 \theta'_1}(z_0) = 0$), and
- $|\mu|$ is not in a finite set in \mathbb{R} (required in the case $k_1 = 0$).

These conditions can be written by using polar coordinates of a and b , according to the change of variables explained in the beginning of Section 3.

4. Proofs of Theorem 1.2 and the second assertion in Theorem 1.1.

4.1. Case $ab \neq 0$. First we study the case $ab \neq 0$. As in the previous section, we set $P(u, v; \mu) = \mu(u^p + \bar{u}) + v^q + \bar{v}$ with $p, q \geq 2$ and $\mu \in \mathbb{C} \setminus \{0\}$. Then, by Lemma 3.2, the set of singular points of f consists of r parallel curves $C_k, k = 0, \dots, r - 1$, on the torus $\{(u, v) \in \mathbb{C}^2 \mid |u| = A, |v| = B\}$, each of which is parametrized, with parameter $e^{i\theta} \in S^1$, as

$$(4.1) \quad (u, v) = \left(Ae^{\left(\frac{q-1}{r}\theta + c_k\right)i}, Be^{\frac{p-1}{r}\theta i} \right),$$

where $r = \gcd(p - 1, q - 1), A = 1/p^{1/(p-1)}, B = 1/q^{1/(q-1)}$ and $c_k = \frac{1}{p-1}(-2 \arg \mu + 2\pi k)$.

Assume that $P(u, v; \mu)$ is an excellent map and set the map $P_k : C_k \rightarrow \mathbb{C}$ as

$$\begin{aligned} P_k(\theta) &= P \left(Ae^{\left(\frac{q-1}{r}\theta + c_k\right)i}, Be^{\frac{p-1}{r}\theta i} \right) \\ &= \mu \left(A^p e^{\left(\frac{p(q-1)}{r}\theta + pc_k\right)i} + Ae^{-\left(\frac{q-1}{r}\theta + c_k\right)i} \right) + B^q e^{\frac{q(p-1)}{r}\theta i} + Be^{-\frac{p-1}{r}\theta i}. \end{aligned}$$

Since P is an excellent map, the set of cusps of P on C_k corresponds to the zero points of $dP_k/d\theta = 0$. The left hand side is calculated as

$$\frac{dP_k}{d\theta} = -2e^{\frac{(p-1)(q-1)}{2r}\theta i} \Phi(\theta)$$

with

$$\Phi(\theta) = (-1)^k |\mu| \frac{q-1}{r} A \sin \left(\frac{(p+1)(q-1)}{2r} \theta + \frac{p+1}{2} c_k \right) + \frac{p-1}{r} B \sin \left(\frac{(p-1)(q+1)}{2r} \theta \right),$$

where we used the formula

$$(4.2) \quad e^{ix} - e^{iy} = 2ie^{\frac{x+y}{2}i} \sin \frac{x-y}{2}$$

and relations $pA^p = qB^q = 1$.

Now we set

$$T(\theta; |\mu|) = (-1)^k |\mu| \sin(n\theta + c'_k) + C \sin(m\theta),$$

where $m = \frac{(p-1)(q+1)}{2r}, n = \frac{(p+1)(q-1)}{2r}, c'_k = \frac{p+1}{2} c_k$ and $C = \frac{(p-1)B}{r} \frac{r}{(q-1)A} > 0$, and observe how the number of roots of $T(\theta; |\mu|) = 0$ changes according to the change of $|\mu|$. Note that $m > n$ if and only if $p > q$. Since $T(\theta; |\mu|)$ is a smooth function for variables $(\theta, |\mu|)$, the number of roots changes only at the points satisfying

$$\begin{aligned} T &= (-1)^k |\mu| \sin(n\theta + c'_k) + C \sin(m\theta) = 0 \quad \text{and} \\ \frac{dT}{d\theta} &= (-1)^k |\mu| n \cos(n\theta + c'_k) + Cm \cos(m\theta) = 0. \end{aligned}$$

LEMMA 4.1. *If $p > q$ then the number of solutions of $T(\theta; |\mu|) = 0$ is monotone decreasing according to the parameter $|\mu| \in (0, \infty)$.*

PROOF. Suppose that θ_0 and $|\mu_0|$ satisfy $T(\theta_0; |\mu_0|) = \frac{dT}{d\theta}(\theta_0; |\mu_0|) = 0$. Set $U(\theta, |\mu|, t) = T(\theta; |\mu|) - t$ and consider the hypersurface given by $U = 0$. Note that $(\theta_0, |\mu_0|, 0)$ is a point on this surface. Then the gradient of U at $(\theta_0, |\mu_0|, 0)$ is

$$\text{grad}U(\theta_0, |\mu_0|, 0) = (0, (-1)^k \sin(n\theta_0 + c'_k), -1).$$

First we study the case $(-1)^k \sin(n\theta_0 + c'_k) \neq 0$. Assume that $(-1)^k \sin(n\theta_0 + c'_k) > 0$. Since $(-1)^k |\mu| \sin(n\theta_0 + c'_k) + C \sin(m\theta_0) = 0$, we have

$$\begin{aligned} \frac{d^2T}{d\theta^2}(\theta_0; |\mu_0|) &= -(-1)^k |\mu| n^2 \sin(n\theta_0 + c'_k) - C m^2 \sin(m\theta_0) \\ &= (-1)^k |\mu| (m^2 - n^2) \sin(n\theta_0 + c'_k) > 0. \end{aligned}$$

This means that the graph $\{U = 0\}$ on the (θ, t) -plane $\{|\mu| = 0\}$ is downward convex. Observing the position of the hypersurface $U = 0$ near $(\theta_0, |\mu_0|, 0)$ using the gradient, we can conclude that the number of the solutions decreases according to the parameter $|\mu|$. The assertion in the case $(-1)^k \sin(n\theta_0 + c'_k) < 0$ follows by the same argument.

Next we study the case $\sin(n\theta_0 + c'_k) = 0$. Since $\frac{dT}{d\theta}(\theta_0; |\mu_0|) = 0$, we have

$$\begin{aligned} \frac{d^3T}{d\theta^3}(\theta_0; |\mu_0|) &= -(-1)^k |\mu| n^3 \cos(n\theta_0 + c'_k) - C m^3 \cos(m\theta_0) \\ &= (-1)^k |\mu| (m^2 - n^2) n \cos(n\theta_0 + c'_k) \neq 0. \end{aligned}$$

This means that $T(\theta; |\mu_0|) = 0$ has a root at $\theta = \theta_0$ with multiplicity 3. Set $U'(\theta, |\mu|, t) = \frac{\partial T}{\partial \theta}(\theta, |\mu|) - t$ and consider the hypersurface $U' = 0$ near the point $(\theta_0, |\mu_0|, 0)$. Then the gradient of U' at $(\theta_0, |\mu_0|, 0)$ is

$$\text{grad}U'(\theta_0, |\mu_0|, 0) = (0, (-1)^k n \cos(n\theta_0 + c'_k), -1).$$

If $(-1)^k n \cos(n\theta_0 + c'_k) > 0$ then $\frac{d^3T}{d\theta^3}(\theta_0; |\mu_0|) > 0$. By the same argument as before, the number of the solutions decreases near $(\theta_0, |\mu_0|, T(\theta_0; |\mu_0|))$ according to the parameter $|\mu|$. It also decreases even if $(-1)^k n \cos(n\theta_0 + c'_k) < 0$ by the same argument. These mean that the number of solutions of $T(\theta; |\mu|) = 0$ decreases near $(\theta_0, |\mu_0|, T(\theta_0; |\mu_0|))$ according to $|\mu|$. \square

LEMMA 4.2. *Suppose $p = q$. Then the number of the solutions of $T(\theta; |\mu|) = 0$ is $\frac{p^2-1}{r}$ except the finite number of points on $|\mu| = 1$ satisfying $\sin c'_k = 0$.*

PROOF. The condition $p = q$ implies that $m = n$ and $C = 1$. Therefore the number of solutions changes at the point $|\mu|$ satisfying the following two equations:

$$\begin{aligned} T(\theta; |\mu|) &= (-1)^k |\mu| \sin(n\theta + c'_k) + \sin(n\theta) = 0 \\ \frac{dT}{d\theta}(\theta; |\mu|) &= (-1)^k |\mu| n \cos(n\theta + c'_k) + n \cos(n\theta) = 0. \end{aligned}$$

From these equations, we have

$$\begin{aligned} 0 &= (-1)^k |\mu| n \sin(n\theta + c'_k) \cos(n\theta) - (-1)^k |\mu| n \cos(n\theta + c'_k) \sin(n\theta) \\ &= (-1)^k |\mu| n \sin c'_k. \end{aligned}$$

By substituting this to $T(\theta; |\mu|) = 0$ and $\frac{dT}{d\theta}(\theta; |\mu|) = 0$, we obtain $|\mu| = 1$. This means that the number of solutions does not change except at the points with $|\mu| = 1$ and $\sin c'_k = 0$. Since the complement of these points in the μ -plane is connected, the number of solutions is constant outside these points. If $|\mu|$ is sufficiently small then the number of solutions is equal to the solution of $\sin(n\theta) = 0$. Since $n = \frac{(p+1)(q-1)}{2r}$, the number of solutions is $\frac{p^2-1}{r}$. \square

PROOF OF THEOREM 1.2 IN CASE $ab \neq 0$. Suppose that $ab \neq 0$. We use the notation in Section 3.

We first prove the assertion in the case $p > q$. If $|\mu|$ is sufficiently small then the number of cusps of the excellent map P on C_k is equal to the solution of $\sin(m\theta) = 0$, which is $\frac{(p-1)(q+1)}{r}$. If $|\mu|$ is sufficiently large then it is equal to the solution of $\sin(n\theta + c'_k) = 0$, which is $\frac{(p+1)(q-1)}{r}$. Then, by Lemma 4.1, the number of cusps on C_k is between $\frac{(p-1)(q+1)}{r}$ and $\frac{(p+1)(q-1)}{r}$. Since the singular set of P has r components, the assertion follows.

Next we consider the case $p = q$. In this case, we have $m = n$ and $C = 1$. Lemma 4.2 proves the assertion unless $|\mu| = 1$ and $\sin c'_k = 0$. Suppose that $|\mu| = 1$ and $\sin c'_k = 0$. By the second condition we set $c'_k = \ell\pi$ with $\ell \in \mathbb{Z}$. If $k + \ell$ is odd, then $T(\theta; 1) = 0$ for any $e^{i\theta} \in S^1$. Therefore P is not an excellent map. If $k + \ell$ is even, then $T(\theta; 1) = 2 \sin(n\theta) = 2 \sin\left(\frac{p^2-1}{r}\theta\right)$. Hence if P is an excellent map then the number of cusps of P on C_k is $\frac{p^2-1}{r}$ for any $k = 0, 1, \dots, r-1$. Since the set of singular points of P has r components, the total number of cusps of P is $p^2 - 1$. This completes the proof. \square

4.2. Case $ab = 0$.

PROOF OF THEOREM 1.2 IN CASE $ab = 0$. Suppose that either a or b is zero. Without loss of generality, we may assume that $a \neq 0$ and $b = 0$. In this case, by Lemma 3.2, the singular set of f is parametrized as $(u, v) = (Ae^{i\theta}, 0)$. By changing the coefficients of f as before, we obtain the form $P = \mu(u^p + \bar{u}) + v^q$. The real and imaginary part f and g of P is given as

$$\begin{aligned} f &= |\mu||u|^p \cos(p \arg u + \arg \mu) + |\mu||u| \cos(-\arg u + \arg \mu) + \operatorname{Re}(v^q), \\ g &= |\mu||u|^p \sin(p \arg u + \arg \mu) + |\mu||u| \sin(-\arg u + \arg \mu) + \operatorname{Im}(v^q). \end{aligned}$$

Following the calculation in Section 3 in this setting, we obtain

$$\begin{aligned} k_1 &= 2|\mu| \cos \Theta_1 \cos \Theta_2, & k_2 &= -2|\mu||u_0| \sin \Theta_1 \cos \Theta_2, \\ k_3 &= q \operatorname{Re}(v^{q-1}), & k_4 &= q \operatorname{Im}(v^{q-1}). \end{aligned}$$

The strategy is the same as in Section 3.1, though we use the coordinates $(r_1, \theta_1, x_2, y_2)$ instead of $(r_1, \theta_1, r_2, \theta_2)$, where $v = x_2 + iy_2$. We then apply the change of coordinates as in Section 3.1 into $(r'_1, \theta'_1, x'_2, y'_2)$ and calculate the Hessian of \hat{R} at z_0 with variables (θ'_1, x'_2, y'_2) as in Lemma 3.6.

Suppose that $q \geq 3$. Since $v_0 = 0$ at the singular point $z_0 = (u_0, v_0)$, we have $D(z_0) = E(z_0) = F(z_0) = 0$. Therefore we have $\det H(z_0) = 0$. If P is an excellent map, then the

right 4×3 minor of the Hessian of \hat{R} with variables $(r'_1, \theta'_1, x'_2, y'_2)$ has rank 3. Since any singularity z_0 of P satisfies $\det H(z_0) = 0$, any singularity is a cusp, which is a contradiction. Hence P is also not an excellent map.

Suppose that $q = 2$. Since $\gcd(p-1, q-1) = 1$, the singular set of Q is connected and parametrized as $(Ae^{i\theta}, 0)$ with $e^{i\theta} \in S^1$. Applying formula (4.2) and $pA^{p-1} = p|u_0|^{p-1} = 1$ in the first equation in Lemma 3.2, we have

$$\begin{aligned} \frac{d}{d\theta} P(Ae^{i\theta}, 0) &= \mu \frac{d}{d\theta} (A^p e^{p\theta i} + Ae^{-\theta i}) = i\mu (pA^p e^{p\theta i} - Ae^{-\theta i}) \\ &= -2\mu A e^{\frac{p-1}{2}\theta i} \sin \frac{p+1}{2}\theta. \end{aligned}$$

Therefore if P is an excellent map then it has $p+1$ cusps. Thus the assertion follows. \square

4.3. Proof of the second assertion in Theorem 1.1. We now prove the second assertion in Theorem 1.1, which states that any fold singularity appearing in the proof of the first assertion is indefinite. Since any excellent map in Theorem 1.1 has at least one cusp by Theorem 1.2, it is enough to show that the fold singularities in a neighborhood of any cusp are indefinite. We only need to check the case where $k_1 \neq 0$ and $\hat{R}_{\theta'_1 \theta'_1}(z_0) \neq 0$ discussed in Section 3.2 because a cusp appears in the proof only in this case.

For a singular point z_0 of P , let $\tau(z_0)$ be the absolute index of z_0 , which is defined as follows: Fix an orientation along the set of singularities. Then the Hessian of P at z_0 defines a symmetric, bilinear form on $L_{z_0} \otimes L_{z_0}$. Let i be the index of this form. Then the absolute index $\tau(z_0)$ is defined by the maximal number of i and the rank of this Hessian minus i , see [16, p.273].

Let z_0 be a cusp of P . The Hessian at z_0 had been calculated in Lemma 3.11 and we see that the rank is 2 and the index is 1. Therefore the absolute index of z_0 is 1. Now we apply Lemma (2) in [16, p.274]. Since the dimension n of the source manifold is 4, the equality $\tau(z_0) = (n-2)/2$ is satisfied. Therefore the absolute index of a fold in a small neighborhood of z_0 is 2. Since the rank of the Hessian at a fold is 3, we see that the index at the fold is neither 0 nor 3, i.e., the fold is indefinite. This completes the proof. \square

5. Proof of Theorem 1.3.

PROOF OF THEOREM 1.3. As in the previous sections, we set $P(u, v; \mu) = \mu(u^2 + \bar{u}) + v^2 + \bar{v}$. We follow the observation in the beginning of Sections 4.1 and 4.2 with the condition $p = q = 2$. Then we can see that $S(P)$ consists of one circle C_0 . This circle is parametrized as in equation (4.1) with $p = q = 2$, $r = 1$ and $k = 0$. Then the map $P_0 = P|_{C_0}$ is given as

$$\begin{aligned} P_0(\theta) &= \mu \left(\frac{1}{4} e^{(2\theta - 4 \arg \mu)i} + \frac{1}{2} e^{-(\theta - 2 \arg \mu)i} \right) + \frac{1}{4} e^{2\theta i} + \frac{1}{2} e^{-\theta i} \\ &= \frac{|\mu|}{4} e^{(2\theta - 3 \arg \mu)i} + \frac{|\mu|}{2} e^{-(\theta - 3 \arg \mu)i} + \frac{1}{4} e^{2\theta i} + \frac{1}{2} e^{-\theta i} \\ &= \frac{1}{4} \left(|\mu| e^{-3i \arg \mu} + 1 \right) e^{2i\theta} + \frac{1}{2} \left(|\mu| e^{3i \arg \mu} + 1 \right) e^{-i\theta}. \end{aligned}$$

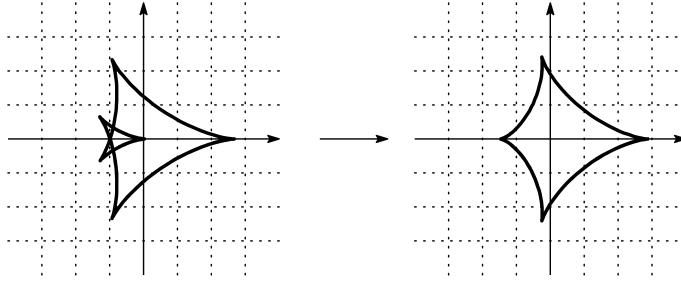


FIGURE 2. Linear deformations of $u^3 + v^2$ into excellent maps.

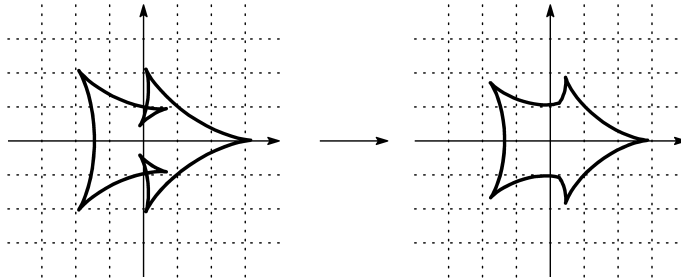


FIGURE 3. Linear deformation of $u^4 + v^2$ into excellent maps.

Set $K = r_\mu e^{-3i \arg \mu} + 1$, then

$$\begin{aligned} P(\theta) &= \frac{1}{4} \left(K e^{2i\theta} + 2\bar{K} e^{-i\theta} \right) = \frac{|K|}{4} \left(e^{i(2\theta + \arg K)} + 2e^{-i(\theta + \arg K)} \right) \\ &= \frac{|K|}{4} e^{-\frac{i}{3} \arg K} h \left(\theta + \frac{2}{3} \arg K \right), \end{aligned}$$

where $h(\theta) = e^{2i\theta} + 2e^{-i\theta}$ is the function in the assertion. Obviously, $h : S^1 \rightarrow \mathbb{R}^2$ is injective. Hence P_0 is also. Solving $\frac{dh}{d\theta} = 0$, we can confirm that the cusps appear at $\theta = 0, \frac{2}{3}\pi$ and $\frac{4}{3}\pi$. Hence the number of cusps is 3. \square

6. Examples. In this section, we give two examples of linear deformations.

EXAMPLE 6.1. We consider the map $P(u, v; \mu) = \mu(u^3 + \bar{u}) + v^2 + \bar{v}$, which is a deformation of $u^3 + v^2$. If μ is generic then $P(u, v; \mu)$ is an excellent map as shown in Theorem 1.1. Figure 2 is the set of critical values of $P(u, v; \mu)$ with $\arg \mu = 0$. The left figure is in the case $|\mu| < \frac{3\sqrt{3}}{2}$ and the right one is when $|\mu| > \frac{3\sqrt{3}}{2}$. By Theorem 1.2, if it is an excellent map then the number of cusps is between 4 and 6. The singularity appearing at $|\mu| = \frac{3\sqrt{3}}{2}$ looks like a “beak to beak”.

EXAMPLE 6.2. We consider the map $P(u, v; \mu) = \mu(u^4 + \bar{u}) + v^2 + \bar{v}$, which is a deformation of $u^4 + v^2$. Figure 3 is the set of critical values of $P(u, v; \mu)$ with $\arg \mu = 0$.

The left figure is in the case $|\mu| < c$ and the right one is when $|\mu| > c$, where $c = 2.615\dots$ By Theorem 1.2, if it is an excellent map then the number of cusps is between 5 and 9. The singularities appearing at $|\mu| = c$ look like “swallowtails”.

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