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CONVERGENCE OF MEASURES PENALIZED BY GENERALIZED FEYNMAN-KAC TRANSFORMS

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Abstract. We prove the existence of limiting laws for symmetric stable-like processes penalized by generalized Feynman-Kac functionals and characterize them by the gauge functions and the ground states of Schr̈odinger type operators.

1. Introduction. In this paper we study limit theorems for symmetric stable-like processes on $\mathbb{R}^d$ penalized by normalized Feynman-Kac functionals as the weight processes.

Penalizing measures by appropriate weight processes can be understood as a change-of-measure phenomenon and such modifications have been studied by many authors ([16, 18, 21, 25]). In penalizations, the weights play a role analogous to a Girsanov transform in which its martingale property allows to define a new probability measure, do not allow immediately to create a new weighted probability measure that may emerge in the limit of weight processes. When such a limit exists, it is called the penalized probability measure associated with the weights.

In [18], Roynette, Vallois and Yor have studied limit theorems for Wiener processes penalized by various weight processes. In [25], the authors studied limit theorems for the one-dimensional symmetric stable process penalized by Feynman-Kac transforms with negative (killing) additive functionals, and they called their limit theorems the Feynman-Kac penalizations. It turns out that their methods are not available in multi-dimensional cases. In [21], Takeda extended their results to Feynman-Kac transforms with positive (creation) continuous additive functionals corresponding to positive smooth measures for multi-dimensional symmetric stable processes on $\mathbb{R}^d$ by classifying associated Schrödinger operators of Feynman-Kac semigroups into the subcritical, critical and supercritical cases, and by characterizing the penalized measures in each cases. Recently, this result was extended to non-local Feynman-Kac transforms by Matsuura [16].

The purpose of this paper is to extend the previous results for penalization problems to the so-called *generalized Feynman-Kac transforms*. More precisely, let $\mathbf{X} = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \mathbf{P}_x)$ be the symmetric α-stable-like process on $\mathbb{R}^d$ with $0 < \alpha < 2$ and $(\mathcal{E}, \mathcal{F})$ the associated

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Dirichlet form of **X** (see §2 for details). For a function $u \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d)$ (see §2 for the definition), let N^u be the continuous additive functional (CAF in abbreviation) of zero quadratic variation appeared in the Fukushima decomposition of $u(X_t) - u(X_0)$ (see (2.4)). Note that N^u is not necessarily of bounded variation in general. Let μ be a positive smooth measure on $\mathbb{R}^d$ and denotes the corresponding positive continuous additive functional (PCAF in abbreviation) of **X** by A^μ to emphsize the Revuz correspondence between μ and A^μ ([10]). Let F be a bounded positive symmetric Borel function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal. Then $A_t^F := \sum_{0<s\le t} F(X_{s-}, X_s)$ is an additive functional of **X**. It is natural to consider the following non-local Feynman-Kac transform by the additive functional $A := N^u + A^\mu + A^F$ of the form

$$e_A(t) := \exp(A_t)\,, \quad t \ge 0\,, \tag{1.1}$$

because the process **X** admits many CAFs which do not have bounded variations, and many discontinuous additive functionals. We call (1.1) the generalized Feynman-Kac transform (weight) in the sense that it involves N^u. Thus, the main contribution is to add the perturbation by AF which is not of bounded variation in general.

Let $\{\mathbf{Q}_{x,t}^A\}_{t\ge 0}$ be the family of normalized probability measures defined as follows: for $B \in \mathcal{F}_s$, $s \ge 0$ and $x \in \mathbb{R}^d$

$$\mathbf{Q}_{x,t}^A(B) := \frac{1}{\mathbf{E}_x[e_A(t)]}\int_B e_A(t)(\omega)\mathbf{P}_x(\mathrm{d}\omega) = \frac{\mathbf{E}_x[e_A(t)\mathbf{1}_B]}{\mathbf{E}_x[e_A(t)]}\,. \tag{1.2}$$

In this paper we are going to study the limiting behaviour of (1.2) when $t \to \infty$. To do this, we will classify the associated formal Schrödinger operator of the generalized Feynman-Kac semigroups induced by (1.1) into the subcritical, critical and supercritical cases. We characterize the penalized measure of (1.2) as $t \to \infty$ in each cases by using the gauge functions and the ground states of the associated Schrödinger operator in the present settings, which are motivated by [21]. However there are difficulties in accomplishing our work, compare to the case of local (or non-local) Feynman-Kac penalizations, because our Feynman-Kac transform is not necessarily of finite variation. Thus we cannot apply directly the methods exposed in [16, 21] to our cases. Nevertheless, we can obtain our results by noting the recent developments of the generalized Feynman-Kac transforms and their related topics studied in [8, 12, 13, 14].

The remainder of the paper is organized as follows. In Section 2, we recall some basic properties on **X** and review the result in [22]. In Section 3, we explain the Girsanov transforms induced by the Doléan-Dade exponential martingale relative to u and F, and give the definitions of Green-tight Kato class measures, including the construction of the ground states of Schödinger operators in our settings. Section 4 is devoted to prove our main results (Theorem 4.1, Theorem 4.2 and Theorem 4.3). In this section, following the ideas from [21], we describe all penalized measures for our Feynman-Kac penalizations. To this end, an analytic characterization of gaugeability and subcriticality for (1.1) and a modified strong Chacon-Ornstein limit-quotient theorem for special additive functionals under time-dependent initial measures play a crucial role (Proposition 4.1 and Proposition 4.2).

Throughout this paper, we use c, C, c_i, C_i $(i = 1, 2, \dots)$ as positive constants which may be different at different occurrences. We denote by $\mathcal{B}(\mathbb{R}^d)$ (resp. $\mathcal{B}_+(\mathbb{R}^d)$, $\mathcal{B}_b(\mathbb{R}^d)$ and $C_b(\mathbb{R}^d)$) the set of measurable (resp. positive measurable, bounded measurable and bounded continuous) functions on $\mathbb{R}^d$.

2. Preliminaries. Let $c(x, y)$ be a symmetric measurable function on $\mathbb{R}^d \times \mathbb{R}^d$ which is bounded between two fixed constants $c_2 > c_1 > 0$, that is, $c_1 \le c(x, y) \le c_2$ for $x, y \in \mathbb{R}^d$. For $0 < \alpha < 2$, define

$$\mathcal{F} = \left\{ f \in L^2(\mathbb{R}^d) \,\middle|\, \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(f(x) - f(y))^2}{|x - y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y < \infty \right\}$$

$$\mathcal{E}(f, g) = \frac{1}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(f(x) - f(y))(g(x) - g(y))c(x, y)}{|x - y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y, \quad f, g \in \mathcal{F}.$$

It is well known that $(\mathcal{E}, \mathcal{F})$ is a regular symmetric Dirichlet form on $L^2(\mathbb{R}^d)$ and hence there is an associated symmetric Hunt process $\mathbf{X} = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \mathbf{P}_x, \zeta)$ on $\mathbb{R}^d$ starting from every point in $\mathbb{R}^d$ except for an exceptional set of zero capacity. The process $\mathbf{X}$ is called a symmetric α-stable-like process. Note that $\mathbf{X}$ is nothing but the rotationally symmetric α-stable process on $\mathbb{R}^d$ when $c(x, y) = \alpha 2^{\alpha-1}\pi^{-d/2}\Gamma(\frac{d+\alpha}{2})/\Gamma(1 - \frac{\alpha}{2})$. It is shown in [5] that $\mathbf{X}$ is irreducible and conservative, and admits a locally Hölder continuous transition density function $p_t(x, y)$ on $]0, \infty[\times\mathbb{R}^d \times \mathbb{R}^d$. The latter in particular implies that $\mathbf{X}$ can be modified to start from every point in $\mathbb{R}^d$ as a Feller process. Moreover, there are constants $C_2 > C_1 > 0$ such that

$$c\left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right) \le p_t(x, y) \le C\left(t^{-d/\alpha} \wedge \frac{t}{|x - y|^{d+\alpha}}\right) \tag{2.1}$$

for all $(t, x, y) \in]0, 1] \times \mathbb{R}^d \times \mathbb{R}^d$. By using the scaling property (see the proof of Proposition 4.1 in [5]), one can show that (2.1) is valid for all $(t, x, y) \in]0, \infty[\times\mathbb{R}^d \times \mathbb{R}^d$.

For $\beta > 0$, we define the β-order resolvent kernel

$$R_\beta(x, y) = \int_0^\infty e^{-\beta t} p_t(x, y)\mathrm{d}t, \quad x, y \in \mathbb{R}^d.$$

When $d > \alpha$ the process $\mathbf{X}$ is transient, hence we can define 0-order resolvent kernel $R(x, y) := R_0(x, y) < \infty$ for $x, y \in \mathbb{R}^d$ with $x \ne y$. $R(x, y)$ is called the Green function of $\mathbf{X}$. By virtue of (2.1), we can immediately check that there exist $C_4 > C_3 > 0$ such that

$$\frac{C_3}{|x - y|^{d-\alpha}} \le R(x, y) \le \frac{C_4}{|x - y|^{d-\alpha}}, \quad x, y \in \mathbb{R}^d. \tag{2.2}$$

For a non-negative Borel measure ν, we write $R_\beta\nu(x) := \int_{\mathbb{R}^d} R_\beta(x, y)\nu(\mathrm{d}y)$, $R\nu(x) := R_0\nu(x)$ and $R_\beta f(x) = R_\beta\nu(x)$ when $\nu = f\mathrm{d}x$ for any $f \in \mathcal{B}_+(\mathbb{R}^d)$ or $f \in \mathcal{B}_b(\mathbb{R}^d)$. Note that the process $\mathbf{X}$ has the resolvent strong Feller property ((**RSF**) in abbreviation), that is, $R_\beta(\mathcal{B}_b(\mathbb{R}^d)) \subset C_b(\mathbb{R}^d)$ for any $\beta > 0$.

A positive Radon measure ν is said to be *of Dynkin class* (resp. *of Kato class*) with respect to $\mathbf{X}$ if $\sup_{x\in\mathbb{R}^d} R_\beta\nu(x) < \infty$ for some $\beta > 0$ (resp. $\lim_{\beta\to\infty} \sup_{x\in\mathbb{R}^d} R_\beta\nu(x) = 0$). Denote by $S^1_D(\mathbf{X})$ (resp. $S^1_K(\mathbf{X})$) the family of measures of Dynkin class (resp. of Kato class). Clearly, $S^1_K(\mathbf{X}) \subset S^1_D(\mathbf{X})$. Note that any positive Radon measure of Dynkin class always belongs to the family of smooth measures in the strict sense in view of Proposition 3.1 in [15]. For a CAF A and $f \in \mathcal{B}_+(\mathbb{R}^d)$, the process $f \cdot A$ is defined by

$$(f \cdot A)_t := \int_0^t f(X_s) \mathrm{d}A_s \tag{2.3}$$

which is also a CAF. We say that a PCAF A^ν of $\mathbf{X}$ and a positive Radon measure ν are in the Revuz correspondence if they satisfy for any $t > 0$, $f \in \mathcal{B}_+(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} f(x)\nu(\mathrm{d}x) =\uparrow \lim_{t\downarrow 0} \frac{1}{t} \int_{\mathbb{R}^d} \mathbf{E}_x \left[\left(f \cdot A^\nu \right)_t \right] \mathrm{d}x \,.$$

It is known that the family of equivalence classes of the set of PCAFs in the strict sense and the family of positive smooth measures in the strict sense are in one to one correspondence under the Revuz correspondence ([10, Theorem 5.1.4]).

Let $(\mathcal{E}, \mathcal{F}_e)$ be *the extended Dirichlet space* of $(\mathcal{E}, \mathcal{F})$ (see [10] for the definition). Any element $f \in \mathcal{F}_e$ admits $\mathcal{E}$-quasi continuous version $\tilde{f}$. Throughout this paper, we always take $\mathcal{E}$-quasi continuous version of the element of $\mathcal{F}_e$, that is, we omit *tilde* from $\tilde{f}$ for $f \in \mathcal{F}_e$. It is known that the process $\mathbf{X}$ has a Lévy system (N, H) given by $H_t = t$ and $N(x, \mathrm{d}y) = 2c(x, y)|x - y|^{-(d+\alpha)}\mathrm{d}y$, that is, for any non-negative Borel function ϕ on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal and any $x \in \mathbb{R}^d$,

$$\mathbf{E}_x \left[\sum_{s \le t} \phi(X_{s-}, X_s) \right] = \mathbf{E}_x \left[\int_0^t \int_{\mathbb{R}^d} \frac{2c(X_s, y)\phi(X_s, y)}{|X_s - y|^{d+\alpha}} \mathrm{d}y \mathrm{d}s \right].$$

To simplify notation, we will write

$$\mu_\phi(\mathrm{d}x) := \left\{ \int_{\mathbb{R}^d} \frac{2c(x, y)\phi(x, y)}{|x - y|^{d+\alpha}} \mathrm{d}y \right\} \mathrm{d}x.$$

Take a function $u \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d)$, where $C_\infty(\mathbb{R}^d)$ denotes the set of continuous functions vanishing at infinity. Then the additive functional $u(X_t) - u(X_0)$ admits the following decomposition: for all $t \in [0, \infty[$

$$u(X_t) - u(X_0) = M^u_t + N^u_t \quad \mathbf{P}_x\text{-a.s.} \quad \text{for q.e. } x \in \mathbb{R}^d, \tag{2.4}$$

where M^u is a square integrable martingale additive functional and N^u is a continuous additive functional (CAF in abbreviation) locally of zero energy. Note that N^u_t is not a process of finite variation in general. The martingale part M^u is given by

$$\begin{aligned} M^u_t = \lim_{n\to\infty} \Bigg\{ & \sum_{s \le t} (u(X_s) - u(X_{s-}))\mathbf{1}_{\{|u(X_s) - u(X_{s-})| > 1/n\}} \\ & - \int_0^t \int_{\{y \in \mathbb{R}^d : |u(y) - u(X_s)| > 1/n\}} \frac{2c(X_s, y)(u(y) - u(X_s))}{|X_s - y|^{d+\alpha}} \mathrm{d}y \mathrm{d}s \Bigg\}. \end{aligned}$$

Let $\mu_{\langle u\rangle}$ be the Revuz measure associated with the quadratic variational processe (or the sharp bracket PCAF) $\langle M^u\rangle$ of M^u. Then

$$\mu_{\langle u\rangle}(\mathrm{d}x) = \int_{\mathbb{R}^d} \frac{2c(x,y)(u(x)-u(y))^2}{|x-y|^{d+\alpha}}\mathrm{d}x\mathrm{d}y\,.$$

On account of [14, Theorem 6.2(2)], we see that under the condition $\mu_{\langle u\rangle} \in S^1_D(\mathbf{X})$ that (2.4) holds for all $x \in \mathbb{R}^d$ as the strict decompositions. Note that $\mathcal{E}(f,f) = \frac{1}{2}\mu_{\langle f\rangle}(\mathbb{R}^d)$ provided $f \in \mathcal{F}_e$.

Let $(\mathcal{A}, \mathcal{D}(\mathcal{A}))$ be a lower bounded symmetric closed form on $L^2(\mathbb{R}^d)$. For positive Radon measures ν_2, ν_1 on $\mathbb{R}^d$, we set

$$\lambda(\nu_2,\nu_1) := \inf\left\{\mathcal{A}^{\nu_2}(f,f) \;\middle|\; f \in \mathcal{D}(\mathcal{A}^{\nu_2}),\ \int_{\mathbb{R}^d} f^2\mathrm{d}\nu_1 = 1\right\},$$

where $\mathcal{A}^{\nu}(f,f) := \mathcal{A}(f,f) + \int_{\mathbb{R}^d} f^2\mathrm{d}\nu$. Then we have the following lemma due to Takeda [22].

LEMMA 2.1 (cf. Lemma 3.1 and Lemma 3.2 in [22]). *Let ν be another positive Radon measure on $\mathbb{R}^d$.*

(1) $\lambda(\nu_2+\nu,\nu_1+\nu) > 1$ *implies* $\lambda(\nu_2,\nu_1) > 1$. *The converse assertion holds if* $\lambda(\nu_2,\nu) > 0$.

(2) $\lambda(\nu_2+\nu,\nu_1+\nu) < 1$ *if and only if* $\lambda(\nu_2,\nu_1) < 1$.

As a consequence, we see that $\lambda(\nu_2+\nu,\nu_1+\nu) = 1$ *if and only if* $\lambda(\nu_2,\nu_1) = 1$ *provided* $\lambda(\nu_2,\nu) > 0$.

Note that Lemma 3.1 and Lemma 3.2 in [22] are stated in the framework of rotationally symmetric α-stable processes. However, their proofs remain valid for general symmetric Markov processes.

3. Construction of ground states. Let F be a bounded positive symmetric function on $\mathbb{R}^d \times \mathbb{R}^d$ vanishing on the diagonal. We say that F is in the *class* $J^1_D(\mathbf{X})$ if μ_F belongs to $S^1_D(\mathbf{X})$. For a bounded $u \in \mathcal{F}_e$ with $\mu_{\langle u\rangle} \in S^1_D(\mathbf{X})$ and $F \in J^1_D(\mathbf{X})$, we set

$$F^u(x,y) := F(x,y) + \{-u(y) - (-u(x))\} = F(x,y) + u(x) - u(y)$$

and $G^u = e^{F^u} - 1$ with identifying $F^0 = F$ and $G^0 = G := e^F - 1$. Since $(F^u)^2 \in J^1_D(\mathbf{X})$, one can consider a purely discontinuous locally square integrable local martingale additive functional M^{F^u} defined by

$$M^{F^u}_t = A^F_t + M^{-u}_t - \int_0^t \int_{\mathbb{R}^d} \frac{2c(X_s,y)F(X_s,y)}{|X_s-y|^{d+\alpha}}\mathrm{d}y\mathrm{d}s\,, \quad t \in [0,\infty[\,. \tag{3.1}$$

Moreover, since $|G^u(x,y) - F^u(x,y)| \le \frac{1}{2}e^{\|F^u\|_\infty}|F^u(x,y)|^2$ and

$$|G^u(x,y)|^2 \le \left(\frac{\|F^u\|_\infty e^{|F^u(x,y)|}}{2}|F^u(x,y)| + |F^u(x,y)|\right)^2$$

$$\leq \left(\frac{\|F^u\|_\infty e^{\|F^u\|_\infty}}{2} + 1 \right)^2 |F^u(x, y)|^2,$$

we see that $G^u - F^u \in J_D^1(\mathbf{X})$ and $(G^u)^2 \in J_D^1(\mathbf{X})$, respectively. Therefore we can also consider a purely discontinuous locally square integrable local martingale additive functional M^{G^u} defined by

$$(3.2)\quad M_t^{G^u} = M_t^{F^u} + \sum_{s \leq t} (G^u - F^u)(X_{s-}, X_s) - \int_0^t \int_{\mathbb{R}^d} \frac{2c(X_s, y)(G^u - F^u)(X_s, y)}{|X_s - y|^{d+\alpha}} \mathrm{d}y\mathrm{d}s$$

for $t \in [0, \infty[$. Let $Y_t := \mathrm{Exp}(M^{G^u})_t$ be the Doléans-Dade exponential of $M_t^{G^u}$, that is, Y_t is the unique solution of $Y_t = 1 + \int_0^t Y_{s-}\mathrm{d}M_s^{G^u}$ for $t \in [0, \infty[$, $\mathbf{P}_x$-a.s. Then Y_t can be represented as

$$(3.3)\qquad Y_t = \exp\left(M_t^{F^u} - \int_0^t \int_{\mathbb{R}^d} \frac{2c(X_s, y)(G^u - F^u)(X_s, y)}{|X_s - y|^{d+\alpha}} \mathrm{d}y\mathrm{d}s \right), \quad t \in [0, \infty[\,.$$

Note that Y_t is a positive local martingale, therefore it is a supermartingale multiplicative functional for all $t \in [0, \infty[$.

Let $\mathbf{Y} = (\Omega, \widetilde{\mathfrak{F}}_\infty, \widetilde{\mathfrak{F}}_t, \widetilde{X}_t, \mathbf{P}_x^Y)$ be the transformed process of $\mathbf{X}$ by Y_t. Note that $\mathbf{Y}$ is an $e^{-2u}\mathrm{d}x$-symmetric Hunt process on $\mathbb{R}^d$. The transition semigroup $\{P_t^Y\}_{t\geq 0}$ and the resolvent $\{R_\alpha^Y\}_{\alpha>0}$ of $\mathbf{Y}$ are defined by $P_t^Y f(x) := \mathbf{E}_x^Y[f(\widetilde{X}_t)] = \mathbf{E}_x[Y_t f(X_t)]$ and $R_\alpha^Y f(x) := \mathbf{E}_x^Y[\int_0^\infty e^{-\alpha t} f(\widetilde{X}_t)\mathrm{d}t] = \mathbf{E}_x[\int_0^\infty e^{-\alpha t} Y_t f(X_t)\mathrm{d}t]$.

Let $(\mathcal{E}^Y, \mathcal{F}^Y)$ be the Dirichlet form of $\mathbf{Y}$ on $L^2(\mathbb{R}^d; e^{-2u}\mathrm{d}x)$. Then $\mathcal{F} = \mathcal{F}^Y$ and

$$(3.4)\quad \mathcal{E}^Y(\varphi, \psi) = \frac{1}{2} \int_{\mathbb{R}^d}\int_{\mathbb{R}^d} \frac{(\varphi(x) - \varphi(y))(\psi(x) - \psi(y))e^{F(x,y)-u(x)-u(y)}c(x, y)}{|x - y|^{d+\alpha}} \mathrm{d}x\mathrm{d}y$$

for any $\varphi, \psi \in \mathcal{F}$ (cf. [12, Theorem 3.2]). In particular, $\mathcal{F}_e = \mathcal{F}_e^Y$. In view of these facts combining with the boundedness of u and F, there exists a constant $C := C(u, F) > 0$ such that $C^{-1}\mathcal{E}(\varphi, \varphi) \leq \mathcal{E}^Y(\varphi, \varphi) \leq C\mathcal{E}(\varphi, \varphi)$ for $\varphi \in \mathcal{F}$ and thus we see that $\mathbf{Y}$ is also a symmetric α-stable-like process on $\mathbb{R}^d$. Hence $\mathbf{Y}$ also admits a locally Hölder continuous transition density function $p_t^Y(x, y)$ satisfying (2.1) for all $(t, x, y) \in\,]0, \infty[\times\mathbb{R}^d \times \mathbb{R}^d$. Moreover it is easy to see that $\mathbf{Y}$ is transient (resp. recurrent) whenever $\mathbf{X}$ is so. Hence the Green kernel $R^Y(x, y)$ also satisfies the following estimate whenever $d > \alpha$:

$$(3.5)\qquad \frac{C_5}{|x - y|^{d-\alpha}} \leq R^Y(x, y) \leq \frac{C_6}{|x - y|^{d-\alpha}}, \quad x, y \in \mathbb{R}^d$$

for some $C_6 > C_5 > 0$.

Now we introduce some notions of Green-tight Kato class measures in the strict sence.

DEFINITION 3.1 (Green-tight Kato class measures). Let ν be a positive Radon measure on $\mathbb{R}^d$ and take an $\alpha \geq 0$.

(1) ν is said to be *α-order Green-tight with respect to* $\mathbf{X}$ if $\nu \in S_K^1(\mathbf{X})$ and for any $\varepsilon > 0$ there exists a compact subset $K = K(\varepsilon)$ of $\mathbb{R}^d$ such that $\sup_{x\in\mathbb{R}^d} R_\alpha(\mathbf{1}_{K^c}\nu)(x) < \varepsilon$.

(2) ν is said to be *α-order Green-tight with respect to* $\mathbf{X}$ *in the sense of Chen* if for any $\varepsilon > 0$ there exists a Borel subset $K = K(\varepsilon)$ of $\mathbb{R}^d$ with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all measurable set $B \subset K$ with $\nu(B) < \delta$, $\sup_{x\in\mathbb{R}^d} R_\alpha(\mathbf{1}_{K^c\cup B}\nu)(x) < \varepsilon$.

In view of the resolvent equation, the α-order Green-tightness is independent of the choice of $\alpha > 0$. Let denote by $S^1_{K^+_\infty}(\mathbf{X})$ (resp. $S^1_{CK^+_\infty}(\mathbf{X})$) the family of positive order Green-tight measures (resp. the family of positive order Green-tight measures in the sense of Chen) of $\mathbf{X}$. The measure having 0-order Green-tightness is suitable to treat transient case. In this case, 0-order Green-tightness always implies the α-order Green-tightness for $\alpha > 0$. Denote by $S^1_{K_\infty}(\mathbf{X})$ (resp. $S^1_{CK_\infty}(\mathbf{X})$) the family of 0-order Green-tight measures (resp. the family of 0-order Green-tight measures in the sense of Chen) of $\mathbf{X}$.

It is shown in [2, Proposition 2.2] that if $\nu \in S^1_{CK_\infty}(\mathbf{X})$, ν is Green-bounded, i.e., $\sup_{x\in\mathbb{R}^d} R\nu(x) < \infty$.

REMARK 3.1.

(1) It is known in general that $S^1_{CK^+_\infty}(\mathbf{X}) \subset S^1_{K^+_\infty}(\mathbf{X})$ and $S^1_{CK_\infty}(\mathbf{X}) \subset S^1_{K_\infty}(\mathbf{X})$ (see [2]). However we have from **(RSF)** for $\mathbf{X}$ the equivalence between Green-tight measures and Green-tight measures in the sense of Chen: $S^1_{K^+_\infty}(\mathbf{X}) = S^1_{CK^+_\infty}(\mathbf{X})$ and $S^1_{K_\infty}(\mathbf{X}) = S^1_{CK_\infty}(\mathbf{X})$ ([12, Lemma 4.1]).

(2) In the transient case, let $R^z(x, y)$ be the Green function of Doob's $R(\cdot, z)$-transformed process $\mathbf{X}^z$ of $\mathbf{X}$ defined by

$$R^z(x, y) := \frac{R(x, y)R(y, z)}{R(x, z)}, \quad x, y \in \mathbb{R}^d \text{ with } x \neq y$$

and $R^z\nu(x) := \int_{\mathbb{R}^d} R^z(x, y)\nu(dy)$. A positive Radon measure ν is said to be conditionally Green-tight in the sense of Chen with respect to $\mathbf{X}$ if for any $\varepsilon > 0$ there exists a Borel subset $K = K(\varepsilon)$ of $\mathbb{R}^d$ with $\nu(K) < \infty$ and a constant $\delta > 0$ such that for all measurable set $B \subset K$ with $\nu(B) < \delta$, $\sup_{(x,z)\in\mathbb{R}^d\times\mathbb{R}^d, x\neq z} R^z(\mathbf{1}_{K^c\cup B}\nu)(x) < \varepsilon$ (see [2]). Let denote by $S^1_{CS_\infty}(\mathbf{X})$ the family of conditionally Green-tight measures in the sense of Chen. It is known in general that $S^1_{CS_\infty}(\mathbf{X}) \subset S^1_{CK_\infty}(\mathbf{X})$ (cf. [2, 7]). For the α-stable-like process, the converse inclusions also holds from the so-called $3R$-inequality: $R^z(x, y) \leq c(R(x, y) + R(y, z))$ in view of (2.2). Hence we see $S^1_{CS_\infty}(\mathbf{X}) = S^1_{CK_\infty}(\mathbf{X}) = S^1_{K_\infty}(\mathbf{X})$ and do not need to distinguish them in the setting of present paper.

The next lemma is an easy consequence of [12, Corollary 5.1 and Corollary 5.2(2)] (cf. [11, Lemma 3.3]).

LEMMA 3.1. *Assume that* $\mathbf{X}$ *is transient (resp. recurrent) and* $\mu_{\langle u\rangle} + \mu + \mu_F \in S^1_{K_\infty}(\mathbf{X})$ *(resp.* $S^1_{K^+_\infty}(\mathbf{X})$*). Then*

(1) *For* $\nu \in S^1_D(\mathbf{X})$, $e^{-2u}\nu \in S^1_D(\mathbf{Y})$.
(2) *For* $\nu \in S^1_K(\mathbf{X})$, $e^{-2u}\nu \in S^1_K(\mathbf{Y})$.

(3) *For* $\nu \in S^1_{K_\infty}(\mathbf{X})$, $e^{-2u}\nu \in S^1_{K_\infty}(\mathbf{Y})$ $\big($*resp.* $\nu \in S^1_{K_\infty^+}(\mathbf{X})$, $e^{-2u}\nu \in S^1_{K_\infty^+}(\mathbf{Y})\big)$.

Consider the generalized non-local Feynman-Kac transform by the additive functionals $A := N^u + A^\mu + A^F$ of the form

$$e_A(t) := \exp(A_t)\,, \quad t \geq 0\,. \tag{3.6}$$

The weight (3.6) defines a Feynman-Kac semigroup $P_t^A f(x) := \mathbf{E}_x[e_A(t) f(X_t)]$ and it has infinitesimal generator of the form $\mathcal{H} := \mathcal{L} + \mathcal{L}u + \mu + d\mathbf{F}$, where $\mathcal{L}$ is the generator of $\mathbf{X}$ and $d\mathbf{F}$ denotes the measure valued operator defined by $d\mathbf{F}f := \mathbf{F}f(x)\mathrm{d}x$ for

$$\mathbf{F}f(x) = \int_{\mathbb{R}^d} \frac{2c(x,y)G(x,y)f(y)}{|x-y|^{d+\alpha}}\mathrm{d}y\,.$$

Define

$$\overline{\mu}(\mathrm{d}x) := \overline{\mu}_{u,\mu,F}(\mathrm{d}x) = \left\{\int_{\mathbb{R}^d} \frac{2c(x,y)(G^u - F^u + F)(x,y)}{|x-y|^{d+\alpha}}\mathrm{d}y\right\}\mathrm{d}x + \mu(\mathrm{d}x)\,.$$

By (3.1) and (3.3), we then see for all $t \in [0, \infty[$,

$$e_A(t) = e^{u(X_t)-u(x)} \exp\big(-M_t^u + A_t^\mu + A_t^F\big) = e^{u(X_t)-u(x)} Y_t \exp\big(A_t^{\overline{\mu}}\big) \tag{3.7}$$

which implies that for any $x \in \mathbb{R}^d$ and $f \in \mathcal{B}_+(\mathbb{R}^d)$,

$$\mathbf{E}_x\big[e_A(t)f(X_t)\big] = e^{-u(x)}\mathbf{E}_x^Y\big[\exp\big(A_t^{\overline{\mu}}\big)(e^u f)(\widetilde{X}_t)\big]\,. \tag{3.8}$$

It is easy to see that $\overline{\mu} \in S^1_{K_\infty^+}(\mathbf{X})$ (resp. $\overline{\mu} \in S^1_{K_\infty}(\mathbf{X})$) whenever $\mu_{\langle u\rangle} \in S^1_{K_\infty^+}(\mathbf{X})$, $\mu \in S^1_{K_\infty^+}(\mathbf{X})$ and $\mu_F \in S^1_{K_\infty^+}(\mathbf{X})$ (resp. $\mu_{\langle u\rangle} \in S^1_{K_\infty}(\mathbf{X})$, $\mu \in S^1_{K_\infty}(\mathbf{X})$ and $\mu_F \in S^1_{K_\infty}(\mathbf{X})$) hold. Moreover we note that if $\nu \in S^1_{K_\infty^+}(\mathbf{X})$ (resp. $\nu \in S^1_{K_\infty}(\mathbf{X})$), then $\int_{\mathbb{R}^d} f^2 \mathrm{d}\nu \leq \|R_1\nu\|_\infty \mathcal{E}_1(f,f)$ for $f \in \mathcal{F}$ (resp. $\int_{\mathbb{R}^d} f^2\mathrm{d}\nu \leq \|R\nu\|_\infty \mathcal{E}(f,f)$ for $f \in \mathcal{F}_e$), hence $\mathcal{F} \subset L^2(\mathbb{R}^d;\nu)$ (resp. $\mathcal{F}_e \subset L^2(\mathbb{R}^d;\nu)$) ([19, Theorem 3.1]).

In the rest of this section, assume that $\mathbf{X}$ is transient. Then there exists a strictly positive continuous function g such that $g\mathrm{d}x \in S^1_{K_\infty}(\mathbf{X})$ in view of [23, Lemma 2.4]. Thus the transformed process $\mathbf{Y}$ is to be a transient $e^{-2u}\mathrm{d}x$-symmetric α-stable-like process having irreducibility, **(RSF)** and $e^{-2u}g\mathrm{d}x \in S^1_{K_\infty}(\mathbf{Y})$ in view of Lemma 3.1(3). Let $\{\tau_t\}_{t\geq 0}$ be the right continuous inverse of the PCAF

$$B_t := \int_0^t g(\widetilde{X}_s)\mathrm{d}s + A_t^{\overline{\mu}}$$

of $\mathbf{Y}$, that is, $\tau_t = \inf\{s > 0 \mid B_s > t\}$. Let $e^{-2u}\eta := e^{-2u}(g\mathrm{d}x + \overline{\mu})$ be the Revuz measure corresponding to B_t. Note that the fine and topological supports of $e^{-2u}\eta$ are equal to $\mathbb{R}^d$. Let $(\check{\mathbf{Y}}, e^{-2u}\eta) := (\Omega, \check{X}_t, \mathbf{P}_x^Y, \check{\zeta})$ be the time changed process of $\mathbf{Y}$ by the PCAF B_t, that is, $\check{X}_t = \widetilde{X}_{\tau_t}$. Then $(\check{\mathbf{Y}}, e^{-2u}\eta)$ is also transient and is an $e^{-2u}\eta$-symmetric Hunt process on $\mathbb{R}^d$ with life time $\check{\zeta} := B_\infty$ ([10, Theorem 6.2.1 and Theorem 6.2.3]).

LEMMA 3.2. *Assume that* $\mathbf{X}$ *is transient and* $\mu_{\langle u\rangle} + \mu + \mu_F \in S^1_{K_\infty}(\mathbf{X})$. *Let* $\eta^* := g\mathrm{d}x + k\overline{\mu}$ *for* $k \geq 0$. *Then the time changed process* $(\check{\mathbf{Y}}, e^{-2u}\eta^*)$ *is irreducible and has* (**RSF**). *Moreover,* $e^{-2u}\eta \in S^1_{K_\infty^+}(\check{\mathbf{Y}}, e^{-2u}\eta^*)$.

PROOF. This lemma can be proved by a nearly same method in [23, Lemma 2.3]. We omit the details. □

Define the quadratic form $\mathcal{Q}$ by

$$\mathcal{Q}(f,g) := \mathcal{E}(f,g) + \mathcal{E}(u, fg) - \int_{\mathbb{R}^d} f(x)g(x)\mu(\mathrm{d}x) - \iint_{\mathbb{R}^d\times\mathbb{R}^d} \frac{2f(x)g(y)c(x,y)G(x,y)}{|x-y|^{d+\alpha}}\mathrm{d}x\mathrm{d}y\,. \tag{3.9}$$

Then it is well-defined for $f, g \in \mathcal{F}$ provided $\mu_{\langle u\rangle} + \mu + \mu_F \in S^1_D(\mathbf{X})$. Moreover, $\mathcal{Q}$ is extended to $\mathcal{F}_e \times \mathcal{F}_e$ with the same expression (3.9) provided $\mu_{\langle u\rangle} + \mu + \mu_F \in S^1_{K_\infty}(\mathbf{X})$. Note that for $f \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d)$, we see $fe^u = f(e^u - 1) + f = f(e^{u_K} - 1) + f \in \mathcal{F}_e$, where $u_K \in \mathcal{F}_e$ such that $u = u_K$ a.e. on supp$[f]$. It follows from (3.3), (3.4) and the Feynman-Kac formula that for $f \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d)$, we have

$$\begin{aligned}\mathcal{E}^Y(fe^u, fe^u) &= \mathcal{Q}(f,f) + \int_{\mathbb{R}^d} f^2\mathrm{d}\mu + \int_{\mathbb{R}^d} f^2 \mathrm{d}\mu_{G^u - F^u + F} \\ &= \mathcal{Q}(f,f) + \int_{\mathbb{R}^d} f^2\mathrm{d}\overline{\mu}\,.\end{aligned} \tag{3.10}$$

Suppose $\mu_{\langle u\rangle} + \mu + \mu_F \in S^1_{K_\infty}(\mathbf{X})$ and let us define the spectral function by

$$\lambda(\overline{\mu}) := \inf\left\{\mathcal{Q}(f,f)\ \middle|\ f \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f^2\mathrm{d}\overline{\mu} = 1\right\}. \tag{3.11}$$

By (3.10), it is easy to see that $\lambda(\overline{\mu}) \geq -1$ and (3.11) is equivalent to

$$\lambda(\overline{\mu}) + 1 := \inf\left\{\mathcal{E}^Y(fe^u, fe^u)\ \middle|\ f \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f^2\mathrm{d}\overline{\mu} = 1\right\}, \tag{3.12}$$

in other words,

$$\inf\left\{\mathcal{E}^Y(fe^u, fe^u)\ \middle|\ f \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d),\ (\lambda(\overline{\mu}) + 1)\int_{\mathbb{R}^d} f^2\mathrm{d}\overline{\mu} = 1\right\} = 1\,. \tag{3.13}$$

Applying Lemma 2.1 to $\nu_1 = (\lambda(\overline{\mu}) + 1)\overline{\mu}$, $\nu = g\mathrm{d}x$ and the following inequality

$$\begin{aligned}&\inf\left\{\mathcal{E}^Y(fe^u, fe^u)\ \middle|\ f \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f^2 g\mathrm{d}x = 1\right\} \\ &\geq \inf\left\{\left\|R^Y\left(e^{-2u}g\mathrm{d}x\right)\right\|_\infty^{-1}\int_{\mathbb{R}^d} f^2 g\mathrm{d}x\ \middle|\ f \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f^2 g\mathrm{d}x = 1\right\} > 0\,,\end{aligned}$$

we see that (3.13) is equivalent to

$$\inf\left\{\mathcal{E}^Y(fe^u, fe^u) + \int_{\mathbb{R}^d} f^2 g\mathrm{d}x\ \middle|\ f \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f^2\mathrm{d}\eta^* = 1\right\} = 1\,, \tag{3.14}$$

where $\eta^* := g\mathrm{d}x + (\lambda(\overline{\mu}) + 1)\overline{\mu}$. Therefore, by applying [23, Lemma 2.1] to the time changed process $(\check{\mathbf{Y}}, e^{-2u}\eta^*)$, there exists a minimizer $\phi_0 \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d)$ uniquely in (3.14) in view of Lemma 3.2, equivalently a minimizer $h \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d)$ in (3.11) given by $h = \phi_0/\|\phi_0\|_{L^2(\mathbb{R}^d;\overline{\mu})}$. Hence $\|h\|_{L^2(\mathbb{R}^d;\overline{\mu})} = 1$ and

$$\lambda(\overline{\mu}) = \mathcal{Q}(h,h) = \mathcal{E}^Y(he^u, he^u) - 1\,. \tag{3.15}$$

The function h on $\mathbb{R}^d$ is called a ground state of the quadratic form $(\mathcal{Q}, \mathcal{F}_e)$. Let denote by g_Y the ground state of $(\mathcal{E}^Y, \mathcal{F}_e)$. Then we see $h = e^{-u} g_Y$ in view of (3.12) and (3.15). Note that g_Y (and hence h) can be taken to be strictly positive in a similar way of [10, Lemma 6.4.5]. Moreover, whenever $\lambda(\overline{\mu}) = 0$, we see that the ground state g_Y (and hence h) is bounded continuous and satisfies the following estimates:

$$\frac{C_7}{|x|^{d-\alpha}} \le g_Y(x) \le \frac{C_8}{|x|^{d-\alpha}} \quad \text{for } |x| > 1 \tag{3.16}$$

(see Lemma 4.9, Proposition 4.16 and (4.19) in [24]).

4. Penalized measures for generalized Feynman-Kac transforms. In this section, we study the limiting behaviour of the measure $\mathbf{Q}^A_{x,t}$ defined by

$$\mathbf{Q}^A_{x,t}(B) := \frac{\mathbf{E}_x[e_A(t)\mathbf{1}_B]}{\mathbf{E}_x[e_A(t)]}\,, \quad B \in \mathcal{F}_s\,, \;\; s \ge 0\,, \;\; x \in \mathbb{R}^d\,, \tag{4.1}$$

where $e_A(t)$ is the generalized Feynman-Kac transform given by (3.6). We shall consider the convergence of the measure (4.1) by classifying the spectral function $\lambda(\overline{\mu})$ into three cases: $\lambda(\overline{\mu}) > 0$, $\lambda(\overline{\mu}) = 0$ and $\lambda(\overline{\mu}) < 0$, which are corresponding to the subcriticality, criticality and supercriticality of the Schrödinger operator $\mathcal{H}$, respectively.

Note that if $\mathbf{X}$ is recurrent (i.e., $d \le \alpha$), then $\lambda(\overline{\mu}) = -1$ in a similar way of [21, Lemma 3.2]. Indeed, we see from (3.12) that $\mathcal{E}^Y(fe^u, fe^u) \ge (\lambda(\overline{\mu}) + 1)\int_{\mathbb{R}^d} f^2 \mathrm{d}\overline{\mu}$ for any $f \in \mathcal{F} \cap C_\infty(\mathbb{R}^d)$. As we mentioned in Section 3, $\mathbf{Y}$ is also recurrent if $\mathbf{X}$ is so. Then there exists a sequence $\{f_n\} \subset \mathcal{F} \cap C_\infty(\mathbb{R}^d)$ satisfying $\lim_{n\to\infty} f_n = 1$ a.e. and $\lim_{n\to\infty} \mathcal{E}^Y(f_n e^u, f_n e^u) = 0$ ([10, Theorem 1.6.3]). Hence if $\lambda(\overline{\mu}) > -1$, then $\overline{\mu} = 0$, which is contradictory. Now, we may assume that $\mathbf{X}$ is transient whenever we consider the cases $\lambda(\overline{\mu}) > 0$ and $\lambda(\overline{\mu}) = 0$.

4.1. The case $\lambda(\overline{\mu}) > 0$. First, we note by [12, Theorem 1.2] that the following proposition holds in view of (3.5) and Remark 3.1.

PROPOSITION 4.1. *Assume $\mu_{\langle u \rangle} + \mu + \mu_F \in S^1_{K_\infty}(\mathbf{X})$. Then the following are equivalent:*

(1) $\lambda(\overline{\mu}) > 0$.
(2) *The functional* (3.6) *is gaugeable, i.e.,* $\sup_{x\in\mathbb{R}^d} \mathbf{E}_x[e_A(\infty)] < \infty$.
(3) *There exists a Green kernel $R^A(x,y) < \infty$ ($x, y \in \mathbb{R}^d$ with $x \ne y$) of the formal Schrödinger operator $\mathcal{H}$.*

The above equivalence still holds for general symmetric Markov processes under more mild conditions on measures (see [12]).

The next result is needed because $e_A(t)$ is not monotone.

LEMMA 4.1. *Assume $\mu_{\langle u\rangle}+\mu+\mu_F\in S^1_{K_\infty}(\mathbf{X})$. Then for any $x\in\mathbb{R}^d$,*

$$\lim_{t\to\infty}\mathbf{E}_x[e_A(t)]=\mathbf{E}_x[e_A(\infty)]\,.$$

PROOF. Since $\lambda(\overline{\mu})>0$, $\mathbf{E}_\cdot[e_A(\infty)]$ is bounded in view of Proposition 4.1. So it is enough to show that there exists $p\ge 1$ such that $\sup_{t\in[0,\infty[}\mathbf{E}_x[e_A(t)^p]<\infty$ for any $x\in\mathbb{R}^d$. For $p\ge 1$, put $F^u_{(p)}:=pF^u$ and $G^u_{(p)}:=e^{F^u_{(p)}}-1$. Define

$$\overline{\mu}_{(p)}(\mathrm{d}x):=\left\{\int_{\mathbb{R}^d}\frac{2c(x,y)\big(G^u_{(p)}-F^u_{(p)}+pF\big)(x,y)}{|x-y|^{d+\alpha}}\mathrm{d}y\right\}\mathrm{d}x+p\mu(\mathrm{d}x)\,.$$

Let

$$Y^{(p)}_t=\exp\left(M_t^{F^u_{(p)}}-\int_0^t\int_{\mathbb{R}^d}\frac{2c(X_s,y)(G^u_{(p)}-F^u_{(p)})(X_s,y)}{|X_s-y|^{d+\alpha}}\mathrm{d}y\mathrm{d}s\right),\qquad t\in[0,\infty[\,.$$

We see from (3.7) that

$$\mathbf{E}_x\left[e_A(t)^p\right]=\mathbf{E}_x^{Y^{(p)}}\left[e^{A_t^{\overline{\mu}_{(p)}}}e^{pu(\widetilde{X}_t)-pu(x)}\right]\le e^{2p\|u\|_\infty}\mathbf{E}_x^{Y^{(p)}}\left[\sup_{t\in[0,\infty[}e^{A_t^{\overline{\mu}_{(p)}}}\right],$$

and thus

$$\sup_{t\in[0,\infty[}\mathbf{E}_x\left[e_A(t)^p\right]\le e^{2p\|u\|_\infty}\mathbf{E}_x^{Y^{(p)}}\left[\sup_{t\in[0,\infty[}e^{A_t^{\overline{\mu}_{(p)}}}\right]\tag{4.2}$$

for any $x\in\mathbb{R}^d$. Set

$$\lambda^{(p)}(\overline{\mu}_{(p)}):=\inf\left\{\mathcal{Q}^{(p)}(f,f)\;\middle|\;f\in\mathcal{F}_e\cap C_\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d}f^2\mathrm{d}\overline{\mu}_{(p)}=1\right\},$$

where $\mathcal{Q}^{(p)}$ is the quadratic form defined for pu, $p\mu$ and pF as in (3.9). Then, by way of [12, Proposition 5.2], there exists a $p_0>1$ sufficiently close to 1 such that $\lambda^{(p)}(\overline{\mu}_{(p)})>0$ for any $p\in[1,p_0]$, hence the function $\mathbf{E}_\cdot^{Y^{(p)}}[\exp(A_\infty^{\overline{\mu}_{(p)}})]$ is bounded for any $p\in[1,p_0]$. This supergaugeability is equivalent to the boundedness of the right hand side of (4.2) by virtue of [12, Lemma 4.8]. □

Let $g_A(x):=\mathbf{E}_x[e_A(\infty)]$ and define a martingale MF $L^{(1)}_t$ by

$$L^{(1)}_t=\frac{g_A(X_t)}{g_A(x)}e_A(t)\,.$$

THEOREM 4.1. *Assume $\mu_{\langle u\rangle}+\mu+\mu_F\in S^1_{K_\infty}(\mathbf{X})$. Then for any $s\ge 0$ and $B\in\mathcal{F}_s$, we have*

$$\lim_{t\to\infty}\frac{\mathbf{E}_x\left[e_A(t)\mathbf{1}_B\right]}{\mathbf{E}_x\left[e_A(t)\right]}=\mathbf{E}_x[L^{(1)}_s\mathbf{1}_B]\,.\tag{4.3}$$

PROOF. The proof is an easy consequence of Lemma 4.1. Indeed,

$$\frac{\mathbf{E}_x\left[e_A(t)\mathbf{1}_B\right]}{\mathbf{E}_x\left[e_A(t)\right]}=\frac{\mathbf{E}_x\left[e_A(s)\mathbf{1}_B\mathbf{E}_{X_s}\left[e_A(t-s)\right]\right]}{\mathbf{E}_x\left[e_A(t)\right]}$$

$$\longrightarrow \frac{\mathbf{E}_x\left[e_A(s)\mathbf{1}_B\mathbf{E}_{X_s}\left[e_A(\infty)\right]\right]}{\mathbf{E}_x\left[e_A(\infty)\right]} = \mathbf{E}_x[L_s^{(1)}\mathbf{1}_B]$$

as $t \to \infty$. □

4.2. The case $\lambda(\overline{\mu}) = 0$. In this case, the following subclass of measures of $S^1_{K_\infty}(\mathbf{X})$ plays a crucial role for the proof of the main theorem (cf. [21, Definition 4.2]).

DEFINITION 4.1. A measure $\nu \in S^1_K(\mathbf{X})$ is said to be in the class $S^1_{SK}(\mathbf{X})$ if

$$\sup_{x\in\mathbb{R}^d}\left(|x|^{d-\alpha}\int_{\mathbb{R}^d}\frac{\nu(\mathrm{d}y)}{|x-y|^{d-\alpha}}\right) < \infty .$$

REMARK 4.1.

(1) The definition of the class $S^1_{SK}(\mathbf{X})$ is invariant under the Girsanov transform Y_t, that is, $\nu \in S^1_{SK}(\mathbf{X})$ implies $e^{-2u}\nu \in S^1_{SK}(\mathbf{Y})$, in view of Lemma 3.1.

(2) A measure $\nu \in S^1_K(\mathbf{X})$ with compact support belongs to the class $S^1_{SK}(\mathbf{X})$. Moreover, $S^1_{SK}(\mathbf{X})$ is a proper subset of $S^1_{K_\infty}(\mathbf{X})$. In fact, it is known that $S^1_{SK}(\mathbf{X})$ is a subset of $S^1_{K_\infty}(\mathbf{X})$ and the measure μ_F induced by the jumping function

$$F(x,y) := \frac{(1\wedge|x-y|^p)}{(1+|x|^2)^{q/2}(1+|y|^2)^{q/2}}, \quad \text{for } p>\alpha,\ \ q>2d-\alpha$$

belongs to $S^1_{SK}(\mathbf{X})$. However the jumping function F itself does not belong to the class $\mathbf{A}_2(\mathbf{X})$ ([16, Example 7.3 and Remark 7.4]). Here $\mathbf{A}_2(\mathbf{X})$ is the class of non-negative bounded Borel functions ϕ on $\mathbb{R}^d\times\mathbb{R}^d$ vanishing on the diagonal such that $\mu_\phi \in S^1_{K_\infty}(\mathbf{X})$ (in our settings) and $\phi \in \mathbf{A}_\infty(\mathbf{X})$ (see [3, Definition 2.3] for details). Now we see that the claim is true because $\mathbf{A}_2(\mathbf{X})$ is a proper subset of the class of ϕ such that $\mu_\phi \in S^1_{K_\infty}(\mathbf{X})$.

We say that $\mathbf{X}$ is Harris recurrent if $\int_D \mathrm{d}x > 0$, then $\mathbf{P}_x(\int_0^\infty \mathbf{1}_D(X_t)\mathrm{d}t = \infty) = 1$ for any $x \in \mathbb{R}^d$ and $D \subset \mathbb{R}^d$.

The following notion of special AF of Harris recurrent process due to [1] and in there the author extended the concept of special function to CAF.

DEFINITION 4.2. Let $\mathbf{X}$ be a Harris recurrent process. A PCAF A of $\mathbf{X}$ is said to be *special* if for any function $g \in \mathcal{B}_+(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} g(x)\mathrm{d}x > 0$, the function $\mathbf{E}_\cdot[\int_0^\infty e^{-\int_0^t g(X_s)\mathrm{d}s}\mathrm{d}A_t]$ is bounded.

Note that if $\mathbf{X}$ is Harris recurrent, then the AF of the form $A^f_t := \int_0^t f(X_s)\mathrm{d}s$ is to be special for a positive bounded Borel funtion f with compact support on $\mathbb{R}^d$.

Let g_Y be the ground state of $(\mathcal{E}^Y, \mathcal{F})$ appeared in (3.16). Define the $\mathbf{P}^Y_x$-martingale MF L by

$$L_t := \frac{g_Y(\widetilde{X}_t)}{g_Y(x)}\exp\left(A^{\overline{\mu}}_t\right). \tag{4.4}$$

Let denote by $\mathbf{Y}^L := (\Omega, \widetilde{\mathcal{F}}_\infty, \widetilde{\mathcal{F}}_t, \widetilde{X}_t, \mathbf{P}_x^{Y,L})$ the transformed process of $\mathbf{Y}$ by L_t: $\mathbf{P}_x^{Y,L}(\mathrm{d}\omega) = L_t(\omega) \cdot \mathbf{P}_x^Y(\mathrm{d}\omega)$.

LEMMA 4.2. *Assume* $\mu_{\langle u\rangle} + \mu + \mu_F \in S_K^1(\mathbf{X})$. *Put* $A_t^{\overline{\mu}/g_Y} := (\frac{1}{g_Y} \cdot A^{\overline{\mu}})_t$.

(1) $\mathbf{Y}^L$ *is an* $e^{-2u} g_Y^2$*-symmetric Harris recurrent process.*

(2) *If* $\mu_{\langle u\rangle} + \mu + \mu_F \in S_{SK}^1(\mathbf{X})$, *then* $A^{\overline{\mu}/g_Y}$ *is a special AF with respect to* $\mathbf{Y}^L$.

PROOF. (1) First, we remark that any irreducible recurrent symmetric Markov process satisfying **(RSF)** is Harris recurrent ([10, Lemma 4.8.1]). By [4, Theorem 2.6(b)] and the positivity of L_t, we see that $\mathbf{Y}^L$ is an irreducible recurrent $e^{-2u} g_Y^2$-symmetric process on $\mathbb{R}^d$. Therefore it is enough to show the **(RSF)** of $\mathbf{Y}^L$ for the Harris recurrence of $\mathbf{Y}^L$. Let $\{R_\alpha^{Y,L}\}_{\alpha>0}$ be the resolvent of $\mathbf{Y}^L$. In a similar way of the proof of [6, Lemma 3.2] we have

$$\lim_{t\to 0} \sup_{x\in\mathbb{R}^d} \mathbf{E}_x^Y \left[|L_t - 1|^p \right] = 0 \tag{4.5}$$

for any $p \geq 1$. Then for $f \in \mathcal{B}_b(\mathbb{R}^d)$ and for any $\alpha, \beta > 0$,

$$\begin{aligned}
&\left\| R_\alpha^{Y,L} f - \beta R_\beta^Y R_\alpha^{Y,L} f \right\|_\infty \\
&\quad \leq \left\| R_\alpha^{Y,L} f - \beta R_\beta^{Y,L} R_\alpha^{Y,L} f \right\|_\infty + \beta \left\| R_\beta^{Y,L} R_\alpha^{Y,L} f - R_\beta^Y R_\alpha^{Y,L} f \right\|_\infty \\
&\quad \leq \left\| R_\beta^{Y,L} f - \alpha R_\beta^{Y,L} R_\alpha^{Y,L} f \right\|_\infty + \beta \left\| R_\alpha^{Y,L} f \right\|_\infty \int_0^\infty e^{-\beta t} \sup_{x\in\mathbb{R}^d} \mathbf{E}_x^Y [|L_t - 1|] \mathrm{d}t \\
&\quad \leq \beta^{-1} \|f\|_\infty + \left\| R_\alpha^{Y,L} f \right\|_\infty \int_0^\infty e^{-t} \sup_{x\in\mathbb{R}^d} \mathbf{E}_x^Y [|L_{t/\beta} - 1|] \mathrm{d}t\,.
\end{aligned}$$

By (4.5), the last term in the right hand side above converges to 0 as $\beta \to \infty$ because $\mathbf{E}_x^Y[|L_t - 1|] \leq 2$ and of the dominated convergence theorem. Now we see $R_\alpha^{Y,L} f \in C_b(\mathbb{R}^d)$ because $\beta R_\beta^Y R_\alpha^{Y,L} f$ is so, which tells us that $\mathbf{Y}^L$ satisfies **(RSF)**.

(2) We prove that the function $\mathbf{E}_\cdot^{Y,L}[\int_0^\infty e^{-\int_0^t g(X_s)\mathrm{d}s} \mathrm{d}A_t^{\overline{\mu}/g_Y}]$ is bounded for any $g \in \mathcal{B}_+(\mathbb{R}^d)$ such that $\int_{\mathbb{R}^d} g(x)\mathrm{d}x > 0$. Let $g_0(x) := g\mathbf{1}_K(x)$ for a compact set K of $\mathbb{R}^d$. Then by Lemma 3.1, we see that $e^{-2u} g_0 \mathrm{d}x \in S_{K_\infty}^1(\mathbf{Y})$ and also $e^{-2u}\overline{\mu} \in S_{K_\infty}^1(\mathbf{Y})$ by the assumption $\mu_{\langle u\rangle} + \mu + \mu_F \in S_{SK}^1(\mathbf{X})$ (hence $\overline{\mu} \in S_{SK}^1(\mathbf{X})$). Since $\overline{\mu}$ satisfies $\lambda(\overline{\mu}) = 0$,

$$\inf \left\{ \mathcal{Q}(f, f) + \int_{\mathbb{R}^d} f^2 g_0 \mathrm{d}x \;\middle|\; f \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f^2 \mathrm{d}\overline{\mu} = 1 \right\} > 0\,,$$

equivalently

$$\inf \left\{ \mathcal{E}^Y(f, f) + \int_{\mathbb{R}^d} f^2 e^{-2u} g_0 \mathrm{d}x \;\middle|\; f \in \mathcal{F}_e \cap C_\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d} f^2 e^{-2u} \mathrm{d}\overline{\mu} = 1 \right\} > 1\,. \tag{4.6}$$

By virtue of [2, Lemma 3.5(2) and Theorem 5.2] with $\mathbf{Y}$ as the underlying process, (4.6) implies that there exists $C > 0$ such that

$$\mathbf{E}_x^Y \left[\int_0^\infty e^{A_t^{\overline{\mu}} - \int_0^t g_0(X_s)ds} \mathrm{d}A_t^{\overline{\mu}} \right] \leq C \int_{\mathbb{R}^d} R^Y(x, y)\overline{\mu}(\mathrm{d}y)\,.$$

Thus we have

$$\begin{aligned}\mathbf{E}_x^{Y,L}\left[\int_0^\infty e^{-\int_0^t g(X_s)\mathrm{d}s}\mathrm{d}A_t^{\overline{\mu}/g_Y}\right] &\le \mathbf{E}_x^{Y,L}\left[\int_0^\infty e^{-\int_0^t g_0(X_s)\mathrm{d}s}\frac{1}{g_Y(X_s)}\mathrm{d}A_t^{\overline{\mu}}\right]\\ &\le \frac{1}{g_Y(x)}\mathbf{E}_x^{Y}\left[\int_0^\infty e^{A_t^{\overline{\mu}}-\int_0^t g_0(X_s)\mathrm{d}s}\mathrm{d}A_t^{\overline{\mu}}\right]\\ &\le C\frac{1}{g_Y(x)}\int_{\mathbb{R}^d} R^Y(x,y)\overline{\mu}(\mathrm{d}y).\end{aligned}$$

Now the proof is finished by (3.5), (3.16) and the definition of the class $S^1_{SK}(\mathbf{X})$. □

For any $B \in \mathcal{F}_s$ ($s \ge 0$), define positive finite measures ν_t, ν, θ_t and θ:

$$\nu_t(\cdot) := \mathbf{E}_x^Y\left[e^{A_s^{\overline{\mu}}}\mathbf{1}_B e^{u(\widetilde{X}_t)-u(x)};\, \widetilde{X}_s\in\cdot\right] \quad\text{and}\quad \nu(\cdot) := \mathbf{E}_x^Y\left[e^{A_s^{\overline{\mu}}}\mathbf{1}_B;\, \widetilde{X}_s\in\cdot\right]. \tag{4.7}$$

$$\theta_t(\cdot) := \mathbf{E}_x^Y\left[e^{A_s^{\overline{\mu}}} e^{u(\widetilde{X}_t)-u(x)};\, \widetilde{X}_s\in\cdot\right] \quad\text{and}\quad \theta(\cdot) := \mathbf{E}_x^Y\left[e^{A_s^{\overline{\mu}}};\, \widetilde{X}_s\in\cdot\right]. \tag{4.8}$$

Also we set

$$\eta_t := \frac{g_Y\cdot\nu_t}{\int_{\mathbb{R}^d} g_Y\mathrm{d}\nu_t},\quad \eta := \frac{g_Y\cdot\nu}{\int_{\mathbb{R}^d} g_Y\mathrm{d}\nu},\quad \kappa_t := \frac{g_Y\cdot\theta_t}{\int_{\mathbb{R}^d} g_Y\mathrm{d}\theta_t},\quad\text{and}\quad \kappa := \frac{g_Y\cdot\theta}{\int_{\mathbb{R}^d} g_Y\mathrm{d}\theta}.$$

By the boundedness of u, it is clear that

$$e^{-4\|u\|_\infty}\eta(\cdot) \le \eta_t(\cdot) \le e^{4\|u\|_\infty}\eta(\cdot),\quad\text{and}\quad e^{-4\|u\|_\infty}\kappa(\cdot) \le \kappa_t(\cdot) \le e^{4\|u\|_\infty}\kappa(\cdot). \tag{4.9}$$

For any PCAFs A and B of $\mathbf{Y}^L$, define the operator

$$U_A^B f(x) := \mathbf{E}_x^{Y,L}\left[\int_0^\infty e^{-B_t} f(\widetilde{X}_t)\mathrm{d}A_t\right]$$

for any $f \in \mathcal{B}_+(\mathbb{R}^d)$. In the particular case where $A_t \equiv t$ we simply write U^B. For any function $v \in \mathcal{B}_+(\mathbb{R}^d)$, let M_v be the operator on $\mathcal{B}_b(\mathbb{R}^d)$ defined by $M_v(f) = vf$. If A and B are respectively the form of $\int_0^t a(X_s)\mathrm{d}s$ and $\int_0^t v(X_s)\mathrm{d}s$, the U_A^B can be written as $U^v M_a$, and whenever B is obtained from a function v, we write U_A^v instead of U_A^B. We note that there exists a function $v \in \mathcal{B}_+(\mathbb{R}^d)$ such that $0 < v(x) \le 1$ for any $x \in \mathbb{R}^d$ and

$$U^v(f) \ge \int_{\mathbb{R}^d} f(x)\mathrm{d}x \tag{4.10}$$

for $f \in \mathcal{B}_+(\mathbb{R}^d)$ ([17, Proposition 4.3 of Chapter 6]).

We need the following strong Chacon-Ornstein type's limit-quotient theorem for special AFs under time-dependent initial probability measures.

PROPOSITION 4.2. *Assume $\mu_{\langle u\rangle} + \mu + \mu_F \in S^1_{SK}(\mathbf{X})$. Let ℓ be a positive bounded Borel function with compact support on $\mathbb{R}^d$ satisfying* (4.10)*. Then*

$$\left|\frac{\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{\overline{\mu}/g_Y}\right]}{\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{\ell}\right]} - \frac{\int_{\mathbb{R}^d} e^{-2u(x)} g_Y(x)\overline{\mu}(\mathrm{d}x)}{\int_{\mathbb{R}^d} e^{-2u(x)} g_Y^2(x)\ell(x)\mathrm{d}x}\right| \longrightarrow 0,\quad as\quad t\to\infty. \tag{4.11}$$

Furthermore,

$$\frac{\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{\ell}\right]}{\mathbf{E}_{\kappa_t}^{Y,L}\left[A_t^{\ell}\right]} \longrightarrow 1, \quad as \quad t\to\infty. \tag{4.12}$$

PROOF. For notational convenience, let $\nu(1) := \int_{\mathbb{R}^d}\nu(\mathrm{d}x)$. We put

$$\nu_1(\mathrm{d}x) := e^{-2u(x)}g_Y(x)\overline{\mu}(\mathrm{d}x) \quad \text{and} \quad \nu_2(\mathrm{d}x) := e^{-2u(x)}g_Y^2(x)\ell(x)\mathrm{d}x\,.$$

Note that $\nu_1(1) < \infty$ and $\nu_2(1) < \infty$ in view of (3.16) and [2, Proposition 2.2]. First, we prove (4.11). Set $A_t^{(1)} := \nu_2(1)A_t^{\overline{\mu}/g_Y}$ and $A_t^{(2)} := \nu_1(1)A_t^{\ell}$. Then $A^{(1)}$ and $A^{(2)}$ are also special AFs with respect to $\mathbf{Y}^L$ and $\nu_{A^{(1)}}(1) = \nu_{A^{(2)}}(1) = \nu_1(1)\nu_2(1)$. Here $\nu_{A^{(1)}}$ and $\nu_{A^{(2)}}$ stand for the corresponding Revuz measures of $A^{(1)}$ and $A^{(2)}$, respectively. Then there is, by virtue of [1, Proposition 3.19], a constant $M > 0$ such that $\left|\mathbf{E}_x^{Y,L}[A_t^{(1)}] - \mathbf{E}_x^{Y,L}[A_t^{(2)}]\right| \le M$ for any $t > 0$ and $x \in \mathbb{R}^d$. Therefore we have for any $t > 0$

$$\left|\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{(1)}\right] - \mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{(2)}\right]\right| \le \int_{\mathbb{R}^d}\left|\mathbf{E}_x^{Y,L}\left[A_t^{(1)}\right] - \mathbf{E}_x^{Y,L}\left[A_t^{(2)}\right]\right|\eta_t(\mathrm{d}x) \le M\,,$$

equivalently

$$\left|\nu_2(1)\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{\overline{\mu}/g_Y}\right] - \nu_1(1)\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{\ell}\right]\right| \le M\,.$$

Dividing by $\nu_2(1)\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{\ell}\right]$ on both side above, we have

$$\left|\frac{\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{\overline{\mu}/g_Y}\right]}{\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{\ell}\right]} - \frac{\nu_1(1)}{\nu_2(1)}\right| \le \frac{M}{\nu_2(1)\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{\ell}\right]}\,. \tag{4.13}$$

By (4.9) and the Harris recurrence of $\mathbf{Y}^L$,

$$\mathbf{E}_{\eta_t}^{Y,L}\left[A_t^{\ell}\right] = \int_{\mathbb{R}^d}\mathbf{E}_x^{Y,L}\left[A_t^{\ell}\right]\eta_t(\mathrm{d}x) \ge e^{-4\|u\|_\infty}\int_{\mathbb{R}^d}\mathbf{E}_x^{Y,L}\left[A_t^{\ell}\right]\eta(\mathrm{d}x) \longrightarrow +\infty \tag{4.14}$$

as $t\to\infty$, and hence the right hand side of (4.13) converges to 0.

Now, we prove (4.12). This can be similarly induced through a series of lemmas to prove [17, Theorem 6.5 of Chapter 6]. We address here the proof for reader's convenience. Let $\lambda(\mathrm{d}x) := e^{-2u(x)}g_Y^2(x)\ell(x)\mathrm{d}x$ and $\overline{\lambda}(\mathrm{d}x) := \lambda(1)^{-1}\lambda(\mathrm{d}x)$. To prove (4.12), it is enough to show that for any probability measures $r_t, (t \ge 0)$ and r such that $c^{-1}r \le r_t \le cr$ for a constant $c > 0$ and

$$\lim_{t\to\infty} R_t := \lim_{t\to\infty}\frac{\mathbf{E}_{r_t}^{Y,L}\left[A_t^{\ell}\right]}{\mathbf{E}_{\overline{\lambda}}^{Y,L}\left[A_t^{\ell}\right]} = 1. \tag{4.15}$$

First, we prove $\overline{\lim}_{t\to\infty}R_t \le 1$. Put $\varphi_x(t) := \mathbf{E}_x^{Y,L}\left[A_t^{\ell}\right]$ and $\varphi(t) := \mathbf{E}_{\overline{\lambda}}^{Y,L}\left[A_t^{\ell}\right]$. For any $\varepsilon > 0$, set $D_t := \{x\in\mathbb{R}^d \mid \varphi_x(t) < (1+\varepsilon)\varphi(t)\}$. Integrating the two terms of the inequality $\varphi_x(t) \ge (1+\varepsilon)\varphi(t)\mathbf{1}_{D_t^c}(x)$ by λ yields $\lambda(1)\varphi(t) \ge (1+\varepsilon)\varphi(t)\lambda(D_t^c)$. Then we see $\lambda(D_t) \ge \lambda(1)\varepsilon/(1+\varepsilon)$. By using the integration by parts, it is easy to derive that for any $f \in \mathcal{B}_+(\mathbb{R}^d)$

$$\varphi_x(t) + U^f\left(\mathbf{E}_{\cdot}^{Y,L}[\ell(X_t)]\right)(x) = U^f\left(M_f\varphi_\cdot(t)\right)(x) + U^f\ell(x). \tag{4.16}$$

Applying (4.16) to the function $f = \ell \mathbf{1}_{D_t}$, we have $\varphi_x(t) \le U^{\ell \mathbf{1}_{D_t}}(\ell \mathbf{1}_{D_t} \varphi_\cdot(t))(x) + U^{\ell \mathbf{1}_{D_t}} \ell(x)$. But $(\ell \mathbf{1}_{D_t} \varphi_\cdot(t))(x) < (1+\varepsilon)\varphi(t) \ell \mathbf{1}_{D_t}$ and since $U^{\ell \mathbf{1}_{D_t}}(\ell \mathbf{1}_{D_t}) = 1$, we get

$$\varphi_x(t) \le (1+\varepsilon)\varphi(t) + U^{\ell \mathbf{1}_{D_t}} \ell(x). \tag{4.17}$$

By the resolvent equation and (4.10)

$$\begin{aligned} U^{\ell \mathbf{1}_{D_t}} \ell(x) &= \sum_{n \ge 0} \left(U^\ell M_{\ell - \ell \mathbf{1}_{D_t}} \right)^n U^\ell \ell(x) \\ &\le \sum_{n \ge 0} (1 - \lambda(D_t))^n = \lambda(D_t)^{-1} \le \frac{1+\varepsilon}{\varepsilon \lambda(1)} := c_\varepsilon \,. \end{aligned} \tag{4.18}$$

Therefore we get

$$\mathbf{E}^{Y,L}_{r_t}\left[A^\ell_t \right] = \int_{\mathbb{R}^d} \varphi_x(t) r_t(\mathrm{d}x) \le (1+\varepsilon)\varphi(t) + c_\varepsilon,$$

which implies the desired result because $\varphi(t) \to \infty$ as $t \to \infty$.

Now we turn to the proof of $\underline{\lim}_{t\to\infty} R_t \ge 1$. For any $0 < \delta < 1$, set $E_t := \{x \in \mathbb{R}^d \mid \varphi_x(t) > (1-\delta)\varphi(t)\}$. By virtue of (4.16), it holds that

$$\begin{aligned} \varphi_x(t) &\ge U^f(M_f \varphi_\cdot(t))(x) - U^f\big(\mathbf{E}^{Y,L}_\cdot[\ell(X_t)]\big)(x) \\ &\ge (1-\delta)\varphi(t) U^f(f \mathbf{1}_{E_t})(x) - U^f\big(\mathbf{E}^{Y,L}_\cdot[\ell(X_t)]\big)(x). \end{aligned} \tag{4.19}$$

In a similar way of [17, Lemma 6.3 of Chapter 6], one can prove that there exist $p, q > 0$ such that $U^{p\ell}(p\ell \mathbf{1}_{E_t}) \ge (1-\delta)\lambda(1)^{-1}\lambda(E_t)$ and $\int_{\mathbb{R}^d}(U^{p\ell}(p\ell \mathbf{1}_{E_t}) - U^{p\ell+q}(p\ell \mathbf{1}_{E_t}))\mathrm{d}r_t \le \delta$. For $f = p\ell + q$ it then follows that

$$\int_{\mathbb{R}^d} U^f(f \mathbf{1}_{E_t}) \mathrm{d}r_t \ge (1-\delta)\lambda(1)^{-1}\lambda(E_t) - \delta \,. \tag{4.20}$$

Note that $\varphi_x(t) \le (1-\delta)\varphi(t)$ on E_t^c and $\varphi_x(t) \le (1+\varepsilon)\varphi(t) + c_\varepsilon$ (hence, it also holds on E_t) in view of (4.17) and (4.18). Integrating $\varphi_x(t)$ with respect to λ, we see

$$\begin{aligned} \lambda(1)\varphi(t) &\le (1-\delta)\varphi(t)\lambda(E_t^c) + ((1+\varepsilon)\varphi(t) + c_\varepsilon)\lambda(E_t) \\ &= (1-\delta)\varphi(t)(\lambda(1) - \lambda(E_t)) + ((1+\varepsilon)\varphi(t) + c_\varepsilon)\lambda(E_t), \end{aligned}$$

hence $\delta\lambda(1)\varphi(t) \le ((\delta+\varepsilon)\varphi(t) + c_\varepsilon)\lambda(E_t)$. Since $\varphi(t) \to \infty$ as $t \to \infty$, $\underline{\lim}_{t\to\infty}\lambda(E_t) \ge \delta\lambda(1)/(\delta+\varepsilon)$, and thus $\underline{\lim}_{t\to\infty}\lambda(E_t) \ge \lambda(1)$. Hence $\lim_{t\to\infty}\lambda(E_t) = \lambda(1)$. It then follows from (4.20) that $\underline{\lim}_{t\to\infty}\int_{\mathbb{R}^d} U^f(f\mathbf{1}_{E_t})\mathrm{d}r_t \ge 1 - 2\delta$. Since $\mathbf{E}^{Y,L}_\cdot[\ell(X_t)] \le 1$ for any $t \ge 0$, $U^f(\mathbf{E}^{Y,L}_\cdot[\ell(X_t)]) \le U^q 1 = q^{-1}$. Now, returning to (4.19), we have $\underline{\lim}_{t\to\infty} R_t \ge (1-\delta)(1-2\delta)$. Since δ is arbitrary, the proof is complete. □

Before stating our main result in this subsection, we note that the $\mathbf{P}^Y_x$-martingale MF L defined in (4.4) can be rewritten as $L_t = \frac{h(\widetilde{X}_t)}{h(x)} e_A(t)(Y_t)^{-1}$, where $h(= e^{-u} g_Y)$ is the ground state of the quadratic form $(\mathcal{Q}, \mathcal{F}_e)$. Thus we can easily see that

$$L^{(2)}_t := \frac{h(X_t)}{h(x)} e_A(t)$$

is a $\mathbf{P}_x$-martingale MF.

THEOREM 4.2. *Assume* $\mu_{\langle u\rangle}+\mu+\mu_F\in S^1_{SK}(\mathbf{X})$. *Then for any* $s\geq 0$ *and* $B\in\mathcal{F}_s$, *we have*

$$\lim_{t\to\infty}\frac{\mathbf{E}_x\left[e_A(t)\mathbf{1}_B\right]}{\mathbf{E}_x\left[e_A(t)\right]}=\mathbf{E}_x[L_s^{(2)}\mathbf{1}_B]. \tag{4.21}$$

PROOF. First, we note that for any $B\in\mathcal{F}_s,\ s\geq 0$

$$\mathbf{E}_x^Y\left[e^{A_t^{\overline{\mu}}+u(\widetilde{X}_t)-u(x)}\mathbf{1}_B\right]=\mathbf{E}_x^Y\left[e^{A_s^{\overline{\mu}}}\mathbf{1}_B e^{u(\widetilde{X}_t)-u(x)}\mathbf{E}^Y_{\widetilde{X}_s}\left[e^{A_{t-s}^{\overline{\mu}}}\right]\right].$$

Indeed, by the boundedness of u, the difference between the left hand and right hand side above is dominated by

$$e^{2\|u\|_\infty}\left|\mathbf{E}_x^Y\left[e^{A_t^{\overline{\mu}}}\mathbf{1}_B\right]-\mathbf{E}_x^Y\left[e^{A_s^{\overline{\mu}}}\mathbf{1}_B\mathbf{E}^Y_{\widetilde{X}_s}\left[e^{A_{t-s}^{\overline{\mu}}}\right]\right]\right|=0.$$

By virtue of (3.7), we then have

$$\begin{aligned}\mathbf{E}_x\left[e_A(t)\mathbf{1}_B\right]&=\mathbf{E}_x^Y\left[e^{A_t^{\overline{\mu}}+u(\widetilde{X}_t)-u(x)}\mathbf{1}_B\right]\\&=\mathbf{E}_x^Y\left[e^{A_s^{\overline{\mu}}}\mathbf{1}_B e^{u(\widetilde{X}_t)-u(x)}\mathbf{E}^Y_{\widetilde{X}_s}\left[e^{A_{t-s}^{\overline{\mu}}}\right]\right]\\&=\mathbf{E}^Y_{\nu_t}\left[e^{A_{t-s}^{\overline{\mu}}}\right].\end{aligned} \tag{4.22}$$

The right hand side of (4.22) can be represented by the special AF $A_t^{\overline{\mu}/g_Y}:=(\frac{1}{g_Y}\cdot A^{\overline{\mu}})_t$ with respect to $\mathbf{Y}^L$ appeared in Proposition 4.2:

$$\begin{aligned}\mathbf{E}^Y_{\nu_t}\left[e^{A_{t-s}^{\overline{\mu}}}\right]&=\nu_t(\mathbb{R}^d)+\mathbf{E}^Y_{\nu_t}\left[\int_0^{t-s}e^{A_{s'}^{\overline{\mu}}}\mathrm{d}A_{s'}^{\overline{\mu}}\right]\\&=\nu_t(\mathbb{R}^d)+\left\{\int_{\mathbb{R}^d}g_Y\mathrm{d}\nu_t\right\}\mathbf{E}^{Y,L}_{\eta_t}\left[A_{t-s}^{\overline{\mu}/g_Y}\right].\end{aligned} \tag{4.23}$$

For a positive Borel function ℓ with compact support on $\mathbb{R}^d$, put $\psi(t):=\mathbf{E}^{Y,L}_{\eta_t}\left[A_t^{\ell}\right]$. As we showed in (4.14), $\lim_{t\to\infty}\psi(t)=+\infty$ and $\lim_{t\to\infty}\psi(t-s)/\psi(t)=1$ for any $s\geq 0$, by the Harris recurrence of $\mathbf{Y}^L$. By the transience of $\mathbf{Y}$ and the dominated convergence theorem, as $t\to\infty$

$$\int_{\mathbb{R}^d}g_Y(x)\nu_t(\mathrm{d}x)\ \longrightarrow\ \int_{\mathbb{R}^d}g_Y(x)\nu(\mathrm{d}x)=e^{-u(x)}\mathbf{E}_x^Y\left[e^{A_s^{\overline{\mu}}}\mathbf{1}_B g_Y(\widetilde{X}_s)\right]$$

and

$$\frac{\nu_t(\mathbb{R}^d)}{\psi(t-s)}\leq\frac{e^{-2\|u\|_\infty}\nu(\mathbb{R}^d)}{\psi(t-s)}\ \longrightarrow\ 0,$$

we have

$$\begin{aligned}\lim_{t\to\infty}\frac{\mathbf{E}_x\left[e_A(t)\mathbf{1}_B\right]}{\psi(t)}&=\lim_{t\to\infty}\frac{\mathbf{E}^Y_{\nu_t}\left[e^{A_{t-s}^{\overline{\mu}}}\right]}{\psi(t)}=\lim_{t\to\infty}\frac{\mathbf{E}^Y_{\nu_t}\left[e^{A_{t-s}^{\overline{\mu}}}\right]}{\psi(t-s)}\frac{\psi(t-s)}{\psi(t)}\\&=e^{-u(x)}\mathbf{E}_x^Y\left[e^{A_s^{\overline{\mu}}}\mathbf{1}_B g_Y(\widetilde{X}_s)\right]\frac{\int_{\mathbb{R}^d}e^{-2u(x)}g_Y(x)\overline{\mu}(\mathrm{d}x)}{\int_{\mathbb{R}^d}e^{-2u(x)}g_Y^2(x)\ell(x)\mathrm{d}x}\end{aligned} \tag{4.24}$$

in view of (4.11) and (4.23).

Next, let $\phi(t) := \mathbf{E}^{Y,L}_{\kappa_t}\left[A^{\ell}_t\right]$. In a similar way of (4.22), (4.23) and (4.24), we also have

$$
\begin{aligned}
\lim_{t\to\infty}\frac{\mathbf{E}_x\left[e_A(t)\right]}{\phi(t)} &= \lim_{t\to\infty}\frac{\mathbf{E}^{Y}_{\theta_t}\left[e^{A^{\overline{\mu}}_{t-s}}\right]}{\phi(t-s)}\frac{\phi(t-s)}{\phi(t)} \\
&= e^{-u(x)}\mathbf{E}^Y_x\left[e^{A^{\overline{\mu}}_s}g_Y(\widetilde{X}_s)\right]\frac{\int_{\mathbb{R}^d}e^{-2u(x)}g_Y(x)\overline{\mu}(\mathrm{d}x)}{\int_{\mathbb{R}^d}e^{-2u(x)}g_Y^2(x)\ell(x)\mathrm{d}x} \qquad (4.25)\\
&= e^{-u(x)}g_Y(x)\frac{\int_{\mathbb{R}^d}e^{-2u(x)}g_Y(x)\overline{\mu}(\mathrm{d}x)}{\int_{\mathbb{R}^d}e^{-2u(x)}g_Y^2(x)\ell(x)\mathrm{d}x}.
\end{aligned}
$$

Hence we have by (4.12), (4.24) and (4.25) that

$$
\begin{aligned}
\lim_{t\to\infty}\frac{\mathbf{E}_x\left[e_A(t)\mathbf{1}_B\right]}{\mathbf{E}_x\left[e_A(t)\right]} &= \lim_{t\to\infty}\frac{\mathbf{E}_x\left[e_A(t)\mathbf{1}_B\right]}{\psi(t)}\left(\frac{\mathbf{E}_x\left[e_A(t)\right]}{\phi(t)}\right)^{-1}\frac{\psi(t)}{\phi(t)} \\
&= \frac{1}{g_Y(x)}\mathbf{E}^Y_x\left[e^{A^{\overline{\mu}}_s}\mathbf{1}_B g_Y(\widetilde{X}_s)\right] = \mathbf{E}^Y_x[L_s\mathbf{1}_B] = \mathbf{E}_x[L^{(2)}_s\mathbf{1}_B]\,.
\end{aligned}
$$

The proof is complete. □

4.3. The case $\lambda(\overline{\mu}) < 0$. In this case, **X** can be transient or recurrent. As we mentioned in the beginning of Section 4, $\lambda(\overline{\mu}) = -1$ whenever **X** is recurrent. We will treat in this subsection the recurrent case. The transient case can be treated in the same way of the recurrent case.

For $\mu_{\langle u\rangle} + \mu + \mu_F \in S^1_{K^+_\infty}(\mathbf{X})$ and $\beta \geq 0$, we define

$$
\lambda_\beta(\overline{\mu}) := \inf\left\{\mathcal{Q}(f,f) + \beta\int_{\mathbb{R}^d}f^2\mathrm{d}x \;\middle|\; f\in\mathcal{F}\cap C_\infty(\mathbb{R}^d),\ \int_{\mathbb{R}^d}f^2\mathrm{d}\overline{\mu} = 1\right\}. \qquad (4.26)
$$

It is easy to see that the function $\lambda_\cdot(\overline{\mu})$ is increasing and concave. Moreover it satisfies $\lim_{\beta\to\infty}\lambda_\beta(\overline{\mu}) = \infty$. Indeed, by the Stollmann-Voigt's inequality ([19, Theorem 3.1]) with respect to $(\mathcal{E}^Y, \mathcal{F})$ and the relation (3.10), we see that for $f \in \mathcal{F}\cap C_\infty(\mathbb{R}^d)$ with $\int_{\mathbb{R}^d} f^2\mathrm{d}\overline{\mu} = 1$

$$
\begin{aligned}
\mathcal{Q}(f,f) + \beta\int_{\mathbb{R}^d}f^2\mathrm{d}x &= \mathcal{E}^Y(fe^u, fe^u) + \beta\int_{\mathbb{R}^d}f^2\mathrm{d}x - 1 \\
&\geq \left\|R^Y_\beta\big(e^{-2u}\overline{\mu}\big)\right\|^{-1}_\infty - 1 \longrightarrow +\infty\,,\quad \text{as } \beta\to\infty
\end{aligned}
$$

because of $e^{-2u}\overline{\mu} \in S^1_{K^+_\infty}(\mathbf{Y}) \subset S^1_K(\mathbf{Y})$ by Lemma 3.1(2). Then there exists $\beta_0 > 0$ such that

$$
\lambda_{\beta_0}(\overline{\mu}) := \inf\left\{\mathcal{Q}(f,f) + \beta_0\int_{\mathbb{R}^d}f^2\mathrm{d}x \;\middle|\; f\in\mathcal{F}\cap C_\infty(\mathbb{R}^d)\ \int_{\mathbb{R}^d}f^2\mathrm{d}\overline{\mu} = 1\right\} = 0\,. \qquad (4.27)
$$

By the same reason as we proved in the end of Section 3 there exists a minimizer $h \in \mathcal{F}\cap C_\infty(\mathbb{R}^d)(\subset L^2(\mathbb{R}^d))$ of (4.27), that is, $\lambda_{\beta_0}(\overline{\mu}) = \mathcal{Q}(h,h) + \beta_0\int_{\mathbb{R}^d}h^2\mathrm{d}x = \mathcal{E}^Y(he^u, he^u) + \beta_0\int_{\mathbb{R}^d}h^2\mathrm{d}x - 1 = 0$. Note that h is a harmonic function in the sense that $\mathcal{H}h = \beta_0 h$. So h

satisfies $P_t^A h := \mathbf{E}_x[e_A(t)h(X_t)] = e^{t\mathcal{H}}h = e^{\beta_0 t}h$, equivalently, $e^{-\beta_0 t}P_t^A h = h$. Therefore we see that the MF $L^{(3)}$ defined by

$$L_t^{(3)} := e^{-\beta_0 t}\frac{h(X_t)}{h(x)}e_A(t)$$

is a $\mathbf{P}_x$-martingale MF.

Let $\mathbf{X}^h = (\Omega, \mathcal{F}_\infty, \mathcal{F}_t, X_t, \mathbf{P}_x^h)$ be the transformed process of $\mathbf{X}$ by $L^{(3)}$. In view of [4, Theorem 2.6(b)], $\mathbf{X}^h$ is an $h^2\mathrm{d}x$-symmetric recurrent process on $\mathbb{R}^d$.

LEMMA 4.3. *Assume $\mu_{\langle u\rangle} + \mu + \mu_F \in S^1_{K_\infty^+}(\mathbf{X})$. Let h and $\beta_0 > 0$ be the minimizer and the constant appeared in* (4.27)*, respectively. Then*

$$\lim_{t\to\infty} e^{-\beta_0 t}\mathbf{E}_x[e_A(t)] = h(x)\int_{\mathbb{R}^d} h(x)\mathrm{d}x \tag{4.28}$$

for all $x \in \mathbb{R}^d$.

PROOF. Note that $e^{-\beta_0 t}\mathbf{E}_x[e_A(t)] = h(x)\mathbf{E}_x^h[1/h(X_t)]$. Then we see from [9, Theorem 1] that the assertion (4.28) holds for a.e. $x \in \mathbb{R}^d$. So, it suffices to show that (4.28) holds for all $x \in \mathbb{R}^d$. This can be deduced through a series of lemmas to prove [20, Corollary 4.7] in a similar way. To this end, one may need the upper estimate of (2.1) and the fact that the function $g_Y := he^u$ satisfies for any $q > 1$ and $C_{10} > C_9 > 0$

$$\frac{C_9}{|x|^{d+\alpha}} \le g_Y(x) \le \frac{C_{10}}{|x|^{(d+\alpha)/q}} \quad \text{for } |x| > 1\,.$$

We omit the details. □

LEMMA 4.4. *Let $\mathbf{Q}_{x,t}^A$ be the probability measure defined by* (1.2) *with the Feynman-Kac transforms* (1.1)*. Assume for any $s \ge 0$ and $x \in \mathbb{R}^d$,*

$$\frac{\mathbf{E}_x[e_A(t) \mid \mathcal{F}_s]}{\mathbf{E}_x[e_A(t)]} \quad \overset{a.e.}{\longrightarrow} \quad L_s^{(3)}\,, \quad \text{as } t \to \infty\,. \tag{4.29}$$

Then for any $B \in \mathcal{F}_s$, $\mathbf{Q}_{x,t}^A(B) \longrightarrow \mathbf{Q}_{x,\infty}^A(B) := \mathbf{E}_x[L_s^{(3)}\mathbf{1}_B]$ as $t \to \infty$.

The above result is a direct consequence of Scheffe's lemma (cf. [18, Theorem 2.1]). Thus under the condition (4.29) the convergence of $\mathbf{E}_x[e_A(t)]^{-1}\mathbf{E}_x[e_A(t) \mid \mathcal{F}_s]$ also holds in $L^1(\mathbb{R}^d)$.

THEOREM 4.3. *Assume $\mu_{\langle u\rangle} + \mu + \mu_F \in S^1_{K_\infty^+}(\mathbf{X})$ ($S^1_{K_\infty}(\mathbf{X})$ in the transient case). Then for any $s \ge 0$ and $B \in \mathcal{F}_s$, we have*

$$\lim_{t\to\infty}\frac{\mathbf{E}_x\,[e_A(t)\mathbf{1}_B]}{\mathbf{E}_x\,[e_A(t)]} = \mathbf{E}_x[L_s^{(3)}\mathbf{1}_B]\,. \tag{4.30}$$

PROOF. The proof is an easy consequence of Lemma 4.3 and Lemma 4.4. Indeed, by noting

$$\frac{\mathbf{E}_x[e_A(t) \mid \mathcal{F}_s]}{\mathbf{E}_x[e_A(t)]} = \frac{e^{-\beta_0 s}e_A(s)e^{-\beta_0(t-s)}\mathbf{E}_{X_s}[e_A(t-s)]}{e^{-\beta_0 t}\mathbf{E}_x[e_A(t)]}$$

$$\longrightarrow \frac{e^{-\beta_0 s} e_A(s) h(X_s) \int_{\mathbb{R}^d} h(x) \mathrm{d}x}{h(x) \int_{\mathbb{R}^d} h(x) \mathrm{d}x} = L_s^{(3)} \quad \text{as } t \to \infty,$$

and $\mathbf{E}_x[e_A(t)\mathbf{1}_B]/\mathbf{E}_x[e_A(t)] = \mathbf{E}_x[\mathbf{1}_B \mathbf{E}_x[e_A(t) \mid \mathcal{F}_s]/\mathbf{E}_x[e_A(t)]]$, we get (4.30). □

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