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SKT AND TAMED SYMPLECTIC STRUCTURES ON SOLVMANIFOLDS

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Abstract. We study the existence of strong Kähler with torsion (SKT) metrics and of symplectic forms taming invariant complex structures J on solvmanifolds G/Γ providing some negative results for some classes of solvmanifolds. In particular, we show that if either J is invariant under the action of a nilpotent complement of the nilradical of G or J is abelian or G is almost abelian (not of type (I)), then the solvmanifold G/Γ cannot admit any symplectic form taming the complex structure J, unless G/Γ is Kähler. As a consequence, we show that the family of non-Kähler complex manifolds constructed by Oeljeklaus and Toma cannot admit any symplectic form taming the complex structure.

1. Introduction. A symplectic form Ω on a complex manifold (M, J) is said *taming* the complex structure J if

$$\Omega(X, JX) > 0$$

for any non-zero vector field X on M or, equivalently, if the $(1, 1)$-part of Ω is positive. The pair (Ω, J) was called in [28] a *Hermitian-symplectic* structure and it was shown that these structures appear as static solutions of the so-called *pluriclosed flow*. By [22, 28] a compact complex surface admitting a Hermitian-symplectic structure is necessarily Kähler (see also Proposition 3.3 in [13]) and it follows from [26] that non-Kähler Moishezon complex structures on compact manifolds cannot be tamed by a symplectic form (see also [31]). However, it is still an open problem to find out an example of a compact Hermitian-symplectic manifold non admitting Kähler structures. It is well known that Hermitian-symplectic structures can be viewed as special strong Kähler with torsion structures ([15]) and that their existence can be characterized in terms of currents ([29]). Here we recall that a Hermitian metric is called *strong Kähler with torsion* (SKT) if its fundamental form is $\partial\overline{\partial}$-closed (see for instance [17, 7] and the references therein). SKT nilmanifolds were first studied in [16] in six dimension and recently in [15] in any dimension, where by *nilmanifold* we mean a compact quotient of a simply connected nilpotent Lie group G by a co-compact lattice Γ. Very few results are known for the existence of SKT metrics on solvmanifolds endowed with an invariant complex structure. By *solvmanifold* G/Γ we mean a compact quotient of a simply connected solvable Lie group G by a lattice Γ and by *invariant complex structure* on G/Γ we mean a complex structure induced by a left invariant complex structure on G. We will call a solvmanifold endowed with an invariant complex structure a *complex solvmanifold*.

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From [15] it is known that a nilmanifold G/Γ endowed with an invariant complex structure J cannot admit any symplectic form taming J unless it admits a Kähler structure (or equivalently G/Γ is a complex torus). Then it is quite natural trying to extend the result to complex solvmanifolds.

By [18] a solvmanifold G/Γ admits a Kähler structure if and only if it is a finite quotient of a complex torus. This in particular implies that when G is not of type (I) and non abelian, then G/Γ is not Kähler. We recall that being of type (I) means that for any $X \in \mathfrak{g}$ all eigenvalues of the adjoint operator ad_X are pure imaginary.

Given a solvable Lie algebra $\mathfrak{g}$ we denote by $\mathfrak{n}$ its *nilradical* which is defined as the *maximal nilpotent ideal* of $\mathfrak{g}$. It is well known that there always exists a *nilpotent complement* $\mathfrak{c}$ of $\mathfrak{n}$ in $\mathfrak{g}$, i.e., there exists a nilpotent subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{c} + \mathfrak{n}$ (see [10, Theorem 2.2]). In general the complement $\mathfrak{c}$ is not unique and we do not expect to have a direct sum between $\mathfrak{c}$ and $\mathfrak{n}$.

The first main result of the paper consists in proving the following theorem about the non-existence of Hermitian-symplectic and SKT structures on homogeneous spaces of splitting Lie groups.

THEOREM 1.1. *Let G be a Lie group endowed with a left-invariant complex structure J and suppose that*

1) *the Lie algebra $\mathfrak{g}$ of G is a semidirect product $\mathfrak{g} = \mathfrak{s} \ltimes_\phi \mathfrak{h}$, where $\mathfrak{s}$ is a solvable Lie algebra and $\mathfrak{h}$ a Lie algebra;*
2) *$\phi : \mathfrak{s} \to \mathrm{Der}\,(\mathfrak{h})$ is a representation on the space of derivations of $\mathfrak{h}$;*
3) *ϕ is not of type* (I) *and the image $\phi(\mathfrak{s})$ is a nilpotent subalgebra of* $\mathrm{Der}\,(\mathfrak{h})$*;*
4) *$J(\mathfrak{h}) \subset \mathfrak{h}$;*
5) *$J_{|\mathfrak{h}} \circ \phi(X) = \phi(X) \circ J_{|\mathfrak{h}}$ for any $X \in \mathfrak{s}$.*

Then $\mathfrak{g}$ does not admit any symplectic structure taming J. Moreover if $\mathfrak{s}$ is nilpotent and $J(\mathfrak{s}) \subset \mathfrak{s}$, then $\mathfrak{g}$ does not admit any J-Hermitian SKT metric.

The previous theorem can be in particular applied to compact homogeneous complex spaces of the form $(G/\Gamma, J)$, where (G, J) satisfies conditions 1), . . . , 5) in the theorem and Γ is a discrete subgroup of G. This type of homogeneous spaces covers a large class of examples including the so-called Oeljeklaus-Toma manifolds (see [23]).

In general a simply connected solvable Lie group is not of splitting type (i.e., its Lie algebra does not satisfy conditions 1), 2), 3) of Theorem 1.1). The following theorem provides a non-existence result in the non-splitting case.

THEOREM 1.2. *Let $(G/\Gamma, J)$ be a complex solvmanifold. Assume that J is invariant under the action of a nilpotent complement of the nilradical $\mathfrak{n}$. Then G/Γ admits a symplectic form taming J if and only if $(G/\Gamma, J)$ is Kähler.*

A special class of invariant complex structures on solvmanifolds is provided by *abelian complex structures* (see [4]). A complex structure J on a Lie algebra $\mathfrak{g}$ is called *abelian* if $[JX, JY] = [X, Y]$ for every $X, Y \in \mathfrak{g}$. In the abelian case the Lie subalgebra $\mathfrak{g}^{1,0}$ of the complexification $\mathfrak{g}_{\mathbb{C}}$ of $\mathfrak{g}$ is abelian and that motivates the name. In Section 6 we will prove the following

THEOREM 1.3. *Let $(G/\Gamma, J)$ be a solvmanifold endowed with an invariant* abelian *complex structure J. Then $(G/\Gamma, J)$ doesn't admit a symplectic form taming J unless it is a complex torus.*

In the last section of the paper we take into account solvmanifolds G/Γ with G *almost-abelian*. The almost-abelian condition means that the nilradical $\mathfrak{n}$ of the Lie algebra $\mathfrak{g}$ of G has codimension 1 and $\mathfrak{n}$ is abelian. About this case we will prove the following

THEOREM 1.4. *Let $(G/\Gamma, J)$ be a complex solvmanifold with G almost-abelian. Assume $\mathfrak{g}$ being either not of type* (I) *or* 6*-dimensional. Then $(G/\Gamma, J)$ does not admit any symplectic form taming J.*

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2. Preliminary results on representations of Lie algebras. In this section we prove some preliminary results which will be useful in the sequel.

2.1. Representations of solvable Lie algebras. Let $\mathfrak{g}$ be a solvable Lie algebra and let $\rho : \mathfrak{g} \to \mathrm{End}(V)$ be a representation on a real vector space V whose image $\rho(\mathfrak{g})$ is a nilpotent subalgebra of $\mathrm{End}(V)$. For every $X \in \mathfrak{g}$ we can consider the Jordan decomposition

$$\rho(X) = (\rho(X))_s + (\rho(X))_n$$

which induces two maps ρ_s and ρ_n from $\mathfrak{g}$ onto $\mathrm{End}(V)$. The following facts can be easily deduced from [11]:

- The maps $\rho_s : \mathfrak{g} \ni X \mapsto (\rho(X))_s \in \mathrm{End}(V)$ and $\rho_n : \mathfrak{g} \ni X \mapsto (\rho(X))_n \in \mathrm{End}(V)$ are Lie algebra homomorphisms.
- The images $\rho_s(\mathfrak{g})$ and $\rho_n(\mathfrak{g})$ are subalgebras of $\mathrm{End}(V)$ satisfying $[\rho_s(\mathfrak{g}), \rho_n(\mathfrak{g})] = 0$.

For a real-valued character α of $\mathfrak{g}$, we denote

$$V_\alpha(V) = \{v \in V \ : \ \rho_s(X)v = \alpha(X)v \text{ for every } X \in \mathfrak{g}\}\,,$$

and for a complex-valued character α of $\mathfrak{g}$ we set

$$V_\alpha(V_{\mathbb{C}}) = \{v \in V_{\mathbb{C}} \ : \ \rho_s(X)v = \alpha(X)v \text{ for every } X \in \mathfrak{g}\}\,.$$

When α is real we have $V_\alpha(V_{\mathbb{C}}) = V_\alpha(V) \otimes \mathbb{C}$. From the condition $[\rho_s(\mathfrak{g}), \rho_n(\mathfrak{g})] = 0$, we get

$$\rho(X)\big(V_\alpha(V_{\mathbb{C}})\big) \subset V_\alpha(V_{\mathbb{C}})\,,$$

for any $X \in \mathfrak{g}$ (see [25]). Moreover, as a consequence of the Lie theorem, there exits a basis of $V_\alpha(V_{\mathbb{C}})$ such that for any $X \in \mathfrak{c}$ the map $\rho(X)$ is represented by an upper triangular matrix

$$\begin{pmatrix} \alpha & & * \\ & \ddots & \\ 0 & & \alpha \end{pmatrix}.$$

Therefore we obtain a decomposition

$$V_{\mathbb{C}} = V_{\alpha_1}(V_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}(V_{\mathbb{C}})$$

with $\alpha_1, \ldots, \alpha_n$ characters of $\mathfrak{g}$. Since ρ is a real-valued representation, the set $\{\alpha_1, \ldots, \alpha_n\}$ is invariant under complex conjugation (i.e., $\overline{\alpha}_i \in \{\alpha_1, \ldots, \alpha_n\}$). We recall the following

DEFINITION 2.1. A representation ρ of $\mathfrak{g}$ is of *type* (I) if for any $X \in \mathfrak{g}$ all the eigenvalues of $\rho(X)$ are pure imaginary.

The following lemma will be very useful in the sequel:

LEMMA 2.2. *Let $\mathfrak{h}$ and $\mathfrak{g}$ be Lie algebras with $\mathfrak{g}$ solvable. Let $\rho : \mathfrak{g} \to D(\mathfrak{h})$ be a representation on the space of derivations on $\mathfrak{h}$ which we assume to not be of type* (I). *Then there exists a complex character α of $\mathfrak{g}$ satisfying*

$$\text{(2.1)} \qquad \mathrm{Re}(\alpha) \neq 0\,, \quad V_\alpha(\mathfrak{h}_{\mathbb{C}}) \neq 0 \text{ and } [V_\alpha(\mathfrak{h}_{\mathbb{C}}), V_{\overline{\alpha}}(\mathfrak{h}_{\mathbb{C}})] = 0\,.$$

PROOF. Since ρ is assumed to be not of type (I), then there exits a complex character α_1 such that $\mathrm{Re}(\alpha_1) \neq 0$ and $V_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) \neq 0$. If $[V_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}), V_{\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}})] = 0$, then α_1 satisfies the three conditions required. Otherwise, since $\rho_s : \mathfrak{g} \ni X \mapsto (ad_X)_s \in D(\mathfrak{h})$, we have $0 \neq [V_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}), V_{\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}})] \subset V_{\alpha_1+\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}}) \neq 0$ and we take $\alpha_2 = \alpha_1 + \overline{\alpha}_1 = 2\,\mathrm{Re}(\alpha_1)$. Again if $[V_{\alpha_2}(\mathfrak{h}_{\mathbb{C}}), V_{\overline{\alpha}_2}(\mathfrak{h}_{\mathbb{C}})] = 0$, then α_2 satisfies all the conditions required, otherwise we have $0 \neq [V_{\alpha_2}(\mathfrak{h}_{\mathbb{C}}), V_{\alpha_2}(\mathfrak{h}_{\mathbb{C}})] \subset V_{2\alpha_2}(\mathfrak{h}_{\mathbb{C}}) \neq 0$ and we consider $\alpha_3 = 2\alpha_2$. We claim that we can iterate this operation until we get a character α_k satisfying (2.1). Indeed, since $\mathfrak{h}$ is finite dimensional, we have a sequence of characters

$$\alpha_2,\ \alpha_3 = 2\alpha_2,\ \alpha_4 = 2\alpha_3, \ldots, \alpha_k = 2\alpha_{k-1}$$

such that $V_{\alpha_s}(\mathfrak{h}_{\mathbb{C}}) \neq 0$ and $[V_{\alpha_s}(\mathfrak{h}_{\mathbb{C}}), V_{\alpha_s}(\mathfrak{h}_{\mathbb{C}})] \neq 0$ for $2 \leq s \leq k-1$, and $V_{\alpha_k}(\mathfrak{h}_{\mathbb{C}}) \neq 0$ and $[V_{\alpha_k}(\mathfrak{h}_{\mathbb{C}}), V_{\alpha_k}(\mathfrak{h}_{\mathbb{C}})] = 0$. Hence the claim follows. □

2.2. Nilpotent complements of nilradicals of solvable Lie algebras. Let $\mathfrak{g}$ be a solvable Lie algebra with nilradical $\mathfrak{n}$. As remarked in the introduction there always exists a nilpotent subalgebra $\mathfrak{c}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{c} + \mathfrak{n}$ (not necessarily a direct sum) (see [10, Theorem 2.2]). Such a nilpotent subalgebra $\mathfrak{c}$ is called a *nilpotent complement* of $\mathfrak{n}$. Let us consider $ad : \mathfrak{c} \to \mathrm{Der}(\mathfrak{g})$ and the semisimple $ad_s : \mathfrak{c} \ni C \mapsto (ad_C)_s \in \mathrm{Der}(\mathfrak{g})$ and the nilpotent part $ad_n : \mathfrak{c} \ni C \mapsto (ad_C)_n \in \mathrm{Der}(\mathfrak{g})$ of ad. Then ad_s and ad_n are homomorphisms from $\mathfrak{c}$. Since $\ker ad_s = \mathfrak{c}/\mathfrak{c} \cap \mathfrak{n} \cong \mathfrak{g}/\mathfrak{n}$, ad_s can be regarded as a homomorphism from $\mathfrak{g}$. For a real-valued character α of $\mathfrak{g}$, we denote

$$V_\alpha(\mathfrak{g}) = \{X \in \mathfrak{g} \ : \ ad_{sY} X = \alpha(Y)X \text{ for every } Y \in \mathfrak{g}\}\,,$$

and for a complex-valued character α,

$$V_\alpha(\mathfrak{g}_{\mathbb{C}}) = \{X \in \mathfrak{g}_{\mathbb{C}} \ : \ ad_{sY} X = \alpha(Y)X \text{ for every } Y \in \mathfrak{g}\}\,.$$

If α is real valued we have $V_\alpha(\mathfrak{g}_{\mathbb{C}}) = V_\alpha(\mathfrak{g}) \otimes \mathbb{C}$. Since $\mathfrak{c}$ is nilpotent, we have $ad_C(V_\alpha(\mathfrak{g}_{\mathbb{C}})) \subset V_\alpha(\mathfrak{g}_{\mathbb{C}})$ for any $C \in \mathfrak{c}$. We can take a basis of $V_\alpha(\mathfrak{g}_{\mathbb{C}})$ such that ad_C is represented as an upper triangular matrix

$$\begin{pmatrix} \alpha & & * \\ & \ddots & \\ 0 & & \alpha \end{pmatrix},$$

for any $C \in \mathfrak{c}$. Then we obtain a decomposition

$$\mathfrak{g}_{\mathbb{C}} = V_{\mathbf{0}}(\mathfrak{g}_{\mathbb{C}}) \oplus V_{\alpha_1}(\mathfrak{g}_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}(\mathfrak{g}_{\mathbb{C}})$$

where $\mathbf{0}$ is the trivial character and $\alpha_1, \dots, \alpha_n$ are some non-trivial characters. We also consider

$$\mathfrak{n}_{\mathbb{C}} = V_{\mathbf{0}}(\mathfrak{n}_{\mathbb{C}}) \oplus V_{\alpha_1}(\mathfrak{n}_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}(\mathfrak{n}_{\mathbb{C}}) .$$

Since $\mathfrak{c}$ is nilpotent, $\mathfrak{c}$ acts nilpotently on itself via ad. Hence we have $\mathfrak{c} \subset V_{\mathbf{0}}(\mathfrak{g}_{\mathbb{C}})$ and $V_{\alpha_i}(\mathfrak{n}_{\mathbb{C}}) = V_{\alpha_i}(\mathfrak{g}_{\mathbb{C}})$ by $\mathfrak{g} = \mathfrak{c} + \mathfrak{n}$ for each i and we get the decomposition

$$\mathfrak{n}_{\mathbb{C}} = V_{\mathbf{0}}(\mathfrak{n}_{\mathbb{C}}) \oplus V_{\alpha_1}(\mathfrak{g}_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}(\mathfrak{g}_{\mathbb{C}}) .$$

DEFINITION 2.3. We say that a solvable Lie algebra $\mathfrak{g}$ is of type (I) if for any $X \in \mathfrak{g}$ all the eigenvalues of the adjoint operator ad_X are pure imaginary.

Note that if we write $\mathfrak{g} = \mathfrak{c} + \mathfrak{n}$, where $\mathfrak{c}$ is an abelian complement of the nilradical $\mathfrak{n}$, then $\mathfrak{g}$ is of type (I) if and only if the representation $ad : \mathfrak{c} \to \mathrm{Der}(\mathfrak{n})$ is of type (I). The following lemma is readily implied by Lemma 2.2.

LEMMA 2.4. *If $\mathfrak{g}$ is a solvable Lie algebra which is not of type* (I). *Then there exists a character α satisfying*

$$\mathrm{Re}(\alpha) \neq 0 , \quad V_\alpha(\mathfrak{g}_{\mathbb{C}}) \neq 0 , \ \textit{and} \ [V_\alpha(\mathfrak{g}_{\mathbb{C}}), V_{\overline{\alpha}}(\mathfrak{g}_{\mathbb{C}})] = 0 .$$

3. Proof of Theorem 1.1. In this section we provide a proof of Theorem 1.1. The following easy-proof lemma will be useful in the sequel:

LEMMA 3.1. *Let $\mathfrak{g}$ be a nilpotent Lie algebra and let θ be a closed* 1*-form on $\mathfrak{g}$. Then a* 1*-form η solves $d\eta - \eta \wedge \theta = 0$ if and only if it is multiple of θ.*

PROOF. Consider the differential operator $d + \theta\wedge$ acting on $\bigwedge \mathfrak{g}^*$. Then it is known that the cohomology of $\bigwedge \mathfrak{g}^*$ with respect to $(d + \theta\wedge)$ is trivial (see [12]). Hence if $\eta \in \bigwedge^1 \mathfrak{g}^*$ solves $d\eta - \eta \wedge \theta = 0$, then η is $(d + \theta\wedge)$-exact and so $\eta \in \mathrm{span}_{\mathbb{R}}\langle\theta\rangle$, as required. □

Now we are ready to prove Theorem 1.1.

Proof of Theorem 1.1. Firstly we have

$$\mathfrak{h}_{\mathbb{C}} = V_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}(\mathfrak{h}_{\mathbb{C}})$$

where $\alpha_1, \dots, \alpha_n$ are some characters of $\mathfrak{s}$. Therefore $\mathfrak{g}_{\mathbb{C}}$ splits as

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{s}_{\mathbb{C}} \oplus V_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}(\mathfrak{h}_{\mathbb{C}}) .$$

Then we get

$$[\mathfrak{s}, V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})] \subset V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})$$

and

$$J V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}) \subset V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})$$

since $J_{|\mathfrak{h}} \circ \phi(X) = \phi(X) \circ J_{|\mathfrak{h}}$ for any $X \in \mathfrak{s}$. In view of Lemma 2.4, we may assume that α_1 satisfies

$$\mathrm{Re}(\alpha_1) \neq 0\,, \quad V_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) \neq 0\,, \text{ and } [V_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}), V_{\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}})] = 0$$

and we can write

$$\bigwedge \mathfrak{g}^*_{\mathbb{C}} = \bigwedge \left(\mathfrak{s}^*_{\mathbb{C}} \oplus V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) \oplus \cdots \oplus V^*_{\alpha_n}(\mathfrak{h}_{\mathbb{C}})\right)\,.$$

Then we have

$$d(\mathfrak{s}^*_{\mathbb{C}}) = \mathfrak{s}^*_{\mathbb{C}} \wedge \mathfrak{s}^*_{\mathbb{C}}\,,$$

and by $[\mathfrak{s}, V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})] \subset V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})$ and $[V_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}), V_{\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}})] = 0$, we obtain

$$d(V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})) \subset \mathfrak{s}^*_{\mathbb{C}} \wedge V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}) \oplus \bigoplus_{(\alpha_k,\alpha_l)\neq(\alpha_1,\overline{\alpha}_1)} V^*_{\alpha_k}(\mathfrak{h}_{\mathbb{C}}) \wedge V^*_{\alpha_l}(\mathfrak{h}_{\mathbb{C}})\,.$$

Moreover

$$d(\mathfrak{s}^*_{\mathbb{C}} \wedge V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}) \subset \mathfrak{s}^*_{\mathbb{C}} \wedge \mathfrak{s}^*_{\mathbb{C}} \wedge V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}) \oplus \bigoplus_{(\alpha_k,\alpha_l)\neq(\alpha_1,\overline{\alpha}_1)} \mathfrak{s}^*_{\mathbb{C}} \wedge V^*_{\alpha_k}(\mathfrak{h}_{\mathbb{C}}) \wedge V^*_{\alpha_l}(\mathfrak{h}_{\mathbb{C}})$$

and

$$d(V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}) \wedge V^*_{\alpha_j}(\mathfrak{h}_{\mathbb{C}})) \subset \mathfrak{s}^*_{\mathbb{C}} \wedge V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}) \wedge V^*_{\alpha_j}(\mathfrak{h}_{\mathbb{C}}) + \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}}\,.$$

By these relations, we deduce:

($\star_1$) *the* 3*-forms which belongs to the space* $\mathfrak{s}^*_{\mathbb{C}} \wedge V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) \wedge V^*_{\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}})$ *cannot appear in the spaces* $d(\mathfrak{s}^*_{\mathbb{C}}\wedge\mathfrak{s}^*_{\mathbb{C}})$, $d(\mathfrak{s}^*_{\mathbb{C}}\wedge V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}))$ *and* $d(V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})\wedge V^*_{\alpha_j}(\mathfrak{h}_{\mathbb{C}}))$*, excepting* $d(V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}})\wedge V^*_{\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}}))$.

Consider the operator $d^c = J^{-1}dJ$. Then, assuming $J\mathfrak{s} \subset \mathfrak{s}$, we have

$$dd^c(\mathfrak{s}^*_{\mathbb{C}} \wedge V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})) \subset \mathfrak{s}^*_{\mathbb{C}}\wedge\mathfrak{s}^*_{\mathbb{C}}\wedge\mathfrak{s}^*_{\mathbb{C}}\wedge V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}) \bigoplus_{(\alpha_k,\alpha_l)\neq(\alpha_1,\overline{\alpha}_1)} \mathfrak{s}^*_{\mathbb{C}}\wedge\mathfrak{s}^*_{\mathbb{C}}\wedge V^*_{\alpha_k}(\mathfrak{h}_{\mathbb{C}})\wedge V^*_{\alpha_l}(\mathfrak{h}_{\mathbb{C}}) \oplus \mathfrak{s}^*_{\mathbb{C}}\wedge\mathfrak{h}^*_{\mathbb{C}}\wedge\mathfrak{h}^*_{\mathbb{C}}\wedge h^*_{\mathbb{C}}$$

and

$$dd^c(V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}) \wedge V^*_{\alpha_j}(\mathfrak{h}_{\mathbb{C}})) \subset \mathfrak{s}^*_{\mathbb{C}} \wedge \mathfrak{s}^*_{\mathbb{C}} \wedge V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}) \wedge V^*_{\alpha_j}(\mathfrak{h}_{\mathbb{C}}) \oplus \mathfrak{s}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \oplus \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}}\,.$$

By these relations, we have:

($\star_2$) if $J\mathfrak{s} \subset \mathfrak{s}$, then 4-forms in $\mathfrak{s}^*_{\mathbb{C}}\wedge\mathfrak{s}^*_{\mathbb{C}}\wedge V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}})\wedge V^*_{\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}})$ do not appear in $dd^c(\mathfrak{s}^*_{\mathbb{C}}\wedge\mathfrak{s}^*_{\mathbb{C}})$, $dd^c(\mathfrak{s}^*_{\mathbb{C}}\wedge V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}))$ and $dd^c(V^*_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})\wedge V^*_{\alpha_j}(\mathfrak{h}_{\mathbb{C}}))$, excepting $dd^c(V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}})\wedge V^*_{\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}}))$.

We are going to prove the non-existence of taming symplectic (resp. SKT) structures by showing that for any d-closed (resp. dd^c-closed) 2-form Ω there exists a non-zero $X \in \mathfrak{g}$ such that $\Omega(X, JX) = 0$. We treat the cases $\mathrm{Im}(\alpha_1) \neq 0$ and $Im(\alpha_1) = 0$, separately.

Case 1 : $\mathrm{Im}(\alpha_1) \neq 0$. In this case, we have $V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) \neq V^*_{\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}})$. The condition $J(V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}})) \subset V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}})$ together the assumption $\phi \circ J = J \circ \phi$ implies the existence of a basis $\{e_1, \dots, e_p\}$ of $V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}})$ triangularizing the action of $\mathfrak{s}$ on $V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}})$ and diagonalizing J. The dual basis $\{e^1, \dots, e^p\}$ satisfies

$$de^i = \delta \wedge e^i \mod \mathfrak{s}^*_{\mathbb{C}} \wedge \langle e^1, \dots, e^{i-1}\rangle \oplus \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}}$$

for a closed 1-form $\delta \in \mathfrak{s}^*$. Each e^i could be either a $(1, 0)$-form or a $(0, 1)$-form; therefore $\sqrt{-1}e^i \wedge \overline{e}^i$ is a real $(1, 1)$-form. Since

$$d(e^i \wedge \overline{e}^j) = (\delta + \overline{\delta}) \wedge e^i \wedge \overline{e}^j$$
$$\mod \mathfrak{s}^*_{\mathbb{C}} \wedge \langle e^1, \dots, e^{i-1}\rangle \wedge \langle \overline{e}^j\rangle + \mathfrak{s}^*_{\mathbb{C}} \wedge \langle e^i\rangle \wedge \langle \overline{e}^1, \dots, \overline{e}^{j-1}\rangle + \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}}$$

condition $(\star_1)$, then implies that every closed 2-form has no component along $e^p \wedge \overline{e}^p$. Therefore J cannot be tamed by any symplectic form.

Suppose now that J preserves $\mathfrak{s}$ and $\mathfrak{s}$ is nilpotent. Then we get

$$dd^c(e^i \wedge \overline{e}^j) = (dJ(\delta + \overline{\delta}) - J(\delta + \overline{\delta}) \wedge (\delta + \overline{\delta})) \wedge e^i \wedge \overline{e}^j$$
$$\mod \mathfrak{s}^*_{\mathbb{C}} \wedge \mathfrak{s}^*_{\mathbb{C}} \wedge \langle e^1, \dots, e^{i-1}\rangle \wedge \langle \overline{e}^1, \dots, \overline{e}^j\rangle + \mathfrak{s}^*_{\mathbb{C}} \wedge \mathfrak{s}^*_{\mathbb{C}} \wedge \langle e^1, \dots, e^i\rangle \wedge \langle \overline{e}^1, \dots, \overline{e}^{j-1}\rangle$$
$$+ \mathfrak{s}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} + \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}}.$$

By $\mathrm{Re}(\alpha_1) \neq 0$, we have $\delta + \overline{\delta} \neq 0$ and $d(\delta + \overline{\delta}) = 0$. Hence Lemma 3.1 ensures

$$dJ(\delta + \overline{\delta}) - J(\delta + \overline{\delta}) \wedge (\delta + \overline{\delta}) \neq 0\,.$$

By $(\star_2)$, it follows that every dd^c-closed $(1, 1)$-form has no component along $e^p \wedge \overline{e}^p$ and that consequently J doesn't admit any compatible SKT metric.

Case 2 : $\mathrm{Im}(\alpha_1) = 0$. In this case, we have $V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) = V^*_{\overline{\alpha}_1}(\mathfrak{h}_{\mathbb{C}})$. Since α_1 is real-valued, we have $V^*_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) = V^*_{\alpha_1}(\mathfrak{h}) \otimes \mathbb{C}$. By using $JV^*_{\alpha_1}(\mathfrak{h}) \subset V^*_{\alpha_1}(\mathfrak{h})$ and $\phi \circ J = J \circ \phi$, we can construct a bais $\{e_1, \dots, e_{2p}\}$ such that the action of $\mathfrak{s}$ on $V^*_{\alpha_1}(\mathfrak{h})$ is trigonalized and $Je^{2k-1} = e^{2k}$ for every $k = 1, \dots, p$. For the dual basis $\{e^1, \dots, e^{2p}\}$, we have

$$de^i = \delta \wedge e^i \mod \mathfrak{s}^*_{\mathbb{C}} \wedge \langle e^1, \dots, e^{i-1}\rangle \oplus \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}}$$

for a closed real 1-form $\delta \in \mathfrak{s}^*$. Thus

$$d(e^i \wedge e^j) = 2\delta \wedge e^i \wedge e^j \mod \mathfrak{s}^*_{\mathbb{C}} \wedge \langle e^1, \dots, e^{i-1}\rangle \wedge \langle e^j\rangle + \mathfrak{s}^*_{\mathbb{C}} \wedge \langle e^i\rangle \wedge \langle e^1, \dots, e^{j-1}\rangle + \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}}.$$

By the condition $\mathrm{Re}(\alpha_1) \neq 0$, we obtain $\delta \neq 0$ and every closed 2-form Ω cannot have component along $e^{2p-1} \wedge e^{2p}$. Using $(\star_1)$, we obtain

$$\Omega^{1,1}(e_{2p-1}, J(e_{2p-1})) = \Omega^{1,1}(e_{2p-1}, e_{2p}) = 0$$

and J cannot be tamed by any symplectic form, as required.

Suppose now that J preserves $\mathfrak{s}$ and $\mathfrak{s}$ is nilpotent. Then we get

$$dd^c(e^i \wedge e^j) = 2(dJ\delta - 2J\delta \wedge \delta) \wedge e^i \wedge e^j$$
$$\mod \mathfrak{s}^*_{\mathbb{C}} \wedge \mathfrak{s}^*_{\mathbb{C}} \wedge \langle e^1, \dots, e^{i-1} \rangle \wedge \langle e^1, \dots, e^j \rangle + \mathfrak{s}^*_{\mathbb{C}} \wedge \mathfrak{s}^*_{\mathbb{C}} \wedge \langle e^1, \dots, e^i \rangle \wedge \langle e^1, \dots, e^{j-1} \rangle$$
$$+ \mathfrak{s}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} + \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} \wedge \mathfrak{h}^*_{\mathbb{C}} .$$

By $\mathrm{Re}(\alpha_1) \neq 0$, we have $\delta \neq 0$ and $d\delta = 0$. Hence by Lemma 3.1, we have $dJ\delta - 2J\delta \wedge \delta \neq 0$ and from ($\star_1$) it follows that every dd^c-closed $(1, 1)$-form has no component along $e^{2p-1} \wedge e^{2p}$. Therefore J doesn't admit any compatible SKT metric and the claim follows. □

As a consequence we get the following

COROLLARY 3.2. *Let G/Γ be a complex parallelizable solvmanifold* (*i.e.*, *G is a complex Lie group*). *Suppose that G is non-nilpotent. Then G/Γ does not admit any SKT-structure.*

PROOF. Let $\mathfrak{n}$ be the nilradical of the Lie algebra $\mathfrak{g}$ of G. Take a complex 1-dimensional subspace $\mathfrak{a} \subset \mathfrak{g}$ such that $\mathfrak{a} \cap \mathfrak{n} = \{0\}$ and consider a subspace $\mathfrak{h} \subset \mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{h}$ and $\mathfrak{n} \subset \mathfrak{h}$. Since $\mathfrak{a}$ is a subalgebra of $\mathfrak{g}$ and $\mathfrak{n} \supset [\mathfrak{g}, \mathfrak{g}]$, $\mathfrak{h}$ is an ideal of $\mathfrak{g}$ and we have $\mathfrak{g} = \mathfrak{a} \ltimes \mathfrak{h}$. By $\mathfrak{a} \cap \mathfrak{n} = \{0\}$, the action of $\mathfrak{a}$ on $\mathfrak{h}$ is non-nilpotent and so the action is not of type (I). Hence the corollary follows from Theorem 1.1. □

4. Examples.

In this section we apply Theorem 1.1 to some examples.

EXAMPLE 1. Let $G = \mathbb{C} \ltimes_\phi \mathbb{C}^{2m}$ where

$$\phi(x + \sqrt{-1}y)(w_1, w_2, \dots, w_{2m-1}, w_{2m}) = (e^{a_1 x} w_1, e^{-a_1 x} w_2, \dots, e^{a_m x} w_{2m-1}, e^{-a_m x} w_{2m})$$

for some integers $a_i \neq 0$. We denote by J the natural complex structure on G. Then G admits the left-invariant pseudo-Kähler structure

$$\omega = \sqrt{-1} dz \wedge d\bar{z} + \sum_{i=1}^{m} (dw_{2i-1} \wedge d\overline{w}_{2i} + d\overline{w}_{2i-1} \wedge dw_{2i}) .$$

Moreover G has a co-compact lattice Γ such that $(G/\Gamma, J)$ satisfies the Hodge symmetry and decomposition (see [21]). In view of Theorem 1.1, $(G/\Gamma, J)$ does not admit neither a taming symplectic structure nor an SKT structure. Moreover by Theorem 1.4, G/Γ does not admit an invariant complex structure tamed by any symplectic form.

EXAMPLE 2 (Oeljeklaus-Toma manifolds). Theorem 1.1 can be applied to the family of non-Kähler complex manifolds constructed by Oeljeklaus and Toma in [23]. We brightly describe the construction of these manifolds:

Let K be a finite extension field of $\mathbb{Q}$ with the degree $s + 2t$ for positive integers s, t. Suppose K admits embeddings $\sigma_1, \dots, \sigma_s, \sigma_{s+1}, \dots, \sigma_{s+2t}$ into $\mathbb{C}$ such that $\sigma_1, \dots, \sigma_s$ are real embeddings and $\sigma_{s+1}, \dots, \sigma_{s+2t}$ are complex ones satisfying $\sigma_{s+i} = \overline{\sigma}_{s+i+t}$ for $1 \leq i \leq t$. We can choose K admitting such embeddings (see [23]). Denote $\mathcal{O}_K$ the ring of algebraic integers of K, $\mathcal{O}_K^*$ the group of units in $\mathcal{O}_K$ and

$$\mathcal{O}_K^{*+} = \{a \in \mathcal{O}_K^* : \sigma_i > 0 \text{ for all } 1 \leq i \leq s\} .$$

Define $l : \mathcal{O}_K^{*+} \to \mathbb{R}^{s+t}$ by

$$l(a) = (\log|\sigma_1(a)|, \ldots, \log|\sigma_s(a)|, 2\log|\sigma_{s+1}(a)|, \ldots, 2\log|\sigma_{s+t}(a)|)$$

for $a \in \mathcal{O}_K^{*+}$. Then by Dirichlet's units theorem, $l(\mathcal{O}_K^{*+})$ is a lattice in the vector space $L = \{x \in \mathbb{R}^{s+t} : \sum_{i=1}^{s+t} x_i = 0\}$. Let $p : L \to \mathbb{R}^s$ be the projection given by the first s coordinate functions. Then there exists a subgroup U of $\mathcal{O}_K^{*+}$ of rank s such that $p(l(U))$ is a lattice in $\mathbb{R}^s$. We have the action of $U \ltimes \mathcal{O}_K$ on $H^s \times \mathbb{C}^t$ such that

$$\begin{aligned}(a, b) \cdot (x_1 + \sqrt{-1}y_1, \ldots, x_s + \sqrt{-1}y_s, z_1, \ldots, z_t)& \\ = (\sigma_1(a)x_1 + \sigma_1(b) + \sqrt{-1}\sigma_1(a)y_1, \ldots, \sigma_s(a)x_s + \sigma_s(b) + \sqrt{-1}\sigma_s(a)y_s,& \\ \sigma_{s+1}(a)z_1 + \sigma_{s+1}(b), \ldots, \sigma_{s+t}(a)z_t + \sigma_{s+t}(b))\,.&\end{aligned}$$

In [23] it is proved that the quotient $X(K, U) = H^s \times \mathbb{C}^t / U \ltimes \mathcal{O}_K$ is compact. We call one of these complex manifolds a Oeljeklaus-Toma manifold of type (s, t).

Consider the Lie group $G = \mathbb{R}^s \ltimes_\phi (\mathbb{R}^s \times \mathbb{C}^t)$ with

$$\phi(t_1, \ldots, t_s) = \operatorname{diag}(e^{t_1}, \ldots, e^{t_s}, e^{\psi_1 + \sqrt{-1}\varphi_1}, \ldots, e^{\psi_t + \sqrt{-1}\varphi_t})$$

where $\psi_k = \frac{1}{2}\sum_{i=1}^s b_{ik}t_i$ and $\varphi_k = \sum_{i=1}^s c_{ik}t_i$ for some $b_{ik}, c_{ik} \in \mathbb{R}$. Let $\mathfrak{g}$ be the Lie algebra of G. Then $\bigwedge \mathfrak{g}^*$ is generated by basis $\{\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_s, \gamma_1, \gamma_2, \ldots, \gamma_{2t-1}, \gamma_{2t}\}$ satisfying

$$d\alpha_i = 0,\ d\beta = -\alpha_i \wedge \beta_i\,,$$
$$d\gamma_{2i-1} = \overline{\psi}_i \wedge \gamma_{2i-1} + \overline{\varphi}_i \wedge \gamma_{2i},\ d\gamma_{2i} = -\overline{\varphi}_i \wedge \gamma_{2i-1} + \overline{\psi}_i \wedge \gamma_{2i}\,,$$

where $\overline{\psi}_i = \frac{1}{2}\sum_{i=1}^s b_{ik}\alpha_i$ and $\overline{\varphi}_i = \sum_{i=1}^s c_{ik}\alpha_i$. Consider $w_i = \alpha_i + \sqrt{-1}\beta_i$ for $1 \le i \le s$ and $w_{s+i} = \gamma_{2i-1} + \sqrt{-1}\gamma_{2i}$ as $(1, 0)$-forms. Then $w_1, \ldots, w_{s+t}$ gives a left-invariant complex structure J on G. In [20], it is proved that any Oeljeklaus-Toma manifold of type (s, t) can be regarded as a complex solvmanifold $(G/\Gamma, J)$.

Consider the 2-dimensional Lie algebra $\mathfrak{r}_2 = \operatorname{span}_{\mathbb{R}}\langle A, B\rangle$ such that $[A, B] = B$ and the complex structure $J_{\mathfrak{r}_2}$ on $\mathfrak{r}_2$ defined by the relation $JA = B$. Then the Lie algebra $\mathfrak{g}$ of G splits as $\mathfrak{g} = (\mathfrak{r}_2)^s \ltimes \mathbb{C}^t$ and $J = J_{(\mathfrak{r}_2)^s} \oplus J_{\mathbb{C}^t}$. Hence the first part of Theorem 1.1 implies that G/Γ does not admit Hermitian-symplectic structures.

On the other hand, $(\mathfrak{r}_2)^s$ is not nilpotent and we cannot apply the second part of Theorem 1.1 about the existence of SKT structures. Actually, in the case $s = t = 1$, the corresponding Oeljeklaus-Toma manifold M is a 4-dimensional solvmanifold and by the unimodularity any invariant 3-form is closed forcing M to be SKT. For $s \neq 1$ things work differently:

PROPOSITION 4.1. *Let $s \ge 2$. Then every Oeljeklaus-Toma manifold of type $(s, 1)$ does not admit a SKT structure.*

PROOF. In case $t = 1$, we have $G = \mathbb{R}^s \ltimes_\phi (\mathbb{R}^s \times \mathbb{C})$ where

$$\phi(t_1, \ldots, t_s) = \operatorname{diag}(e^{t_1}, \ldots, e^{t_s}, e^{-\frac{1}{2}(t_1 + \cdots + t_s) + \sqrt{-1}\varphi_1})\,.$$

Then $\bigwedge \mathfrak{g}^*$ is generated by a basis $\{\alpha_1, \ldots, \alpha_s, \beta_1, \ldots, \beta_s, \gamma_1, \gamma_2\}$ satisfying

$$d\alpha_i = 0,\ d\beta = -\alpha_i \wedge \beta_i\,,$$

$$d\gamma_1 = \frac{1}{2}\theta \wedge \gamma_1 + \overline{\varphi}_1 \wedge \gamma_2,\ d\gamma_2 = -\overline{\varphi}_1 \wedge \gamma_1 + \frac{1}{2}\theta \wedge \gamma_2\,,$$

where $\theta = \alpha_1 + \cdots + \alpha_s$ (see [20]). Let us consider the left-invariant $(1,0)$ coframe

$$\begin{aligned} w_i &= \alpha_i + \sqrt{-1}\beta_i\,,\ \text{ for } 1 \le i \le s \\ w_{s+1} &= \gamma_1 + \sqrt{-1}\gamma_2\,. \end{aligned}$$

This coframe induces a global left-invariant coframe on the corresponding Oeljeklaus-Toma manifold $M = G/\Gamma$. We have

$$dd^c(w_{s+1} \wedge \overline{w}_{s+1}) = (dJ\theta - J\theta \wedge \theta) \wedge w_{s+1} \wedge \overline{w}_{s+1}$$

and

$$dJ\theta - J\theta \wedge \theta = -(\alpha_1 \wedge \beta_1 + \cdots + \alpha_s \wedge \beta_s) - (\beta_1 + \cdots + \beta_s) \wedge (\alpha_1 + \cdots + \alpha_s) \neq 0\,.$$

It follows that if Ω is a $(1,1)$-form satisfying $dd^c\Omega = 0$, then Ω has no component along $w_{s+1} \wedge \overline{w}_{s+1}$. This implies that every dd^c-closed $(1,1)$-form on M is degenerate, as require. Hence the proposition follows. □

EXAMPLE 3. In [30] it was introduced the following Lie algebra admitting pseudo-Kähler structures:
Let $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\langle A_i, W_i, X_j, Y_j, Z_j, X'_j, Y'_j, Z'_j\rangle_{i=1,2,\,j=1,2,3,4}$ where

$$\begin{gathered} [A_1, A_2] = W_1\,, \\ [X_1, Y_1] = Z_1, \quad [X_3, Y_3] = Z_3\,, \\ [A_1, X_1] = t_0X_1,\ [A_1, X_2] = t_0X_2\,,\ [A_1, X_3 = -t_0X_3,\ [A_1, X_4] = -t_0X_4\,, \\ [A_1, Y_1] = -2t_0Y_1,\ [A_1, Y_2] = -2t_0Y_2,\ [A_1, Y_3] = 2t_0Y_3,\ [A_1, Y_4] = 2t_0Y_4\,, \\ [A_1, Z_1] = -t_0Z_1,\ [A_1, Z_2] = -t_0Z_2,\ [A_1, Z_3] = t_0Z_3,\ [A_1, Z_4] = t_0Z_4\,, \\ [X_2, Y_1] = Z_2,\ [X_4, Y_3] = Z_4\,, \\ [X'_1, Y'_1] = Z'_1, \quad [X'_3, Y'_3] = Z'_3\,, \\ [A_2, X'_1] = t_0X'_1,\ [A_2, X'_2] = t_0X'_2\,,\ [A_2, X'_3] = -t_0X'_3,\ [A_2, X'_4] = -t_0X'_4\,, \\ [A_2, Y'_1] = -2t_0Y'_1,\ [A_2, Y'_2] = -2t_0Y'_2,\ [A_2, Y'_3] = 2t_0Y'_3,\ [A_2, Y'_4] = 2t_0Y'_4\,, \\ [A_2, Z'_1] = -t_0Z'_1,\ [A_2, Z'_2] = -t_0Z'_2,\ [A_2, Z'_3] = t_0Z'_3,\ [A_2, Z'_4] = t_0Z'_4\,, \\ [X'_2, Y'_1] = Z'_2,\ [X'_4, Y'_3] = Z'_4 \end{gathered}$$

and the other brackets vanish. Then the simply connected solvable Lie group G corresponding to $\mathfrak{g}$ has a lattice (see [30]). We can write $\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\langle A_i, W_i\rangle_{i=1,2} \ltimes \mathrm{span}_{\mathbb{R}}\langle X_j, Y_j, Z_j, X'_j, Y'_j, Z'_j\rangle_{j=1,2,3,4}$ and G has the left-invariant complex structure J defined as

$$\begin{gathered} JA_1 = A_2,\ JW_1 = W_2\,, \\ JX_1 = X_2,\ JY_1 = Y_2,\ JZ_1 = Z_2,\ JX_3 = X_4,\ JY_3 = Y_4,\ JZ_3 = Z_4\,, \\ JX'_1 = X'_2,\ JY'_1 = Y'_2,\ JZ'_1 = Z'_2,\ JX'_3 = X'_4,\ JY_3 = Y_4,\ JZ'_3 = Z'_4\,. \end{gathered}$$

In view of Theorem 1.1, G/Γ does not admit any SKT structure compatible with J.

5. Proof of Theorem 1.2. The proof of Theorem 1.2 is mainly based on the following proposition which is interesting in its own.

PROPOSITION 5.1. *Let G be a simply-connected solvable Lie group whose Lie algebra $\mathfrak{g}$ is not of type* (I). *Let J be a left-invariant complex structure on G satisfying*

$$ad_C \circ J = J \circ ad_C$$

for every C belonging to a nilpotent complement $\mathfrak{c}$ of the nilradical of $\mathfrak{g}$. Then G does not admit any left-invariant symplectic form taming J.

PROOF. By Section 2.2, we have

$$\mathfrak{g}_{\mathbb{C}} = V_{\mathbf{0}}(\mathfrak{g}_{\mathbb{C}}) \oplus V_{\alpha_1}(\mathfrak{g}_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}(\mathfrak{g}_{\mathbb{C}})$$

where $\mathbf{0}$ is the trivial character and $\alpha_1, \ldots, \alpha_n$ are some non-trivial characters. Take a subspace $\mathfrak{a} \subset \mathfrak{c}$ such that $\mathfrak{g} = \mathfrak{a} \oplus \mathfrak{n}$. Then we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{a}_{\mathbb{C}} \oplus V_{\mathbf{0}}(\mathfrak{n}_{\mathbb{C}}) \oplus V_{\alpha_1}(\mathfrak{g}_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}(\mathfrak{g}_{\mathbb{C}})\,.$$

So we obtain

$$[\mathfrak{a}_{\mathbb{C}}, V_{\mathbf{0}}(\mathfrak{n}_{\mathbb{C}})] \subset V_{\mathbf{0}}(\mathfrak{n}_{\mathbb{C}}), \qquad [\mathfrak{a}_{\mathbb{C}}, V_{\alpha_i}(\mathfrak{g}_{\mathbb{C}})] \subset V_{\alpha_i}(\mathfrak{g}_{\mathbb{C}})$$

and

$$J V_{\alpha_i}(\mathfrak{g}_{\mathbb{C}}) \subset V_{\alpha_i}(\mathfrak{g}_{\mathbb{C}}).$$

By Lemma 2.4, we may assume that α_1 satisfies

$$\mathrm{Re}(\alpha_1) \neq 0\,, \quad V_{\alpha_1}(\mathfrak{g}_{\mathbb{C}}) \neq 0 \text{ and } [V_{\alpha_1}(\mathfrak{g}_{\mathbb{C}}), V_{\overline{\alpha}_1}(\mathfrak{g}_{\mathbb{C}})] = 0\,.$$

Consider the natural splitting

$$\bigwedge \mathfrak{g}_{\mathbb{C}}^* = \bigwedge \left(\mathfrak{a}_{\mathbb{C}}^* \oplus V_{\mathbf{0}}^*(\mathfrak{n}_{\mathbb{C}}) \oplus V_{\alpha_1}^*(\mathfrak{g}_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}^*(\mathfrak{g}_{\mathbb{C}})\right)\,.$$

Then we have

$$d(\mathfrak{a}_{\mathbb{C}}^*) = 0$$

and, by taking into account $[\mathfrak{a}, V_{\mathbf{0}}(\mathfrak{n}_{\mathbb{C}})] \subset V_{\mathbf{0}}(\mathfrak{n}_{\mathbb{C}})$, $[\mathfrak{a}, V_{\alpha_i}(\mathfrak{g}_{\mathbb{C}})] \subset V_{\alpha_i}(\mathfrak{g}_{\mathbb{C}})$ and $[V_{\alpha_1}(\mathfrak{g}_{\mathbb{C}}), V_{\overline{\alpha}_1}(\mathfrak{g}_{\mathbb{C}})] = 0$, we get

$$d(V_{\mathbf{0}}^*(\mathfrak{n}_{\mathbb{C}})) \subset \mathfrak{a}_{\mathbb{C}}^* \wedge \mathfrak{a}_{\mathbb{C}}^* \oplus \mathfrak{a}_{\mathbb{C}}^* \wedge V_{\mathbf{0}}^*(\mathfrak{n}_{\mathbb{C}}) \oplus \bigoplus_{(\alpha_k,\alpha_l)\neq(\alpha_1,\overline{\alpha}_1)} V_{\alpha_k}^*(\mathfrak{g}_{\mathbb{C}}) \wedge V_{\alpha_l}^*(\mathfrak{g}_{\mathbb{C}}) \oplus \bigoplus V_{\alpha_m}^*(\mathfrak{g}_{\mathbb{C}}) \wedge V_{\mathbf{0}}^*(\mathfrak{n}_{\mathbb{C}})\,,$$

and

$$d(V_{\alpha_i}^*(\mathfrak{g}_{\mathbb{C}})) \subset \mathfrak{a}_{\mathbb{C}}^* \wedge \mathfrak{a}_{\mathbb{C}}^* \oplus \mathfrak{a}_{\mathbb{C}}^* \wedge V_{\alpha_i}^*(\mathfrak{g}_{\mathbb{C}}) \oplus \bigoplus_{(\alpha_k,\alpha_l)\neq(\alpha_1,\overline{\alpha}_1)} V_{\alpha_k}^*(\mathfrak{g}_{\mathbb{C}}) \wedge V_{\alpha_l}^*(\mathfrak{g}_{\mathbb{C}}) \oplus \bigoplus V_{\beta_m}^*(\mathfrak{g}_{\mathbb{C}}) \wedge V_{\mathbf{0}}^*(\mathfrak{n}_{\mathbb{C}})\,.$$

Hence we have

$$d(\mathfrak{a}_{\mathbb{C}}^* \wedge \mathfrak{a}_{\mathbb{C}}^*) = 0,$$

and

$$d(\mathfrak{a}^*_{\mathbb{C}}\wedge V^*_{\alpha_i}(\mathfrak{g}_{\mathbb{C}}))\subset \mathfrak{a}^*_{\mathbb{C}}\wedge\mathfrak{a}^*_{\mathbb{C}}\wedge\mathfrak{a}^*_{\mathbb{C}}\oplus\mathfrak{a}^*_{\mathbb{C}}\wedge\mathfrak{a}^*_{\mathbb{C}}\wedge V^*_{\alpha_i}(\mathfrak{g}_{\mathbb{C}})$$
$$\oplus\bigoplus_{(\alpha_k,\alpha_l)\neq(\alpha_1,\overline{\alpha}_1)}\mathfrak{a}^*_{\mathbb{C}}\wedge V^*_{\alpha_k}(\mathfrak{g}_{\mathbb{C}})\wedge V^*_{\alpha_l}(\mathfrak{g}_{\mathbb{C}})\oplus\bigoplus\mathfrak{a}^*_{\mathbb{C}}\wedge V^*_{\alpha_m}(\mathfrak{g}_{\mathbb{C}})\wedge V^*_{\mathbf{0}}(\mathfrak{n}_{\mathbb{C}})$$

and

$$d(V^*_{\alpha_i}(\mathfrak{g}_{\mathbb{C}})\wedge V^*_{\alpha_j}(\mathfrak{g}_{\mathbb{C}}))\subset\mathfrak{a}^*_{\mathbb{C}}\wedge\mathfrak{a}^*_{\mathbb{C}}\wedge V^*_{\alpha_i}(\mathfrak{g}_{\mathbb{C}})\oplus\mathfrak{a}^*_{\mathbb{C}}\wedge\mathfrak{a}^*_{\mathbb{C}}\wedge V^*_{\alpha_j}(\mathfrak{g}_{\mathbb{C}})$$
$$\oplus\,\mathfrak{a}^*_{\mathbb{C}}\wedge V^*_{\alpha_i}(\mathfrak{g}_{\mathbb{C}})\wedge V^*_{\alpha_j}(\mathfrak{g}_{\mathbb{C}})\oplus\mathfrak{n}^*_{\mathbb{C}}\wedge\mathfrak{n}^*_{\mathbb{C}}\wedge\mathfrak{n}^*_{\mathbb{C}}\,.$$

Combining these relations we have:

($\diamond$) 3-forms in $\mathfrak{a}^*_{\mathbb{C}}\wedge V^*_{\alpha_1}(\mathfrak{g}_{\mathbb{C}})\wedge V^*_{\overline{\alpha}_1}(\mathfrak{g}_{\mathbb{C}})$ do not appear in $d(\mathfrak{a}^*_{\mathbb{C}}\wedge\mathfrak{a}^*_{\mathbb{C}})$, $d(\mathfrak{a}^*_{\mathbb{C}}\wedge V^*_{\alpha_i}(\mathfrak{g}_{\mathbb{C}}))$ and $d(V^*_{\alpha_i}(\mathfrak{g}_{\mathbb{C}})\wedge V^*_{\alpha_j}(\mathfrak{g}_{\mathbb{C}}))$, excepting $d(V^*_{\alpha_1}(\mathfrak{g}_{\mathbb{C}})\wedge V^*_{\overline{\alpha}_1}(\mathfrak{g}_{\mathbb{C}}))$.

The non-existence of taming symplectic structures will be obtained by showing that for any d-closed 2-form Ω there exists a non-trivial $X\in\mathfrak{g}$ such that $\Omega(X,JX)=0$. From now on, we distinguishe the case where $\mathrm{Im}(\alpha_1)\neq 0$ from the case $\mathrm{Im}(\alpha_1)=0$.

Case 1 : $\mathrm{Im}(\alpha_1)\neq 0$. In this case we have $V^*_{\alpha_1}(\mathfrak{g}_{\mathbb{C}})\neq V^*_{\overline{\alpha}_1}(\mathfrak{g}_{\mathbb{C}})$. Since $J(V^*_{\alpha_1}(\mathfrak{g}_{\mathbb{C}}))\subset V^*_{\alpha_1}(\mathfrak{g}_{\mathbb{C}})$, there exists a basis $\{e_1,\dots,e_p\}$ such that the action of $\mathfrak{c}$ onto $V^*_{\alpha_1}(\mathfrak{g}\otimes\mathbb{C})$ is trigonalized and J is diagonalized. The dual basis $\{e^1,\dots,e^p\}$ satisfies

$$de^i=\delta\wedge e^i\mod \mathfrak{a}^*_{\mathbb{C}}\wedge\langle e^1,\dots,e^{i-1}\rangle\oplus\mathfrak{n}^*_{\mathbb{C}}\wedge\mathfrak{n}^*_{\mathbb{C}}$$

for a complex closed form $\delta\in\mathfrak{a}^*_{\mathbb{C}}$. Each e^i is either a $(1,0)$ or a $(0,1)$-form and so $\sqrt{-1}e^i\wedge\overline{e}^i$ is a real $(1,1)$-form. Therefore

$$d(e^i\wedge\overline{e}^j)=(\delta+\overline{\delta})\wedge e^i\wedge e^j$$
$$\mod \mathfrak{a}^*_{\mathbb{C}}\wedge\langle e^1,\dots,e^{i-1}\rangle\wedge\langle\overline{e}^j\rangle+\mathfrak{a}^*_{\mathbb{C}}\wedge\langle e^i\rangle\wedge\langle\overline{e}^1,\dots,\overline{e}^{j-1}\rangle+\mathfrak{n}^*_{\mathbb{C}}\wedge\mathfrak{n}^*_{\mathbb{C}}\wedge\mathfrak{n}^*_{\mathbb{C}}\,.$$

By $\mathrm{Re}(\alpha_1)\neq 0$, we have $\delta+\overline{\delta}\neq 0$. Hence ($\diamond$) implies that every closed 2-form Ω has no component along $e^p\wedge\overline{e}^p$. Hence

$$\Omega^{1,1}(e_p+\overline{e}_p,J(e_p+\overline{e}_p))=\Omega^{1,1}(e_p+\overline{e}_p,\sqrt{-1}(e_p-\overline{e}_p))=0$$

and J cannot be tamed by any symplectic form.

Case 2 : $\mathrm{Im}(\alpha_1)=0$. In this case we have $V^*_{\alpha_1}(\mathfrak{g}_{\mathbb{C}})=V^*_{\overline{\alpha}_1}(\mathfrak{g}_{\mathbb{C}})$. Since α_1 is real-valued, we have $V^*_{\alpha_1}(\mathfrak{g}_{\mathbb{C}})=V^*_{\alpha_1}(\mathfrak{g})\otimes\mathbb{C}$. Since $JV^*_{\alpha_1}(\mathfrak{g})\subset V^*_{\alpha_1}(\mathfrak{g})$ and $ad_C\circ J=J\circ ad_C$ for any $C\in\mathfrak{c}$ there exists a basis $\{e_1,\dots,e_{2p}\}$ such that the action of $\mathfrak{c}$ on $V^*_{\alpha_1}(\mathfrak{g})$ is trigonalized and $Je^{2k-1}=e^{2k}$ for each k. Let $\{e^1,\dots,e^{2p}\}$ be the dual basis. Then

$$de^i=\delta\wedge e^i\mod \mathfrak{a}^*_{\mathbb{C}}\wedge\langle e^1,\dots,e^{i-1}\rangle\oplus\mathfrak{n}^*_{\mathbb{C}}\wedge\mathfrak{n}^*_{\mathbb{C}}$$

for a real closed form $\delta\in\bigwedge\mathfrak{a}^*$. Hence we have

$$d(e^i\wedge e^j)=2\delta\wedge e^i\wedge e^j\mod \mathfrak{a}^*_{\mathbb{C}}\wedge\langle e^1,\dots,e^{i-1}\rangle\wedge\langle e^j\rangle+\mathfrak{a}^*_{\mathbb{C}}\wedge\langle e^i\rangle\wedge\langle e^1,\dots,e^{j-1}\rangle+\mathfrak{n}^*_{\mathbb{C}}\wedge\mathfrak{n}^*_{\mathbb{C}}\wedge\mathfrak{n}^*_{\mathbb{C}}.$$

By $\mathrm{Re}(\alpha_1) \neq 0$, we have $\delta \neq 0$. Hence by ($\diamond$), every closed 2-form Ω has no component along $e^{2p-1} \wedge e^{2p}$. Hence we have

$$\Omega^{1,1}(e_{2p-1}, Je_{2p-1}) = \Omega^{1,1}(e_{2p-1}, e_{2p}) = 0$$

and J cannot be tamed by any symplectic form, as required. □

Now we are ready to prove Theorem 1.2.

Proof of Theorem 1.2. In view of [15] the existence of a symplectic form taming J implies the existence of an *invariant* symplectic form taming J. Hence it is enough to prove that there are no invariant symplectic forms taming J. By Proposition 5.1, the Lie algebra $\mathfrak{g}$ is not of type (I). Given a nilpotent complement $\mathfrak{c} \subset \mathfrak{g}$, we define the diagonal representation

$$ad_s : \mathfrak{g} = \mathfrak{c} + \mathfrak{n} \ni C + X \mapsto (ad_C)_s \in D(\mathfrak{g}) .$$

Consider the extension $Ad_s : G \to Aut(\mathfrak{g})$. Then the Zariski-closure $T = \mathcal{A}(Ad_s(G))$ in $Aut(\mathfrak{g})$ is a maximal torus of the Zariski-closure $\mathcal{A}(Ad(G))$ (see [19] and [9]). It is known that there exists a simply-connected nilpotent Lie group U_G, called the *nilshadow* of G, which is independent on the choice of T and satisfies $T \ltimes G = T \ltimes U_G$. From [9] it follows that if J is a left-invariant complex structure on G satisfying $J \circ Ad_s = Ad_s \circ J$, then U_G inherits a left-invariant complex structure $\tilde{J}$ such that $(U_G, \tilde{J})$ is bi-holomorphic to (G, J). Now every lattice of G induces a discrete subgroup Γ in $T \ltimes U_G$ such that $\tilde{\Gamma} = U_G \cap \Gamma$ is a lattice of U_G and has finite index in Γ (see [3, Chapter V-5]). There follows that $(G/\tilde{\Gamma}, J)$ is bi-holomorphic to $(U_G/\tilde{\Gamma}, \tilde{J})$. Hence $U_G/\tilde{\Gamma}$ is a finite covering of a Hermitian-symplectic manifold and, consequently, it inherits an invariant symplectic form $\tilde{\Omega}$ taming $\tilde{J}$. By the main result of [15] it follows that $U_G/\tilde{\Gamma}$ is a torus. Hence $(G/\Gamma, J)$ is a finite quotient of a complex torus $U_G/\tilde{\Gamma}$ by a finite group of holomorphic automorphisms and by [5], $(G/\Gamma, J)$ admits a Kähler metric. □

6. Abelian complex structures.

In this section we consider abelian complex structures providing a proof of Theorem 1.3.

Theorem 1.3 is mainly motivated by the research in [2] where it is showed that a Lie group with a left-invariant abelian complex structure admits a compatible left-invariant Kähler structure if and only if it is a direct product of several copies of the real hyperbolic plane by an Euclidean factor. Moreover, from [2, Lemma 2.1] it follows that a Lie algebra $\mathfrak{g}$ with an abelian complex structure J has the following properties:

1. the center $\xi(\mathfrak{g})$ of $\mathfrak{g}$ is J-invariant;
2. for any $X \in \mathfrak{g}$, $ad_{JX} = -ad_X J$;
3. the commutator $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ is abelian or, equivalently, $\mathfrak{g}$ is 2-step solvable;
4. $J\mathfrak{g}^1$ is an abelian subalgebra of $\mathfrak{g}$;
5. $\mathfrak{g}^1 \cap J\mathfrak{g}^1$ is contained in the center of the subalgebra $\mathfrak{g}^1 + J\mathfrak{g}^1$.

Our Theorem 1.3 can be easily deduced in dimensions 4 and 6 by using the classification of Lie algebras admitting an abelian complex structure. Indeed, by the classifications in dimensions 4 ([27]) and 6 ([1]) we know that if $(\mathfrak{g}, J)$ is a unimodular Lie algebra with an abelian complex structure, then the existence of a symplectic form taming J implies that $\mathfrak{g}$ is abelian. In dimension 4 this fact follows from [14]. In dimension 6 we use that the only unimodular (non-nilpotent) Lie algebra admitting an abelian complex structure is holomorphically isomorphic to $(\mathfrak{s}_{(-1,0)}, J)$, where $\mathfrak{s}_{(-1,0)}$ is the solvable Lie algebra with Lie brackets

$$
\begin{aligned}
&[f_1, e_1] = [f_2, e_2] = e_1\,, \quad [f_1, e_2] = -[f_2, e_1] = e_2\,,\\
&[f_1, e_3] = [f_2, e_4] = -e_3\,, \quad [f_1, e_4] = -[f_2, e_3] = -e_4
\end{aligned}
$$

and the abelian complex structure J is given by

$$
Jf_1 = f_2,\ Je_1 = e_2,\ Je_3 = e_4\,.
$$

This Lie algebra has nilradical $\mathfrak{n} = \mathrm{span}_{\mathbb{R}}\langle e_1, e_2, e_3, e_4\rangle$ and $ad_c \circ J = J \circ ad_c$, for every $c \in \mathfrak{c} = \langle f_1, f_2\rangle$. Since $\mathfrak{c}$ is an abelian complement of $\mathfrak{n}$, Theorem 5.1 implies that $(\mathfrak{s}_{(-1,0)}, J)$ does not admit any symplectic form taming J.

Theorem 1.3 follows from the following

PROPOSITION 6.1. *Let $(\mathfrak{g}, J)$ be a unimodular Lie algebra with an abelian complex structure. Assume that there exists a symplectic form Ω on $\mathfrak{g}$ taming J. Then $\mathfrak{g}$ is abelian.*

PROOF. Since the pair (J, Ω) induces a Hermitian symplectic structure on every J-invariant subalgebra of $\mathfrak{g}$ and $\mathfrak{g}^1$ and $J\mathfrak{g}^1$ are both abelian Lie subalgebras of $\mathfrak{g}$, it is quite natural to work with $\mathfrak{g}^1 + J\mathfrak{g}^1$. We have the following two cases which we will treat separately:

Case A : $\mathfrak{g}^1 + J\mathfrak{g}^1 = \mathfrak{g}$

Case B : $\mathfrak{g}^1 + J\mathfrak{g}^1 \neq \mathfrak{g}$.

In the Case A we necessary have $\mathfrak{g}^1 \cap J\mathfrak{g}^1 = \{0\}$, since otherwise by using that $\mathfrak{g}^1 \cap J\mathfrak{g}^1 \subseteq \xi(\mathfrak{g})$, it should exist a non-zero $X \in J\xi(\mathfrak{g}) \cap \mathfrak{g}^1$, but this contradicts Lemma 3.1 in [15]. Therefore

$$
\mathfrak{g} = \mathfrak{g}^1 \oplus J\mathfrak{g}^1\,,
$$

or equivalently $\mathfrak{g}$ is an abelian double product. As a consequence of Corollary 3.3 in [2] the Lie bracket in $\mathfrak{g}$ induces a structure of commutative and associative algebra on $\mathfrak{g}^1$ given by

$$
X \cdot Y = [JX, Y]\,.
$$

Let $\mathcal{A} := (\mathfrak{g}^1, \cdot)$. Then $\mathcal{A}^2 = \mathcal{A}$ and $(\mathfrak{g}, J)$ is holomorphically isomorphic to $\mathrm{aff}(\mathcal{A}) = \mathcal{A} \oplus \mathcal{A}$ with the standard complex structure

$$
J(X, Y) = (Y, -X)\,.
$$

Note that in general the Lie bracket on the affine Lie algebra $\mathrm{aff}(\mathcal{A})$ associated to a commutative associative algebra $(\mathcal{A}, \cdot)$ is given by

$$
[(x, y), (x', y')] = (0, x \cdot y' - x' \cdot y)\,,
$$

for every $(x, y), (x', y') \in \mathrm{aff}(\mathcal{A})$. Moreover, $\mathrm{aff}(\mathcal{A})$ is nilpotent if and only if $\mathcal{A}$ is nilpotent as associative algebra. We are going to show now that when $\mathrm{aff}(\mathcal{A})$ is unimodular and it is endowed with a symplectic form taming J, then the Lie algebra $\mathrm{aff}(\mathcal{A})$ is forced to be abelian.

Since we know that this is true in dimension 4 and 6 we can prove the assertion by induction on the dimension of $\mathcal{A}$. We may assume that $\mathcal{A}$ is not a direct sum of proper non-trivial ideals, since otherwise if $\mathcal{A} = \mathcal{A}_1 \oplus \cdots \oplus A_k$, then $\mathrm{aff}(\mathcal{A}) = \mathrm{aff}(\mathcal{A}_1) \oplus \cdots \oplus \mathrm{aff}(\mathcal{A}_k)$ and by induction we obtain that any $\mathrm{aff}(\mathcal{A}_k)$ is abelian. Since $\mathcal{A}$ is a commutative associative algebra over $\mathbb{R}$, by applying Lemma 3.1 in [6], we get that $\mathcal{A}$ is either

(i) nilpotent, or
(ii) equal to $\tilde{\mathcal{B}} = \mathcal{B} \oplus \mathbb{R}\langle 1 \rangle$ for a nilpotent commutative associative algebra $\mathcal{B}$, where by 1 we denote the unit of $\mathcal{A}$ or
(iii) equal to $\mathbb{C} \oplus \mathcal{R}$, where $\mathcal{R}$ is the radical of $\mathcal{A}$.

Since $\mathrm{aff}(\mathbb{C})$ is not unimodular then we can exclude the case (iii). Moreover, in the case (ii) $\mathrm{aff}(\mathcal{A})$ cannot be unimodular, since

$$[(1,0),(x',y')] = (0,y')\,,$$

for every $(x',y') \in \mathrm{aff}(\mathcal{A})$. In particular, $[(1,0),(0,1)] = (0,1)$ and then trace $(ad_{(1,0)}) \neq 0$. We conclude then that the Lie algebra $\mathrm{aff}(\mathcal{A})$ has to be nilpotent and by [15] $\mathrm{aff}(\mathcal{A})$ has to be abelian, since it is Hermitian-symplectic.

Let us consider now the Case B in which $\mathfrak{g}^1 + J\mathfrak{g}^1$ is a proper ideal of $\mathfrak{g}$. By induction on the dimension we may assume that $\mathfrak{g}^1 + J\mathfrak{g}^1$ is abelian. Fix an arbitrary J-invariant complement $\mathfrak{h}$ of $\mathfrak{g}^1 + J\mathfrak{g}^1$. We show that $[\mathfrak{h}, \mathfrak{g}^1 + J\mathfrak{g}^1] = 0$ proving in this way that $\mathfrak{g}$ is nilpotent. Fix $X \in \mathfrak{h}$ and consider the following two bilinear forms on $\mathfrak{g}^1 + J\mathfrak{g}^1$

$$B_X(Y,Z) := \Omega([X,Y],Z)\,, \quad B'_X(Y,Z) := \Omega([JX,Y],Z)\,.$$

Since Ω is closed and $\mathfrak{g}^1 + J\mathfrak{g}^1$ is abelian, the two bilinear forms B_X and B'_X are both symmetric. On the other hand the abelian condition on J ensures that

$$B'_X(Y,Z) = -B_X(JY,Z)\,,$$

for every $Y, Z \in \mathfrak{g}^1 + J\mathfrak{g}^1$. Thus

$$\begin{aligned} B_X(JY,JZ) &= \Omega([X,JY],JZ) = -\Omega([JX,Y],JZ) \\ &= -B'_X(Y,JZ) = -B'_X(JZ,Y) = -\Omega([JX,JZ],Y) \\ &= -\Omega([X,Z],Y) = -B_X(Y,Z)\,, \end{aligned}$$

for every $Y, Z \in \mathfrak{g}^1 + J\mathfrak{g}^1$ or, equivalently,

$$\Omega([X,JY],JZ) = -\Omega([X,Y],Z)\,, \quad \forall\, Y,Z \in \mathfrak{g}^1 + J\mathfrak{g}^1\,.$$

In particular

$$\Omega([X,JY],J[X,JY]) = \Omega([X,Y],[JX,Y])\,, \quad \forall\, Y,Z \in \mathfrak{g}^1 + J\mathfrak{g}^1\,.$$

We finally show that $\Omega([X,Y],[JX,Y]) = 0$ obtaining in this way $[X,JY] = 0$.

Indeed,

$$\begin{aligned} \Omega([X,Y],[JX,Y]) &= \Omega([X,[JX,Y]],Y) \\ &= -\Omega([Y,[X,JX]],Y) - \Omega([JX,[Y,X]],Y) \\ &= -\Omega([X,Y],[JX,Y]), \end{aligned}$$

which implies $\Omega([X,Y],[JX,Y]) = 0$, as required. Therefore $[\mathfrak{h}, \mathfrak{g}^1 + J\mathfrak{g}^1] = 0$ and $\mathfrak{g}$ is nilpotent. Finally Theorem 1.3 in [15] implies that $\mathfrak{g}$ is abelian, as required. □

7. Almost-abelian solvmanifolds. By [24] a 4-dimensional unimodular Hermitian symplectic Lie algebra $\mathfrak{g}$ is Kähler and it is isomorphic to the almost abelian Lie algebra $\tau\tau'_{3,0}$ with structure equations

$$[e_1, e_2] = -e_3, \quad [e_1, e_3] = e_2 .$$

Note that indeed a 4-dimensional unimodular (non abelian) Lie algebra $\mathfrak{g}$ is symplectic if and only if it is isomorphic either to the 3-step 4-dimensional nilpotent Lie algebra or to a direct product of $\mathbb{R}$ with a 3-dimensional unimodular solvable Lie algebra.

The proof of Theorem 1.4 is implied by the two subsequent propositions. The first one implies the statement of Theorem 1.4 when $\mathfrak{g}$ is not of type (I).

PROPOSITION 7.1. *Let J be a complex structure on a unimodular almost abelian* (*non-abelian*) *Lie algebra* $\mathfrak{g}$. *If* $\mathfrak{g}$ *is not of type* (I), *then* $\mathfrak{g}$ *does not admit a symplectic structure taming J.*

PROOF. Let $\mathfrak{n}$ be the nilradical of $\mathfrak{g}$. Since $\mathfrak{g}$ is almost abelian we have that $\mathfrak{n}$ has codimension 1 and $\mathfrak{n}$ is abelian. Let Ω be a symplectic form taming J and g the associated J-Hermitian metric. We recall that this metric is defined as the Hermitian metric induced by $(1,1)$-component $\Omega^{1,1}$ of Ω. With respect to the Hermitian metric g we have the orthogonal decomposition

$$\mathfrak{g} = \mathfrak{n} \oplus \mathrm{span}_{\mathbb{R}}\langle X\rangle .$$

Since JX is orthogonal to X, JX belongs to $\mathfrak{g}^1$ and thus $JX \in \mathfrak{n}$. By the unimodularity of $\mathfrak{g}$, we get that $[X, JX]$ belongs to the orthogonal complement of $\mathrm{span}_{\mathbb{R}}\langle X, JX\rangle$ with respect to g, i.e., to the J-invariant abelian Lie subalgebra

$$\mathfrak{h} = \mathrm{span}_{\mathbb{R}}\langle X, JX\rangle^{\perp} .$$

Since $\mathfrak{n}$ is abelian, by using the integrability of J we obtain

$$ad_X(JY) = J ad_X(Y) ,$$

for every $Y \in \mathfrak{h}$. We can show that $\mathfrak{h}$ is ad_X-invariant. Indeed, we know that

$$g([X,Y],X) = 0, \text{ for every } Y \in \mathfrak{h} ,$$

or equivalently

$$\Omega(J[X,Y],X) = \Omega([X,Y],JX), \text{ for every } Y \in \mathfrak{h} . \tag{7.1}$$

Using $J(ad_X(Y)) = ad_X(JY)$ we have

$$\Omega(J[X,Y],JX) = \Omega([X,JY],JX)\,.$$

By (7.1) it follows that

$$\Omega([X,JY],JX) = \Omega(J[X,JY],X) = -\Omega([X,Y],X)\,,$$

i.e., $g([X,Y],JX) = 0$, for every $Y \in \mathfrak{h}$. By Section 2.2, we have the decomposition

$$\mathfrak{h}_{\mathbb{C}} = V_{\mathbf{0}}(\mathfrak{h}_{\mathbb{C}}) \oplus V_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}(\mathfrak{h}_{\mathbb{C}})$$

where $\mathbf{0}$ is the trivial character and $\alpha_1,\dots,\alpha_n$ are some non-trivial characters. Therefore

$$\mathfrak{g}_{\mathbb{C}} = \langle X, JX\rangle \oplus V_{\alpha_1}(\mathfrak{h}_{\mathbb{C}}) \oplus \cdots \oplus V_{\alpha_n}(\mathfrak{h}_{\mathbb{C}})$$

with

$$[X, V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})] \subset V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}), \quad [JX, V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})] = 0$$

and

$$JV_{\alpha_i}(\mathfrak{h}_{\mathbb{C}}) \subset V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})\,.$$

Thus

$$\bigwedge \mathfrak{g}^*_{\mathbb{C}} = \Lambda\langle x, Jx\rangle \otimes \Lambda(V^*_{\mathbf{0}}(\mathfrak{n}_{\mathbb{C}}) \oplus V^*_{\alpha_1}(\mathfrak{g}_{\mathbb{C}}) \oplus \cdots \oplus V^*_{\alpha_n}(\mathfrak{g}_{\mathbb{C}}))\,,$$

where x denotes the dual of X. Since $\mathfrak{g}$ is not of type (I), then there exists $\xi \in V_{\alpha_i}$ such that

$$J\xi = i\xi, \quad d\xi = a_i\xi \wedge x + \beta_i \wedge x\,,$$

with $\mathrm{Re}(a_i) \neq 0$ and $\beta_i \in V_{\alpha_i}(\mathfrak{h}_{\mathbb{C}})$ such that $\beta_i \wedge \xi = 0$. Therefore $x \wedge \xi \wedge \overline{\xi}$ can appear only in $d(\xi \wedge \overline{\xi})$, but this implies then that $\Omega(Z, JZ) = 0$, where $Z - iJZ$ is the dual of ξ. □

REMARK 7.2. Theorem 1.4 can be generalized to (I)-type Lie algebras by introducing some extra assumptions on J. Indeed, if (Ω, J) is a Hermitian-symplectic structure on a unimodular almost-abelian Lie algebra $\mathfrak{g}$ of type I, then we still have the orthogonal decomposition with respect to the metric g induced by $\Omega^{1,1}$

$$\mathfrak{g} = \mathrm{span}_{\mathbb{R}}\langle X, JX\rangle \oplus \mathfrak{h}\,, \tag{7.2}$$

with $[X, JX] \in \mathfrak{h}$, $\mathfrak{h}$ abelian and $ad_X(\mathfrak{h}) \subseteq \mathfrak{h}$. So in particular, $\mathfrak{g}^1 \subseteq \mathfrak{h}$ and $dx = 0 = d(Jx)$. Therefore if for instance we require that $[X, JX] = 0$, then $\mathfrak{c} = \langle X\rangle$ is an abelian complement of $\mathfrak{n}$ and J is $\mathfrak{c}$-invariant. So if the associated simply-connected Lie group G has a lattice, we can apply Theorem 1.2 obtaining that $(G/\Gamma, J)$ is Kähler.

Using Proposition 7.1 and the previous remark we can prove the following

THEOREM 7.3. *Let G/Γ be a 6-dimensional solvmanifold endowed with a left-invariant complex structure J. If G is almost abelian and G/Γ admits a symplectic structure taming J, then G/Γ admits a Kähler structure.*

PROOF. If G is not of type (I), then the result follows by Proposition 7.1. Suppose that G is of type (I). By previous remark we have the orthogonal decomposition (7.2) with $[X, JX] \in \mathfrak{h}$, $\mathfrak{h}$ abelian and $ad_X(\mathfrak{h}) \subseteq \mathfrak{h}$.

If $[X, JX] = 0$, the result follows applying Theorem 1.2. Suppose that $Y = [X, JX] \neq 0$. Since $Y \in \mathfrak{h}$, we have that X, JX, Y, JY are linearly independent and they generate a 4-dimensional subspace of $\mathfrak{g}$.

If $[X, Y] \in \mathrm{span}_{\mathbb{R}}\langle Y, JY\rangle$, then $\mathfrak{k} = \mathrm{span}_{\mathbb{R}}\langle X, JX, Y, JY\rangle$ is a 4-dimensional Lie subalgebra of $\mathfrak{g}$. Since $\mathfrak{k}$ is J-invariant, then $\mathfrak{k}$ admits a Hermitian-symplectic structure. The result follows from the fact the $\mathfrak{k}$ is unimodular and then it has to be isomorphic to $\tau\tau'_{3,0}$, but if $[X, JX] \neq 0$ this is not possible.

If $[X, Y]$ does not belong to $\mathrm{span}_{\mathbb{R}}\langle Y, JY\rangle$, then

$$\{X, JX, Y = [X, JX], JY, Z = [X, Y], JZ\}$$

is a basis of $\mathfrak{g}$. Note that $JZ = [X, JY]$. Let $\{x, Jx, y, Jy, z, Jz\}$ be the dual basis of $\{X, JX, YJY, Z, JZ\}$. We have that $\mathfrak{g}$ has structure equations

$$\begin{cases} dx = 0\,, \\ d(Jx) = 0, \\ dy = -x \wedge Jx\,, \\ d(Jy) = x \wedge (az + bJz)\,, \\ dz = -x \wedge y\,, \\ d(Jz) = -x \wedge Jy\,, \end{cases}$$

with $a, b \in \mathbb{R}$. Then, by a direct computation one has that

$$d(z \wedge Jz) = -x \wedge y \wedge Jz + z \wedge x \wedge Jy$$

and that the term $z \wedge x \wedge Jy$ can appear only in $d(z \wedge Jz)$. Therefore, we must have $\Omega(Z, JZ) = 0$. □

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